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STABILITY OF MINIMAL SOLUTION MAPPINGS FOR PARAMETRIC SET OPTIMIZATION PROBLEMS WITH PRE-ORDER RELATIONS*

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Dedicated to Professor Guang-Ya Chen on the occasion of his 80th birthday.

Abstract: In this paper, we establish the continuity and compactness of minimal solution mappings for parametric set optimization problems with the general pre-order relations under some suitable conditions. As applications, we obtain the continuity and compactness of minimal solution mappings for parametric set optimization problems with the order induced by a convex cone.

Key words: *parametric set optimization problem, minimal solution, pre-ordered relation, semi-continuity.*

Mathematics Subject Classification: *90C27, 90C31, 90C48*

1 Introduction

It is well known that the set optimization theory is a very powerful tool to many branches of pure and applied mathematics such as mathematical economics and finance, vector optimization, vector equilibrium, image processing, viability theory, optimal control and fuzzy optimization, see [3–5, 8–10, 12, 14, 15, 19, 28–33, 35, 38–40] and the references therein. Recently, various theoretical results with applications have been studied extensively by many authors for set optimization problems in the literature; for instance, we refer the reader to [2, 13, 16–18, 20–22, 24–27, 34].

One of the most important problems for set optimization problems is to investigate the stability of solution mappings. Among many desirable stability of solution mappings for set optimization problems, the semi-continuity of solution mappings for parametric set optimization problems is of considerable interest. Recently, Xu and Li [36] studied the upper and lower semi-continuity and closedness of the minimal solution set mappings for a parametric set optimization problem with the upper set less order relation under strong conditions. Xu and Li [37] introduced the concept of a u-lower level mapping and obtained the upper and lower semi-continuity for the u-lower level mappings. They also gave some

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suitable assumptions for the continuity of the minimal solution mappings to a parametric set optimization problem via the upper set less order relation. Han and Huang [11] obtained some characterizations of the generalized l-B-well-posedness and u-B-well-posedness for set optimization. They also investigated the upper and lower semi-continuity of the minimal solution mapping for set optimization problems with the upper and lower set less order relation. Very recently, Khoshkhabar-amiranloo [23] gave some sufficient conditions for ensuring the upper semi-continuity, lower semi-continuity and compactness of the minimal solutions to set optimization problems with the lower set less order relation. Han and Huang [13] established the upper and lower semi-continuity of strongly approximate solutions for parametric set optimization problems by using the nonlinear scalarizing function. Chen et al. [6] discussed the upper and lower semi-continuity of the strict l-lower level mappings and established the continuity of the strict minimal solution mappings to parametric set optimization problems under some mild conditions. However, to our best knowledge, there are few papers dealing with the upper and lower semi-continuity of minimal solutions for parametric set optimization problems with the general pre-order relations. The main purpose of this paper is to establish the continuity of the minimal solutions mappings of parametric set optimization problems with the general pre-order relations. Moreover, some applications to parametric set optimization problems with the order induced by a convex cone are also given. The results presented in this paper improve and generalize some corresponding ones in [11, 23, 36, 37].

The rest of this paper is organized as follows: Section 2 gives some necessary definitions and notations. In Section 3, we give some sufficient conditions for ensuring the continuity and compactness of the minimal solution mappings of parametric set optimization problems with the general pre-order relations. As applications, we obtain the stability of minimal solution mappings for parametric set optimization problems with the order induced by a convex cone. This paper ends with conclusions and future work in Section 4.

2 Preliminaries

Throughout this paper, assume that X, Y and Z are normed vector spaces. Let $\mathcal{P}(Y)$ denote the family of all nonempty subsets of Y with a binary relation $\preccurlyeq$. Moreover, define a binary relation $\prec$ on $\mathcal{P}(Y)$ as follows: for any $A, B \in \mathcal{P}(Y)$,

$$A \prec B \Leftrightarrow A \preccurlyeq B \text{ and } A \neq B.$$

Definition 2.1. Let $A, B, C \in \mathcal{P}(Y)$ be arbitrarily given sets. We say that the relation $\preccurlyeq$ is a pre-order if it is

(i) reflexive, that is, $A \preccurlyeq A$;

(ii) transitive, that is, $A \preccurlyeq B$ and $B \preccurlyeq C$ imply $A \preccurlyeq C$.

We say the pre-order $\preccurlyeq$ is a partial order if $A, B \in \mathcal{P}(Y)$, $A \preccurlyeq B$ and $B \preccurlyeq A$ imply that $A = B$.

Definition 2.2 ([20])**.** Let $K \subset Y$ be a convex cone.

(i) The lower set less order relation $\leq^l$ on $\mathcal{P}(Y)$ is defined as follows: for any $A, B \in \mathcal{P}(Y)$,

$$A \leq^l B \Leftrightarrow B \subset A + K.$$

(ii) The upper set less order relation $\leq^u$ on $\mathcal{P}(Y)$ is defined as follows: for any $A, B \in \mathcal{P}(Y)$,

$$A \leq^u B \Leftrightarrow A \subset B - K.$$

(iii) The set less order relation $\leq^s$ on $\mathcal{P}(Y)$ is defined as follows: for any $A, B \in \mathcal{P}(Y)$,

$$A \leq^s B \Leftrightarrow A \leq^l B \text{ and } A \leq^u B.$$

(iv) The certainly less order relation $\leq^c$ on $\mathcal{P}(Y)$ is defined as follows: for any $A, B \in \mathcal{P}(Y)$,

$$A \leq^c B \Leftrightarrow A = B \text{ or } A \neq B, B - A \subset K.$$

It is easy to check that $\leq^l$, $\leq^u$, $\leq^s$ and $\leq^c$ are pre-order relations, respectively. From Definition 2.2 , we know that $A \leq^c B \Rightarrow A \leq^s B \Rightarrow A \leq^l B$ and $A \leq^u B$.

In what follows, we assume that $\mathcal{P}(Y)$ is a pre-order set with the pre-order relation $\preccurlyeq$.

Definition 2.3. Let $\mathcal{S}$ be a nonempty subset of $\mathcal{P}(Y)$. A set $A \in \mathcal{S}$ is said to be

(i) [20] a $\preccurlyeq$-minimal set of $\mathcal{S}$ if, for $B \in \mathcal{S}$, $B \preccurlyeq A$ implies $A \preccurlyeq B$. $\text{Min}(\mathcal{S}, \preccurlyeq)$ denotes the family of all $\preccurlyeq$-minimal sets of $\mathcal{S}$.

(ii) a $\prec$-minimal set of $\mathcal{S}$ if, there is no $B \in \mathcal{S}$ such that $B \prec A$. $\text{Min}(\mathcal{S}, \prec)$ denotes the family of all $\prec$-minimal sets of $\mathcal{S}$.

Definition 2.4. Let $A, B, C \in \mathcal{P}(Y)$. We say that $\mathcal{P}(Y)$ satisfies condition P if $A \preccurlyeq B \prec C$ implies that $A \prec C$.

Remark 2.5. Clearly, if the pre-order $\preccurlyeq$ is a partial order, then $\mathcal{P}(Y)$ satisfies condition P.

Let D be a nonempty subset of X and $F : D \to \mathcal{P}(Y)$ a set-valued mapping. Let us consider the set optimization problem with respect to $\preccurlyeq$ as follows:

$$\text{(SOP)} \quad \text{Min}_{\preccurlyeq} F(x) \quad \text{subject to } \; x \in D.$$

Definition 2.6. A point $x_0 \in D$ is said to be

(i) [20] a $\preccurlyeq$-minimal solution of (SOP) if, for any $x \in D$, $F(x) \preccurlyeq F(x_0)$ implies that $F(x_0) \preccurlyeq F(x)$. $\text{Min}(F(D), \preccurlyeq)$ denotes the family of all $\preccurlyeq$-minimal solutions of (SOP).

(ii) a $\prec$-minimal solution of (SOP) if there is no $x \in D$ such that $F(x) \prec F(x_0)$. $\text{Min}(F(D), \prec)$ denotes the family of all $\prec$-minimal solutions of (SOP).

In particular, if K is a convex cone in Y and the pre-order $\preccurlyeq$ is one of $\leq^l, \leq^u, \leq^s, \leq^c$, then $\text{Min}(F(D), \leq^l)$ (resp., $\text{Min}(F(D), \leq^u)$, $\text{Min}(F(D), \leq^s)$, $\text{Min}(F(D), \leq^c)$) denotes the family of all $\leq^l$(resp., $\leq^u$, $\leq^s$, $\leq^c$)-minimal solutions of (SOP).

Remark 2.7. Obviously, $\text{Min}(F(D), \prec) \subset \text{Min}(F(D), \preccurlyeq)$.

Proposition 2.8. *If $\mathcal{P}(Y)$ satisfies condition P, then $Min(F(D), \preccurlyeq) = Min(F(D), \prec)$.*

Proof. We only need to prove that $\mathrm{Min}(F(D), \preccurlyeq) \subset \mathrm{Min}(F(D), \prec)$. If $\mathrm{Min}(F(D), \preccurlyeq) = \emptyset$, then it is easy to see that the conclusion is true. Let $x_0 \in \mathrm{Min}(F(D), \preccurlyeq)$. Suppose that $x_0 \notin \mathrm{Min}(F(D), \prec)$. Then there exists $x' \in D$ such that $F(x') \prec F(x_0)$. This shows that $F(x_0) \preccurlyeq F(x')$ and so $F(x_0) \preccurlyeq F(x') \prec F(x_0)$. Since $\mathcal{P}(Y)$ satisfies condition P, we have $F(x_0) \prec F(x_0)$, which is a contraction. Therefore, $x_0 \in \mathrm{Min}(F(D), \prec)$. This completes the proof. □

Definition 2.9 ([1, 7, 11, 22, 23])**.** Let $D \subset X$ be a nonempty subset and $K \subset Y$ a convex cone. A set-valued mapping $\Phi : D \to \mathcal{P}(Y)$ is said to be

(i) upper semi-continuous (for short, u.s.c.) at $x_0 \in D$ if, for any neighborhood V satisfying $\Phi(x_0) \subset V$, there exists a neighborhood $U(x_0)$ of x_0 such that

$$\Phi(x) \subset V,\ \forall x \in U(x_0) \cap D;$$

(ii) lower semi-continuous (for short, l.s.c.) at $x_0 \in D$ if, for any neighborhood V satisfying $\Phi(x_0) \cap V \neq \emptyset$, there exists a neighborhood $U(x_0)$ of x_0 such that

$$\Phi(x) \cap V \neq \emptyset,\ \forall x \in U(x_0) \cap D;$$

(iii) K-upper semi-continuous (for short,K-u.s.c.) at $x_0 \in D$ if, for any neighborhood V satisfying $\Phi(x_0) \subset V$, there exists a neighborhood $U(x_0)$ of x_0 such that

$$\Phi(x) \subset V + K,\ \forall x \in U(x_0) \cap D;$$

(iv) K-lower semi-continuous (for short, K-l.s.c.) at $x_0 \in D$ if, for any neighborhood V satisfying $\Phi(x_0) \cap V \neq \emptyset$, there exists a neighborhood $U(x_0)$ of x_0 such that

$$\Phi(x) \cap (V + K) \neq \emptyset,\ \forall x \in U(x_0) \cap D.$$

We say that Φ is u.s.c. (resp., l.s.c., K-u.s.c., K-l.s.c.) on D if Φ is u.s.c. (resp., l.s.c., K-u.s.c., K-l.s.c.) at each $x_0 \in D$. We say that Φ is continuous on D if Φ is both u.s.c. and l.s.c. on D.

Remark 2.10. (i) Φ is l.s.c. at $x_0 \in D$ if and only if for any sequence $\{x_n\} \subset D$ with $x_n \to x_0$, for any $y_0 \in \Phi(x_0)$, there exists $y_n \in \Phi(x_n)$ such that $y_n \to y_0$.

(ii) If Φ has compact-valued at $x_0 \in D$, then Φ is u.s.c. at $x_0 \in D$ if and only if, for any sequence $\{x_n\} \subset D$ with $x_n \to x_0$ and any $y_n \in \Phi(x_n)$, there exist a point $y_0 \in \Phi(x_0)$ and subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y_0$.

Definition 2.11 ([7, 11, 22, 23])**.** Let $D \subset X$ be a nonempty subset and $K \subset Y$ a convex cone. A set-valued mapping $\Phi : D \to \mathcal{P}(Y)$ is said to be

(i) Hausdorff upper semi-continuous (for short, H-u.s.c.) at $x_0 \in D$ if, for any neighborhood V of 0 in Y, there exists a neighborhood $U(x_0)$ of x_0 such that

$$\Phi(x) \subset \Phi(x_0) + V,\ \forall x \in U(x_0) \cap D;$$

(ii) Hausdorff lower semi-continuous (for short, H-l.s.c.) at $x_0 \in D$ if, for any neighborhood V of 0 in Y, there exists a neighborhood $U(x_0)$ of x_0 such that

$$\Phi(x_0) \subset \Phi(x) + V,\ \forall x \in U(x_0) \cap D.$$

(iii) Hausdorff K-upper semi-continuous (for short, H-K-u.s.c.) at $x_0 \in D$ if, for any neighborhood V of 0 in Y, there exists a neighborhood $U(x_0)$ of x_0 such that

$$\Phi(x) \subset \Phi(x_0) + V + K, \ \forall x \in U(x_0) \cap D;$$

(iv) Hausdorff K-lower semi-continuous (for short, H-K-l.s.c.) at $x_0 \in D$ if, for any neighborhood V of 0 in Y, there exists a neighborhood $U(x_0)$ of x_0 such that

$$\Phi(x_0) \subset \Phi(x) + V + K, \ \forall x \in U(x_0) \cap D;$$

(v) H-u.s.c. (resp., H-l.s.c., H-K-u.s.c. and H-K-l.s.c.) on D if Φ is H-u.s.c. (resp., H-l.s.c., H-K-u.s.c. and H-K-l.s.c.) at each $x_0 \in D$;

(vi) H-K-continuous on D if Φ is both H-K-u.s.c. and H-K-l.s.c. on D.

Note that

$$\text{u.s.c.} \Rightarrow K\text{-u.s.c.} \Rightarrow \text{H-}K\text{-u.s.c}; \quad \text{u.s.c.} \Rightarrow \text{H-u.s.c.} \Rightarrow \text{H-}K\text{-u.s.c};$$

and

$$\text{H-l.s.c.} \Rightarrow \text{H-}K\text{-l.s.c.} \Rightarrow K\text{-l.s.c.}; \quad \text{H-l.s.c.} \Rightarrow \text{l.s.c.} \Rightarrow K\text{-l.s.c.}$$

As pointed out in [22], Φ is H-K-l.s.c. (resp., H-l.s.c.) at x_0 if it is K-l.s.c. (resp., l.s.c.) and compact-valued at x_0. Moreover, Φ is u.s.c. (resp., K-u.s.c.) at x_0 if it is H-u.s.c. (resp., H-K-u.s.c.) and compact-valued at x_0.

A set-valued mapping Φ is K-closed-valued on D if $\Phi(x) + K$ is closed for each $x \in D$. It follows from [23] that, if Φ is H-K-u.s.c. and K-closed-valued at x_0, then Φ is K-u.s.c. at x_0.

Definition 2.12. Let $D \subset X$ be a nonempty subset and $F : D \to \mathcal{P}(Y)$ a set-valued mapping. We say that

(i) F has the $\preccurlyeq$-continuous property at $x_0 \in D$ with respect to $y_0 \in D$ if, for any sequence $\{x_n\}, \{y_n\} \subset D$ with $x_n \to x_0$, $y_n \to y_0$ such that $F(y_n) \preccurlyeq F(x_n)$ for n sufficiently large, one has $F(y_0) \preccurlyeq F(x_0)$. F has the $\preccurlyeq$-continuous property on D if F has the $\preccurlyeq$-continuous property at each $x_0 \in D$ with respect to each $y_0 \in D$.

(ii) F has the converse $\preccurlyeq$-continuous property at $x_0 \in D$ with respect to $y_0 \in D$ if, for $F(y_0) \preccurlyeq F(x_0)$ and any sequence $\{x_n\}, \{y_n\} \subset D$ with $x_n \to x_0$, $y_n \to y_0$, one has $F(y_n) \preccurlyeq F(x_n)$ for n sufficiently large. F has the converse $\preccurlyeq$-continuous property on D if F has the converse $\preccurlyeq$-continuous property at each $x_0 \in D$ with respect to each $y_0 \in D$.

The following example illustrates Definition 2.12.

Example 2.13. Let $X = Y = \mathbb{R}$, $D = [1, 2]$ and $K = \{x \in \mathbb{R} : x \geq 0\}$ with $\preccurlyeq$ being the lower set less order relation $\leq^l$. Define a set-valued mapping $F : D \to \mathcal{P}(Y)$ as follows:

$$F(x) = [-1, x], \quad \forall x \in D.$$

Then, it is easy to check that F has the $\leq^l$-continuous property and converse $\leq^l$-continuous property on D.

Proposition 2.14. (i) *If $K \subset Y$ is a convex cone with int$K \neq \emptyset$ (int$K$ denotes interior of K), $\preccurlyeq$ is the lower set less order relation $\leq^l$, F is H-K-continuous and K-closed-valued on D, then F has the $\leq^l$-continuous property and converse $\leq^l$-continuous property on D.*

(ii) *If $K \subset Y$ is a convex cone with int$K \neq \emptyset$, $\preccurlyeq$ is the upper set less order relation $\leq^u$, F is H-$(-K)$-continuous and $(-K)$-closed-valued on D, then F has the $\leq^u$-continuous property and converse $\leq^u$-continuous property on D.*

Proof. Let $e \in \mathrm{int}K$ and $\varepsilon > 0$. Then $-\varepsilon e + \mathrm{int}K$ is an open neighborhood of 0 in Y.

(i) Take two sequences $\{x_n\}$ and $\{y_n\}$ in D with $x_n \to x_0$ and $y_n \to y_0$ such that $F(y_n) \leq^l F(x_n)$ for n sufficiently large. Noting that F is H-K-u.s.c. on D, one has

$$F(y_n) \subset F(y_0) - \varepsilon e + \mathrm{int}K + K$$

for n sufficiently large. Since F is H-K-l.s.c. on D, we have

$$F(x_0) \subset F(x_n) - \varepsilon e + \mathrm{int}K + K$$

for n sufficiently large. Letting $\varepsilon \to 0$, by the K-closedness of F, we know that $F(y_0) \leq^l F(y_n)$ and $F(x_n) \leq^l F(x_0)$ for n sufficiently large. Thus, $F(y_0) \leq^l F(x_0)$ and so F has the $\leq^l$-continuous property on D.

Moreover, for $x_0', y_0' \in D$ with $F(y_0') \leq^l F(x_0')$, take two sequences $\{x_n'\}, \{y_n'\} \subset D$ such that $x_n' \to x_0'$ and $y_n' \to y_0'$. Since F is H-K-u.s.c. on D, one has

$$F(x_n') \subset F(x_0') - \varepsilon e + \mathrm{int}K + K$$

for n sufficiently large. Since F is H-K-l.s.c. on D, we have

$$F(y_0') \subset F(y_n') - \varepsilon e + \mathrm{int}K + K$$

for n sufficiently large. Letting $\varepsilon \to 0$, by the K-closedness of F, we know that $F(x_0') \leq^l F(x_n')$ and $F(y_n') \leq^l F(y_0')$. Thus, $F(y_n') \leq^l F(x_n')$ for n sufficiently large. This shows that F has the converse $\leq^l$-continuous property on D.

(ii) Similar to the proof of (i), we can prove that (ii) holds. □

Remark 2.15. If K is a closed and convex cone with int$K \neq \emptyset$ and F is a compact-valued continuous mapping on D, then Proposition 2.14 is also true. Moreover, if $\preccurlyeq$ is the set less order relation $\leq^s$, then F has the $\leq^s$-continuous property and converse $\leq^s$-continuous property on D.

Proposition 2.14 and Remark 2.15 show that the set-valued mappings satisfying Definition 2.12 can be found easily.

Definition 2.16 ([20])**.** Let $D \subset X$ be a nonempty subset and $F : D \to \mathcal{P}(Y)$ a set-valued mapping. We say that F is $\preccurlyeq$-semi-continuous at $x_0 \in D$ if, for $F(x_0) \in V$, there exists a neighborhood $U(x_0)$ of x_0 such that

$$F(x) \in V, \quad \forall x \in U(x_0) \cap D,$$

where $V = \{T \in \mathcal{P}(Y) : T \not\preccurlyeq Q\}$ for some $Q \in \mathcal{P}(Y)$.

We say that F is $\preccurlyeq$-semi-continuous on D if it is $\preccurlyeq$-semi-continuous at each $x_0 \in D$.

Let $L(F(x)) = \{y \in D : F(y) \preccurlyeq F(x)\}$ for $x \in D$. Then, for any $x \in D$, $L(F(x))$ is closed if and only if F is $\preccurlyeq$-semi-continuous on D (see [20]).

Proposition 2.17. *Let D be a nonempty subset of X and $F : D \to \mathcal{P}(Y)$ a set-valued mapping. If F has the $\preccurlyeq$-continuous property on D, then $L(F(x)) = \{y \in D : F(y) \preccurlyeq F(x)\}$ is closed for $x \in D$.*

Proof. For any given $x \in D$, take a sequence $\{x_n\} \subset L(F(x))$ with $x_n \to x_0$. Then $F(x_n) \preccurlyeq F(x)$. Since F has the $\preccurlyeq$-continuity property, it follows that $F(x_0) \preccurlyeq F(x)$. Thus, $x_0 \in L(F(x))$. □

Lemma 2.18 ([20])**.** *Let D be a nonempty compact subset of X. If $F : D \to \mathcal{P}(Y)$ is a $\preccurlyeq$-semi-continuous set-valued mapping, then (SOP) has a $\preccurlyeq$-minimal solution.*

From Proposition 2.17 and Lemma 2.18, we can get the following lemma.

Lemma 2.19. *Let $D \subset X$ be a nonempty and compact subset and $F : D \to \mathcal{P}(Y)$ a set-valued mapping. If F has the $\preccurlyeq$-continuous property on D, then (SOP) has a $\preccurlyeq$-minimal solution.*

Lemma 2.20 ([2])**.** *Let $D \subset X$ be a nonempty compact subset, $K \subset Y$ be a closed, pointed convex cone and $\preccurlyeq$ be the lower set less order relation $\leq^l$. If $F : D \to \mathcal{P}(Y)$ is K-u.s.c. set-valued mapping on D, then (SOP) has a $\leq^l$-minimal solution.*

From Corollary 24 of [2], we get the following lemma.

Lemma 2.21. *Let $D \subset X$ be a nonempty compact subset, $K \subset Y$ be a closed, pointed convex cone and $\preccurlyeq$ be the upper set less order relation $\leq^u$. If $F : D \to \mathcal{P}(Y)$ is K-l.s.c. set-valued mapping on D with compact values, then (SOP) has a $\leq^u$-minimal solution.*

From Remark 2.15, Proposition 2.17 and Lemma 2.18, we can get the following lemma.

Lemma 2.22. *Let $D \subset X$ be a nonempty compact subset, $K \subset Y$ be a closed, pointed and convex cone with int$K \neq \emptyset$ and $\preccurlyeq$ be the set less order relation $\leq^s$. If $F : D \to \mathcal{P}(Y)$ is continuous set-valued mapping on D with compact values, then (SOP) has a $\leq^s$-minimal solution.*

3 Main Results

Let $D \subset X$ be a nonempty subset and $\Lambda \subset Z$ a nonempty subset. Assume that $Q : \Lambda \to 2^D$ and $F : D \times \Lambda \to \mathcal{P}(Y)$ are two set-valued mappings. We consider the following parametric set optimization problem (PSOP) with respect to $\preccurlyeq$:

$$\mathrm{Min}_{\preccurlyeq} F(x, \lambda) \quad \text{subject to } \ x \in Q(\lambda).$$

Assume that $\mathrm{Min}(F, Q(\cdot), \preccurlyeq)$, $\mathrm{Min}(F, Q(\cdot), \prec) : \Lambda \to 2^X$ are solution set mappings of (PSOP), i.e.,

$$\mathrm{Min}(F, Q(\lambda), \preccurlyeq) = \{x_0 \in Q(\lambda) : x \in Q(\lambda), F(x, \lambda) \preccurlyeq F(x_0, \lambda) \text{ implies } F(x_0, \lambda) \preccurlyeq F(x, \lambda)\};$$

$$\mathrm{Min}(F, Q(\lambda), \prec) = \{x_0 \in Q(\lambda) : \nexists x \in Q(\lambda) \text{ such that } F(x, \lambda) \prec F(x_0, \lambda)\}.$$

Some special cases of (PSOP) are as follows.

(I) If K is a closed, pointed and convex cone in Y and $\preccurlyeq$ is the lower set less order relation $\leq^l$, then the problem (PSOP) reduces to

$$\mathrm{Min}_{\leq^l} F(x,\lambda) \quad \text{subject to } \; x \in Q(\lambda), \tag{3.1}$$

which was considered by Han and Huang [11] and Khoshkhabar-amiranloo [23].

(II) If K is a closed, pointed and convex cone in Y and $\preccurlyeq$ is the upper set less order relation $\leq^u$, then the problem (PSOP) reduces to

$$\mathrm{Min}_{\leq^u} F(x,\lambda) \quad \text{subject to } \; x \in Q(\lambda), \tag{3.2}$$

which was studied by Xu and Li [36, 37].

(III) If K is a closed, pointed and convex cone in Y and $\preccurlyeq$ is the set less order relation $\leq^s$, then the problem (PSOP) reduces to

$$\mathrm{Min}_{\leq^s} F(x,\lambda) \quad \text{subject to } \; x \in Q(\lambda). \tag{3.3}$$

(IV) If K is a closed, pointed and convex cone in Y and $\preccurlyeq$ is the certainly less order relation $\leq^c$, then the problem (PSOP) reduces to

$$\mathrm{Min}_{\leq^c} F(x,\lambda) \quad \text{subject to } \; x \in Q(\lambda). \tag{3.4}$$

3.1 The continuity of the minimal solution mapping of a parametric set optimization problem with the general pre-order realation

In this subsection, we establish the continuity and compactness of the minimal solution mapping for (PSOP) with the general pre-order relation under some suitable conditions.

Theorem 3.1. *Let $\lambda_0 \in \Lambda$. Suppose that the following conditions hold*

(i) *$Q(\cdot)$ is continuous at λ_0 and $Q(\lambda_0)$ is compact;*

(ii) *$F(\cdot,\cdot)$ has the $\preccurlyeq$-continuous property and converse $\preccurlyeq$-continuous property on $Q(\lambda_0) \times \{\lambda_0\}$.*

Then, Min$(F, Q(\cdot), \preccurlyeq)$ is continuous at λ_0 and Min$(F, Q(\lambda_0), \preccurlyeq)$ is compact.

Proof. By Lemma 2.19, we know that $\mathrm{Min}(F, Q(\lambda_0), \preccurlyeq) \neq \emptyset$. Now we show that $\mathrm{Min}(F, Q(\cdot), \preccurlyeq)$ is u.s.c. at λ_0. Suppose on the contrary that $\mathrm{Min}(F, Q(\cdot), \preccurlyeq)$ is not u.s.c. at λ_0. Then, there exist a neighborhood V_0 of $\mathrm{Min}(F, Q(\lambda_0), \preccurlyeq)$ and a sequence $\{\lambda_n\}$ with $\lambda_n \to \lambda_0$ such that $\mathrm{Min}(F, Q(\lambda_n), \preccurlyeq) \not\subset V_0$ for $n = 1, 2, \cdots$. Thus, there exists $x_n \in \mathrm{Min}(F, Q(\lambda_n), \preccurlyeq)$ such that

$$x_n \notin V_0 \quad \text{for} \quad n = 1, 2, \cdots. \tag{3.5}$$

Since $Q(\cdot)$ is u.s.c. at λ_0, $Q(\lambda_0)$ is compact and $x_n \in Q(\lambda_n)$, by Remark 2.10 (ii), there exist $x_0 \in Q(\lambda_0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$. Without loss of generality, we can assume that $x_n \to x_0$. We claim that $x_0 \in \mathrm{Min}(F, Q(\lambda_0), \preccurlyeq)$. In fact, suppose that $x_0 \notin \mathrm{Min}(F, Q(\lambda_0), \preccurlyeq)$. Then there exists $y_0 \in Q(\lambda_0)$ such that $F(y_0, \lambda_0) \preccurlyeq F(x_0, \lambda_0)$ and $F(x_0, \lambda_0) \not\preccurlyeq F(y_0, \lambda_0)$. By the lower semi-continuity of Q, there exists $y_n \in Q(\lambda_n)$ such that $y_n \to y_0$. Since F has the converse $\preccurlyeq$-continuous property on $Q(\lambda_0) \times \{\lambda_0\}$ and

$F(y_0, \lambda_0) \preccurlyeq F(x_0, \lambda_0)$, we have $F(y_n, \lambda_n) \preccurlyeq F(x_n, \lambda_n)$ for n sufficiently large. It follows from $x_n \in \mathrm{Min}(F, Q(\lambda_n), \preccurlyeq)$ that $F(x_n, \lambda_n) \preccurlyeq F(y_n, \lambda_n)$. Since F has the $\preccurlyeq$-continuity property, we obtain $F(x_0, \lambda_0) \preccurlyeq F(y_0, \lambda_0)$, which is a contradiction. Thus, $x_0 \in \mathrm{Min}(F, Q(\lambda_0), \preccurlyeq)$ and so $x_n \in V_0$ for n sufficiently large, which contradicts (3.5). This shows that $\mathrm{Min}(F, Q(\cdot), \preccurlyeq)$ is u.s.c. at λ_0.

Now, we claim that $\mathrm{Min}(F, Q(\cdot), \preccurlyeq)$ is l.s.c. at λ_0. Suppose on the contrary that $\mathrm{Min}(F, Q(\cdot), \preccurlyeq)$ is not l.s.c. at λ_0. Then, there exist $y' \in \mathrm{Min}(F, Q(\lambda_0), \preccurlyeq)$, a neighborhood W_0 of 0 in X and a sequence $\{\lambda'_n\}$ with $\lambda'_n \to \lambda_0$ such that

$$(y' + W_0) \cap \mathrm{Min}(F, Q(\lambda'_n), \preccurlyeq) = \emptyset \quad \text{for} \quad n = 1, 2, \cdots. \tag{3.6}$$

It follows from $y' \in \mathrm{Min}(F, Q(\lambda_0), \preccurlyeq)$ that $y' \in Q(\lambda_0)$. Since $Q(\cdot)$ is l.s.c. at λ_0, by Remark 2.10 (i), there exists $y'_n \in Q(\lambda'_n)$ such that $y'_n \to y'$. Let us show that $y'_n \in \mathrm{Min}(F, Q(\lambda'_n), \preccurlyeq)$ for n sufficiently large. Indeed, if not, there exist a subsequence $\{y'_{n_k}\}$ of $\{y'_n\}$ and a subsequence $\{\lambda'_{n_k}\}$ of $\{\lambda'_n\}$ such that $y'_{n_k} \notin \mathrm{Min}(F, Q(\lambda'_{n_k}), \preccurlyeq)$ for $k = 1, 2, \cdots$. Without loss of generality, we can suppose that $y'_n \notin \mathrm{Min}(F, Q(\lambda'_n), \preccurlyeq)$ for $n = 1, 2, \cdots$. Then, there exists $x'_n \in Q(\lambda'_n)$ such that $F(x'_n, \lambda'_n) \preccurlyeq F(y'_n, \lambda'_n)$ and $F(y'_n, \lambda'_n) \not\preccurlyeq F(x'_n, \lambda'_n)$. Noting that Q is u.s.c. at λ_0 and $Q(\lambda_0)$ is compact, by Remark 2.10 (ii), there exist a point $x' \in Q(\lambda_0)$ and a subsequence $\{x'_{n_k}\}$ of $\{x'_n\}$ such that $x'_{n_k} \to x'$. Without loss of generality, we can suppose that $x'_n \to x'$. It follows from $F(x'_n, \lambda'_n) \preccurlyeq F(y'_n, \lambda'_n)$ and $F(\cdot, \cdot)$ has the $\preccurlyeq$-continuous property on $Q(\lambda_0) \times \{\lambda_0\}$ that $F(x', \lambda_0) \preccurlyeq F(y', \lambda_0)$. Since $y' \in \mathrm{Min}(F, Q(\lambda_0), \preccurlyeq)$, we obtain $F(y', \lambda_0) \preccurlyeq F(x', \lambda_0)$. Since F has the converse $\preccurlyeq$-continuity property, it follows that $F(y'_n, \lambda'_n) \preccurlyeq F(x'_n, \lambda'_n)$ for n sufficiently large, which contradicts the fact that $F(y'_n, \lambda'_n) \not\preccurlyeq F(x'_n, \lambda'_n)$. Thus, $y'_n \in \mathrm{Min}(F, Q(\lambda'_n), \preccurlyeq)$ for n sufficiently large and so $y'_n \in \mathrm{Min}(F, Q(\lambda_n), \preccurlyeq) \cap (y' + W_0)$ for n sufficiently large, which contradicts (3.6). This means that $\mathrm{Min}(F, Q(\cdot), \preccurlyeq)$ is l.s.c. at λ_0. In summary, we show that $\mathrm{Min}(F, Q(\cdot), \preccurlyeq)$ is continuous at λ_0.

Next, we prove that $\mathrm{Min}(F, Q(\lambda_0), \preccurlyeq)$ is compact. Since $\mathrm{Min}(F, Q(\lambda_0), \preccurlyeq) \subset Q(\lambda_0)$, we only need to show that $\mathrm{Min}(F, Q(\lambda_0), \preccurlyeq)$ is closed. For this end, let $\{z_n\} \subset \mathrm{Min}(F, Q(\lambda_0), \preccurlyeq)$ be a sequence such that $z_n \to z_0$. If $z_0 \notin \mathrm{Min}(F, Q(\lambda_0), \preccurlyeq)$, then there exists $\overline{z} \in Q(\lambda_0)$ such that $F(\overline{z}, \lambda_0) \preccurlyeq F(z_0, \lambda_0)$ and $F(z_0, \lambda_0) \not\preccurlyeq F(\overline{z}, \lambda_0)$. By the closedness of $Q(\lambda_0)$, there exists a sequence $\{\overline{z}_n\} \subset Q(\lambda_0)$ such that $\overline{z}_n \to \overline{z}$. Then, for n sufficiently large, $F(\overline{z}_n, \lambda_0) \preccurlyeq F(z_n, \lambda_0)$ due to $F(\cdot, \cdot)$ has the converse $\preccurlyeq$-continuous property on $Q(\lambda_0) \times \{\lambda_0\}$ and $F(\overline{z}, \lambda_0) \preccurlyeq F(z_0, \lambda_0)$. It follows from $z_n \in \mathrm{Min}(F, Q(\lambda_0), \preccurlyeq)$ that $F(z_n, \lambda_0) \preccurlyeq F(\overline{z}_n, \lambda_0)$. Since $F(\cdot, \cdot)$ has the $\preccurlyeq$-continuous property on $Q(\lambda_0) \times \{\lambda_0\}$, we have $F(z_0, \lambda_0) \preccurlyeq F(\overline{z}, \lambda_0)$, which is a contradiction and so $z_0 \in \mathrm{Min}(F, Q(\lambda_0), \preccurlyeq)$. This shows that $\mathrm{Min}(F, Q(\lambda_0), \preccurlyeq)$ is closed and so it is compact. □

Corollary 3.2. *Let $\mathcal{P}(Y)$ satisfy condition P and $\lambda_0 \in \Lambda$. Suppose that the following conditions hold*

(i) *$Q(\cdot)$ is continuous at λ_0 and $Q(\lambda_0)$ is compact;*

(ii) *$F(\cdot, \cdot)$ has the $\preccurlyeq$-continuous property and converse $\preccurlyeq$-continuous property on $Q(\lambda_0) \times \{\lambda_0\}$;*

Then, Min$(F, Q(\cdot), \prec)$ is continuous at λ_0.

Proof. Since $\mathcal{P}(Y)$ satisfies condition P, it follows from Proposition 2.8 that

$$\mathrm{Min}(F, Q(\cdot), \preccurlyeq) = \mathrm{Min}(F, Q(\cdot), \prec).$$

Since all conditions of Theorem 3.1 are satisfied, we know that $\mathrm{Min}(F, Q(\cdot), \prec)$ is continuous at λ_0. □

We give an example to illustrate Theorem 3.1.

Example 3.3. Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $D = [0, 1]$, $\Lambda = [0, 1]$ and

$$K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$$

with $\preccurlyeq$ being the lower set less order relation $\leq^l$. Let $Q(\lambda) = [0, \lambda]$ for $\lambda \in \Lambda$ and

$$F(x, \lambda) = [-1, \lambda x] \times [0, 1], \quad \forall (x, \lambda) \in D \times \Lambda.$$

Let $\lambda_0 = 1$. Then, it is easy to check that all conditions of Theorem 3.1 are satisfied. Moreover, it follows that

$$\mathrm{Min}(F, Q(\lambda), \leq^l) = [0, \lambda].$$

Thus, $\mathrm{Min}(F, Q(\cdot), \leq^l)$ is continuous at 1 and so $\mathrm{Min}(F, Q(\lambda_0), \leq^l) = [0, 1]$ is compact.

3.2 Applications to the parametric set optimization problem with the order induced by a convex cone

In this subsection, as applications of the result presented in previous subsection, we obtain the continuity and compactness of minimal solution mappings for parametric set optimization problems with the order induced by a convex cone.

From Proposition 2.14, Lemma 2.20 and Theorem 3.1, we have the following theorem.

Theorem 3.4. *Let K be a closed, pointed and convex cone in Y with int$K \neq \emptyset$ and $\preccurlyeq$ be the lower set less order relation $\leq^l$. Suppose that the following conditions hold*

(i) *$Q(\cdot)$ is continuous at $\lambda_0 \in \Lambda$ and $Q(\lambda_0)$ is compact;*

(ii) *$F(\cdot, \cdot)$ is a continuous set-valued mapping on $Q(\Lambda) \times \Lambda$ with compact values.*

Then, Min$(F, Q(\cdot), \leq^l)$ is continuous at λ_0 and Min$(F, Q(\lambda_0), \leq^l)$ is compact, where Min$(F, Q(\cdot), \leq^l)$ is a minimal solution mapping from Λ to 2^X.

Remark 3.5. Theorem 3.4 improves Theorems 4.5, 5.4 and 5.6 in [23] and Theorem 5.1 in [11] by removing the cone convexity assumption of F.

From Proposition 2.14, Lemma 2.21 and Theorem 3.1, one has the following theorem.

Theorem 3.6. *Let K be a closed, pointed and convex cone in Y with int$K \neq \emptyset$ and $\preccurlyeq$ be the upper set less order relation $\leq^u$. Suppose that the following conditions hold*

(i) *$Q(\cdot)$ is continuous at $\lambda_0 \in \Lambda$ and $Q(\lambda_0)$ is compact;*

(ii) *$F(\cdot, \cdot)$ is a continuous set-valued mapping on $Q(\Lambda) \times \Lambda$ with compact values.*

Then, Min$(F, Q(\cdot), \leq^u)$ is continuous at λ_0 and Min$(F, Q(\lambda_0), \leq^u)$ is compact, where Min$(F, Q(\cdot), \leq^u)$ is a minimal solution mapping from Λ to 2^X.

Remark 3.7. Theorem 3.6 improves Theorems 3.2, 3.4 in [36] and Theorems 4.1-4.3 in [37] by removing the cone convexity assumption of F.

From Remark 2.15, Lemma 2.22 and Theorem 3.1, we have the following result.

Theorem 3.8. *Let K be a closed, pointed and convex cone in Y with $intK \neq \emptyset$ and $\preccurlyeq$ be the set less order relation $\leq^s$. Suppose that the following conditions hold*

(i) *$Q(\cdot)$ is continuous at $\lambda_0 \in \Lambda$ and $Q(\lambda_0)$ is compact;*

(ii) *$F(\cdot,\cdot)$ is a continuous set-valued mapping on $Q(\Lambda) \times \Lambda$ with compact values.*

Then, $Min(F, Q(\cdot), \leq^s)$ is continuous at λ_0 and $Min(F, Q(\lambda_0), \leq^s)$ is compact, where $Min(F, Q(\cdot), \leq^s)$ is a minimal solution mapping from Λ to 2^X.

When $\preccurlyeq$ is the certainly less order relation $\leq^c$, Theorem 3.1 reduces to the following theorem.

Theorem 3.9. *Let K be a closed, pointed and convex cone in Y and $\preccurlyeq$ be the certainly less order relation $\leq^c$. Suppose that the following conditions hold*

(i) *$Q(\cdot)$ is continuous at $\lambda_0 \in \Lambda$ and $Q(\lambda_0)$ is compact;*

(ii) *$F(\cdot,\cdot)$ has the $\leq^c$-continuous property and converse $\leq^c$-continuous property on $Q(\lambda_0)\times \{\lambda_0\}$.*

Then, $Min(F, Q(\cdot), \leq^c)$ is continuous at λ_0 and $Min(F, Q(\lambda_0), \leq^c)$ is compact, where $Min(F, Q(\cdot), \leq^c)$ is a minimal solution mapping from Λ to 2^X.

4 Conclusions

This paper is devoted to establish the continuity and compactness of minimal solution mappings to parametric set optimization problems under some mild conditions. This paper has the following two main contributions: (i) Some new sufficient conditions are obtained for ensuring the continuity and compactness of the minimal solution mappings of parametric set optimization problems with the general pre-order relations; (ii) The pre-order method developed in this paper provides a new tool for the study of set optimization problems.

It is well known that the well-posedness plays an important role in set (vector) optimization problems. Therefore, it would be interesting to study the well-posedness of set optimization problems with the general pre-order relation. We plan to address such problems as we continue our research.

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