Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 15, No. 3, pp. 477 – 490, 2019

NECESSARY AND SUFFICIENT KKT OPTIMALITY CONDITIONS IN NON-CONVEX MULTI-OBJECTIVE OPTIMIZATION PROBLEMS WITH CONE CONSTRAINTS

Nithirat Sisarat* and Rabian Wangkeeree†

Abstract: This paper deals with a class of differentiable multi-objective optimization problems (MOP) over cone constraints without the convexity of the feasible set, and the cone-convexity of objectives and constraint functions. We present constraint qualifications for these (MOP) problems and establish the relationships among them. We also present necessary and sufficient the Karush-Kuhn-Tucker (KKT) optimality conditions for a weak Pareto minimum as well as a Pareto minimum to (MOP). Our main results improve some recent ones in the literature. Illustrative examples are also provided to guarantee the advantages of each of our results.

Key words: *non-convex multi-objective optimization, cone-convex functions, level set, Karush-Kuhn-Tucker optimality conditions*

Mathematics Subject Classification: *90C26, 90C29, 90C46*

1 Introduction

A Multi-objective (vector-valued) optimization is a subject of mathematical programming problems that extensively studied and applied in various decision-making contexts like economics, human decision making, control engineering, transportation and many others. We refer the reader to [18,19,24]. For the comprehensive treatment of theoretical issues concerning multi-objective optimization can be found in [2, 9, 13, 17]. In the multi-objective setting, the scalar concept of optimality does not apply directly due to the fact that all the objectives can not be simultaneously optimized with a single solution. To this effect, we must decide which objective functions should be improve, and so compromise solutions have to be considered. In this way, we refer to a weak Pareto minimum (resp. a Pareto minimum [15]) which usually uses the coordinate-wise ordering (induced by the positive orthant as ordering cone) to examine the objective vectors. However, in real-world multi-objective problems concerning especially fractional programming problems or even computational aspects of a Pareto optimum, not only the coordinate-wise ordering appears but also the cone defining the lexicographic partial order is of practical interest [7]. This being a reason, the study

*This research was supported by the Thailand Research Fund through the Royal Golden Jubilee Ph.D. Program (Grant No. PHD/0026/2555), and Naresuan University..

†Corresponding author. The second author was partially supported by the Thailand Research Fund, Grant No. RSA6080077.

of multi-objective optimization problems involving general ordering cones has gained attention. Precisely stated, in this paper we will be mainly concerned with the multi-objective optimization problem with cone constraint (MOP) given as

$$\begin{aligned} K - \text{Minimize } \mathbf{f}(\mathbf{x}) \\ \text{subject to } \mathbf{x} \in \mathbb{R}^n,\ -\mathbf{g}(\mathbf{x}) \in Q, \end{aligned} \tag{MOP}$$

where $\mathbf{f} := (f_1, \ldots, f_p)^T : \mathbb{R}^n \to \mathbb{R}^p$ and $\mathbf{g} := (g_1, \ldots, g_m)^T : \mathbb{R}^n \to \mathbb{R}^m$, are differentiable functions, K and Q are closed convex cones with nonempty interiors in $\mathbb{R}^p$ and $\mathbb{R}^m$, respectively. Let

$$\mathcal{X} := \{\mathbf{x} \in \mathbb{R}^n : -\mathbf{g}(\mathbf{x}) \in Q\} \tag{1.1}$$

be the set of all feasible solutions of (MOP). The notation "$K -$ Minimize "refers to the *weak Pareto minimum* (resp. *Pareto minimum*) with respect to the ordering cone K for the problem (MOP), namely a point $\mathbf{x}^* \in \mathcal{X}$ such that for every $\mathbf{x} \in \mathcal{X}$, $\mathbf{f}(\mathbf{x}^*) - \mathbf{f}(\mathbf{x}) \notin \text{int}K$ (resp. $\mathbf{f}(\mathbf{x}^*) - \mathbf{f}(\mathbf{x}) \notin K \backslash \{\mathbf{0}\}$).

Recall that a feasible point $\mathbf{x}^* \in \mathcal{X}$ is said to be a *KKT point* if there exist multipliers $\boldsymbol{\lambda} \in K^* \backslash \{\mathbf{0}\}$ and $\boldsymbol{\mu} \in Q^*$ such that the following Karush-Kuhn-Tucker (KKT) optimality conditions hold:

(i) $\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*) = \mathbf{0}$;

(ii) $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0$,

where K^*, $K^* := \{\mathbf{z} \in \mathbb{R}^p : \mathbf{x}^T \mathbf{z} \geq 0 \text{ for all } \mathbf{x} \in K\}$, stands for the *dual* (*positive polar*) *cone* of K. In this paper, the above feasible point $\mathbf{x}^*$ is also called a *non-trivial* KKT point if the corresponding $\boldsymbol{\mu}$ is a non-zero vector.

As far as we know, the search for weak Pareto minimum (resp. Pareto minimum) to (MOP) has been carried out through the study of the KKT optimality conditions provided that some constraint qualifications hold, and of the convexity of the functions $\mathbf{f}$ and $\mathbf{g}$. In the current work, with the introduction of the scalar convex optimization problem without convexity of constraint functions by Lasserre [12], the studies have been done on establishing KKT optimality conditions for a weak Pareto minimum (resp. a Pareto minimum) of some classes of multi-objective convex optimization problems. In particular, the authors have shown in [21] that even if the convex feasible set is not necessarily described by the cone-convex constraint, the *Slater-type cone constraint qualification* renders the KKT optimality conditions both necessary and sufficient.

The classes of scalar convex optimization problems without convexity of constraint functions have been studied in the literature [6, 12, 14] where apart from [12] in other references inequality constraints are not assumed to be differentiable. A more recent exhaustive treatment of constraint qualifications can be found in [5, 23]. Recently, Ho [8] went further in the case of scalar differentiable problems but moreover without the convexity of the feasible set and of the functions that are involved, and showed that necessary and sufficient KKT optimality conditions are then considered in relation to the presence of convexity of the level sets of objective function. It is therefore of interest to investigate KKT optimality conditions for a weak Pareto minimum and a Pareto minimum of (MOP) without the convexity of the feasible set $\mathcal{X}$ and of the vector-valued functions $\mathbf{f}$ and $\mathbf{g}$. The main purpose of this paper is to make an effort in this direction. Since we now focus our investigations to (MOP) in

The feasible set $\mathcal{X}$ defined as in (1.1) is said to satisfy *Slater-type cone constraint qualification* [9] at $\mathbf{x} \in \mathcal{X}$ if there exists $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that $\mathbf{g}(\mathbf{x}) + \nabla \mathbf{g}(\mathbf{x})(\hat{\mathbf{x}} - \mathbf{x}) \in -\text{int}Q$.

which the feasible set $\mathcal{X}$ is not necessarily convex, we are going to consider the feasible point $\mathbf{x}^*$ under the question satisfying the following property [8]:

$$\forall \mathbf{x} \in \mathcal{X},\ \exists t_n \to 0^+ \text{ such that } \mathbf{x}^* + t_n(\mathbf{x} - \mathbf{x}^*) \in \mathcal{X}, \tag{1.2}$$

which can be seen as a generalized convexity of the feasible set $\mathcal{X}$. Admittedly, some non-convex sets that satisfy the condition (1.2) will illustrate in Example 3.3 in Section 3. It is important to note that Slater's condition together with a mild non-degeneracy condition on the constraints has been shown to guarantee that the KKT conditions are necessary and sufficient for optimality of the scalar problems ([6,8,12,14]). Now, for the problem (MOP), we will assume only a non-degeneracy condition at the point $\mathbf{x}^*$ under consideration (see Assumption 1 in the next section). In what follows the connections among non-degeneracy condition, Slater-type cone constraint qualification, and *Slater's condition* for cone-constraint are also investigated ones. Further, illustrative examples are also provided to demonstrate that our results generalize and improve the corresponding known results obtained in [21] for the problem (MOP) in some appropriate situations.

The rest of the paper is organized as follows. In Sect. 2 we recall some basic definitions and point out important results that will be used later in the paper. Section 3 presents relationships among constraint qualifications of the multi-objective optimization problem (MOP) over cone constraint (1.1) and establishes necessary and sufficient KKT optimality conditions for a feasible point under the question to be a weak Pareto minimum of (MOP). Finally, sufficient conditions for a Pareto minimum of the problem (MOP) are also provided.

2 Preliminaries

In this section, we briefly overview some notations, basic definitions, and preliminary results which will be used throughout the paper. All spaces under consideration are n-dimensional Euclidean space $\mathbb{R}^n$. All vectors are considered to be column vectors which can be transposed to a row vector by the superscript T. A nonempty subset K of $\mathbb{R}^p$ is said to be a *cone* if $tK \subseteq K$ for all $t \geq 0$. For a set A in $\mathbb{R}^n$, we say A is *convex* whenever $t\mathbf{x}_1 + (1-t)\mathbf{x}_2 \in A$ for all $t \in [0,1]$, $\mathbf{x}_1,\ \mathbf{x}_2 \in A$. By int$A$ (resp. co$A$) we will denote the *interior* (resp. *convex hull*) of the set A. The *normal cone* to a closed convex set A at $\mathbf{x} \in A$, denoted by

$$N(A, \mathbf{x}) := \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u}^T(\mathbf{y} - \mathbf{x}) \leq 0,\ \forall \mathbf{y} \in A\}.$$

A set $A \subseteq \mathbb{R}^n$ is called *strictly convex* at $\mathbf{x} \in A$ if $\mathbf{u}^T(\mathbf{y} - \mathbf{x}) < 0$ for every $\mathbf{y} \in A \backslash \{\mathbf{x}\}$ and $\mathbf{u} \in N(A, \mathbf{x}) \backslash \{\mathbf{0}\}$. It is worth noting that the strict convexity of A at some point $\mathbf{x}$ does not guarantee the convexity of A. For instance, the set $A := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 > 0\} \cup \{(0,0)^T\}$ is strictly convex at $(0,0)^T$ while A is not convex.

For a closed convex cone $K \subseteq \mathbb{R}^p$, a vector valued function $\mathbf{f} := (f_1, \ldots, f_p)^T : \mathbb{R}^n \to \mathbb{R}^p$ is said to be K-*convex* (K-*pseudoconvex* [1,22]) at a point $\mathbf{x}^* \in \mathbb{R}^n$ if for every $\mathbf{x} \in \mathbb{R}^n$

$$\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}^*) - \nabla \mathbf{f}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \in K$$

(resp. $-\nabla \mathbf{f}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \notin \text{int}K \Rightarrow \mathbf{f}(\mathbf{x}^*) - \mathbf{f}(\mathbf{x}) \in \text{int}K$), where $\nabla f(\mathbf{x}^*) := (\nabla f_1(\mathbf{x}^*), \ldots, \nabla f_p(\mathbf{x}^*))^T$ is the $p \times n$ Jacobian matrix of $\mathbf{f}$ at $\mathbf{x}^*$ and for each $k = 1, 2, \ldots, p$, $\nabla f_k(\mathbf{x}^*) := \left(\frac{\partial f_k(\mathbf{x}^*)}{\partial x_1}, \frac{\partial f_k(\mathbf{x}^*)}{\partial x_2}, \ldots, \frac{\partial f_k(\mathbf{x}^*)}{\partial x_n}\right)^T$ is the $n \times 1$ gradient vector of f_k at $\mathbf{x}^*$. If $\mathbf{f}$

The feasible set $\mathcal{X}$ defined as in (1.1) is said to satisfy *Slater's condition* if there exists $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that $-\mathbf{g}(\hat{\mathbf{x}}) \in \text{int}Q$.

is K-convex (K-pseudoconvex) at every point $\mathbf{x}^* \in \mathbb{R}^n$, we also say that $\mathbf{f}$ is said to be K-convex (resp. K-pseudoconvex) on $\mathbb{R}^n$.

Now, let us recall the following results which will be useful in the sequel.

Lemma 2.1 ([9, Lemma 3.21, p. 77]). *Let K be a nonempty convex cone in $\mathbb{R}^p$.*

(i) *If K is closed, then*

$$K = \{\boldsymbol{x} \in \mathbb{R}^p : \boldsymbol{x}^T \boldsymbol{z} \geq 0 \text{ for all } \boldsymbol{z} \in K^*\}.$$

(ii) *If $\mathrm{int}K \neq \emptyset$, then*

$$\mathrm{int}K = \{\boldsymbol{x} \in \mathbb{R}^p : \boldsymbol{x}^T \boldsymbol{z} > 0 \text{ for all } \boldsymbol{z} \in K^* \backslash \{\mathbf{0}\}\}.$$

Lemma 2.2 ([20, Lemma 1]). *Consider the problem* (MOP)*, if $\boldsymbol{x}^* \in \mathcal{X}$ is a weak Pareto minimum of* (MOP)*, then there exist $\boldsymbol{\lambda} \in K^*$ and $\boldsymbol{\mu} \in Q^*$ not both zero such that*

$$\Big(\boldsymbol{\lambda}^T \nabla \boldsymbol{f}(\boldsymbol{x}^*) + \boldsymbol{\mu}^T \nabla \boldsymbol{g}(\boldsymbol{x}^*)\Big)(\boldsymbol{x} - \boldsymbol{x}^*) \geq 0, \ \forall \boldsymbol{x} \in \mathbb{R}^n$$

and

$$\boldsymbol{\mu}^T \boldsymbol{g}(\boldsymbol{x}^*) = 0.$$

Now, we recall the following important result which can be found in [11] and will play a key role in deriving a feasible point to be a weak Pareto minimum as well as a Pareto minimum of (MOP).

Proposition 2.3 ([11, Proposition 2.2.]). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\boldsymbol{x}^*$ with $\nabla f(\boldsymbol{x}^*) \neq \mathbf{0}$. Then, the following statements hold:*

(i) $N(L_f^<(\boldsymbol{x}^*), \boldsymbol{x}^*) = \{\boldsymbol{d} \in \mathbb{R}^n : \boldsymbol{d} = r\nabla f(\boldsymbol{x}^*), \text{ for some } r \geq 0\}$ *provided that*

$$L_f^<(\boldsymbol{x}^*) := \{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) < f(\boldsymbol{x}^*)\}$$

is convex.

(ii) $N(L_f(\boldsymbol{x}^*), \boldsymbol{x}^*) = \{\boldsymbol{d} \in \mathbb{R}^n : \boldsymbol{d} = r\nabla f(\boldsymbol{x}^*), \text{ for some } r \geq 0\}$ *provided that*

$$L_f(\boldsymbol{x}^*) := \{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) \leq f(\boldsymbol{x}^*)\}$$

is convex.

We conclude this section by the following useful lemma, which will be crucial in the sequel.

Lemma 2.4. Let $\mathcal{X}$ be defined as in (1.1). Assume that the condition (1.2) is satisfied at a feasible point $\mathbf{x}^* \in \mathcal{X}$. Then for every $\boldsymbol{\mu} \in Q^* \backslash \{\mathbf{0}\}$ such that $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0$, one has

$$\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\mathbf{v} - \mathbf{x}^*) \leq 0 \text{ for all } \mathbf{v} \in \mathcal{X}.$$

Proof. Suppose on contrary that there exists $\mathbf{v} \in \mathcal{X}$ such that $(\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*))(\mathbf{v} - \mathbf{x}^*) > 0$. Then, by the first order approximation together with the condition (1.2), we can find some t_n small enough such that

$$\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^* + t_n(\mathbf{v} - \mathbf{x}^*)) = \boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) + t_n \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\mathbf{v} - \mathbf{x}^*) + o(t_n) > 0, \tag{2.1}$$

where $\frac{o(t)}{t} \to 0$ as $t \to 0^+$, and $\mathbf{x}^* + t_n(\mathbf{v} - \mathbf{x}^*) \in \mathcal{X}$. The latter means that $-\mathbf{g}(\mathbf{x}^* + t_n(\mathbf{v} - \mathbf{x}^*)) \in Q$ and consequently, $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^* + t_n(\mathbf{v} - \mathbf{x}^*)) \leq 0$, which contradicts to (2.1). □

3 Main Results

In this section, we present the constraint qualifications that are used to derive the KKT conditions for (MOP) and their connections. Afterward, we will establish necessary and sufficient KKT optimality conditions for a weak Pareto minimum of (MOP). Moreover, we also establish sufficient conditions for a Pareto minimum of the problem (MOP).

At first, we recall one of constraint qualifications the so-called *non-degeneracy condition* at some feasible point $\mathbf{x}^* \in \mathcal{X}$ in the vector setting, which has been introduced in [21].

Assumption 1: (Non-degeneracy condition [21]) Consider (MOP), for every $\boldsymbol{\mu} \in Q^*\backslash\{\mathbf{0}\}$,

$$\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*) \neq \mathbf{0} \text{ whenever } \boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0.$$

Remark 3.1 (Sufficient condition for non-degeneracy condition to be valid)**.** Note that if the Slater-type cone constraint qualification holds at $\mathbf{x}^*$, then the non-degeneracy condition is satisfied at $\mathbf{x}^*$. Indeed, if there exists $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that $\mathbf{g}(\mathbf{x}^*) + \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) \in -\text{int}Q$, then for every $\boldsymbol{\mu} \in Q^*\backslash\{\mathbf{0}\}$ fulfilling $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0$, one has $\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) = \boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) < 0$ which implies that $\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*) \neq \mathbf{0}$.

Remark 3.2. The Slater's condition can also be guaranteed by the Slater-type cone constraint qualification at some point $\mathbf{x}^*$ as well. To see this, it follows from the Slater-type cone constraint qualification that $\nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) \in -\text{int}Q - \mathbf{g}(\mathbf{x}^*)$ for some $\hat{\mathbf{x}} \in \mathbb{R}^n$. This together with the fact that

$$\frac{\mathbf{g}(\mathbf{x}^* + t(\hat{\mathbf{x}} - \mathbf{x}^*)) - \mathbf{g}(\mathbf{x}^*)}{t} = \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) + o(t),$$

where $\frac{o(t)}{t} \to \mathbf{0}$ as $t \to 0^+$, for some $t_0 > 0$ sufficiently small, it holds

$$\mathbf{g}(\mathbf{x}^* + t_0(\hat{\mathbf{x}} - \mathbf{x}^*)) \in (1 - t_0)\mathbf{g}(\mathbf{x}^*) - t_0 \text{int}Q \subseteq -\text{int}Q.$$

Hence, the Slater's condition has been justified.

Now, we present some sufficient conditions for the Slater-type cone constraint qualification to be valid.

Theorem 3.1. *Let* $\mathcal{X}$ *be defined as in* (1.1)*. Assume that the Slater's condition holds and the condition* (1.2) *is satisfied at a feasible point* $\boldsymbol{x}^* \in \mathcal{X}$*. If the non-degeneracy condition holds at* $\boldsymbol{x}^*$*, then the Slater-type cone constraint qualification also holds at* $\boldsymbol{x}^*$*.*

Proof. Suppose that the non-degeneracy condition holds at $\mathbf{x}^*$. Assume on contrary that for every $\mathbf{x} \in \mathbb{R}^n$, one has $\mathbf{g}(\mathbf{x}^*) + \nabla \mathbf{g}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \notin -\text{int}Q$, or equivalently,

$$-[\mathbf{g}(\mathbf{x}^*) + \nabla \mathbf{g}(\mathbf{x}^*)(\mathbb{R}^n - \mathbf{x}^*)] \cap \text{int}Q = \emptyset.$$

So, by the Eidelheit separation theorem, there exists $\boldsymbol{\mu} \in \mathbb{R}^m\backslash\{\mathbf{0}\}$ such that

$$\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) + \boldsymbol{\mu}^T \mathbf{y} \geq 0, \ \forall \mathbf{x} \in \mathbb{R}^n, \ \forall \mathbf{y} \in Q. \tag{3.1}$$

By taking $\mathbf{x} = \mathbf{x}^*$ and $\mathbf{y} = \mathbf{0}$ in (3.1), we would have $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0$. Hence, with regard to (3.1) with $\mathbf{x} = \mathbf{x}^*$, we get $\boldsymbol{\mu} \in Q$. Therefore, in view of (3.1), we find a vector $\boldsymbol{\mu} \in Q^*\backslash\{\mathbf{0}\}$ with $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0$ such that

$$\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \geq 0, \ \forall \mathbf{x} \in \mathbb{R}^n. \tag{3.2}$$

On the other hand, by assumption, there exists $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that $-\mathbf{g}(\hat{\mathbf{x}}) \in \text{int}Q$. Then,by the continuity of $\mathbf{g}$, there exists $r > 0$ such that $\mathbf{g}(\hat{\mathbf{x}} + r\mathbf{u}) \in -Q$ for all $\mathbf{u} \in \mathbb{B} := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}$. Consequently, $\hat{\mathbf{x}} + r\mathbf{u} \in \mathcal{X}$ for all $\mathbf{u} \in \mathbb{B}$. So, as $\mathbf{x}^* \in \mathcal{X}$ and $\mathbf{x}^*$ satisfies the condition (1.2), we conclude from Lemma 2.4 that

$$\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} + r\mathbf{u} - \mathbf{x}^*) \leq 0, \ \forall \mathbf{u} \in \mathbb{B}. \tag{3.3}$$

In particular, put $\mathbf{u} = \mathbf{0} \in \mathbb{B}$, one has $\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) \leq 0$. Thus, with regard to (3.2), $\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) = 0$, and hence we deduce from (3.3) that

$$\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)\mathbf{u} \leq 0, \ \forall \mathbf{u} \in \mathbb{B}.$$

So, $\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)$ must ultimately be zero vector, which contradicts to the validity of non-degeneracy condition at $\mathbf{x}^*$. □

Remark 3.3. In the absence of the condition (1.2) at $\mathbf{x}^*$, the validity of both Slater and the non-degeneracy conditions at $\mathbf{x}^*$ does not guarantee the validity of Slater-type cone constraint qualification at $\mathbf{x}^*$, for instance, let $\mathbf{x} := (x_1, x_2)^T \in \mathbb{R}^2$, $Q := \{\mathbf{x} \in \mathbb{R}^2 : x_1 \geq 0, \ x_2 \geq 0\}$ and $\mathbf{g}(\mathbf{x}) := (x_2^3 + x_2 - x_1, x_1 - x_2)^T$. We see that $\mathbf{g}(-3, -2) = (-7, -1)^T \in -\text{int}Q$, that is, Slater's condition holds. Also, one has $\nabla \mathbf{g}(\mathbf{x}) = \begin{pmatrix} -1 & 3x_2^2 + 1 \\ 1 & -1 \end{pmatrix}$ and a short calculation shows that the non-degeneracy holds at $\mathbf{x}^* := (0, 0)^T \in \mathcal{X}$, while the condition (1.2) together with the Slater-type cone constraint qualification is invalid at $\mathbf{x}^*$. In fact, let us consider $\mathbf{x}_0 := (-2, -1)^T \in \mathcal{X}$ and arbitrary sequence $\{t_n\}_{n \in \mathbb{N}} \subset (0, +\infty)$ such that $t_n \to 0$ as $n \to +\infty$. So, $t_{n_0} < 1$ for some $n_0 \in \mathbb{N}$ and $\mathbf{x}^* + t_{n_0}(\mathbf{x}_0 - \mathbf{x}^*) = t_{n_0}\mathbf{x}_0 \notin \mathcal{X}$. Otherwise, we have that

$$t_{n_0}(1 - t_{n_0})(1 + t_{n_0}) = (-t_{n_0})^3 + (-t_{n_0}) - (-2t_{n_0}) \leq 0,$$

whence, $1 \leq t_{n_0}$. This contradicts to the fact that $t_{n_0} < 1$. In addition, we can not find out $\hat{\mathbf{x}} := (\hat{x}_1, \hat{x}_2)^T \in \mathbb{R}^2$ such that

$$\begin{pmatrix} -\hat{x}_1 + \hat{x}_2 \\ \hat{x}_1 - \hat{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \mathbf{g}(\mathbf{x}^*) + \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) \in -\text{int}Q.$$

Remark 3.4. (i) It is worth noticing that there is a partial overlapping between Slater's condition and non-degeneracy condition at a given point $\mathbf{x}^*$ in general. For example, it is easy to check that Slater's condition fails to hold for $\mathcal{X} := \{\mathbf{x} \in \mathbb{R}^n : -\mathbf{g}(\mathbf{x}) \in Q\}$, where $Q := \{\mathbf{x} \in \mathbb{R}^2 : x_1 \geq 0, \ x_2 \geq 0\}$ and $\mathbf{g}(\mathbf{x}) := (-x_1 + x_2, x_1 - x_2)^T$ for all $\mathbf{x} \in \mathbb{R}^2$, while non-degeneracy condition holds at $\mathbf{x}^* := (0, 0)^T$. In contrast, redefining $\mathbf{g}(\mathbf{x}) := (x_1^3 - x_2 + 1, -x_1^2 + x_2 - 1)^T$ for all $\mathbf{x} \in \mathbb{R}^2$, we get $-\mathbf{g}(-1, 1) = (1, 1)^T \in \text{int}Q$ and so, Slater's condition holds. Now we see that non-degeneracy does not hold at $\mathbf{x}^*$. Indeed, taking $\boldsymbol{\mu}_0 := (1, 1)^T \in Q^* \backslash \{\mathbf{0}\}$, one has $\boldsymbol{\mu}_0^T \mathbf{g}(\mathbf{x}^*) = 0$ and

$$\boldsymbol{\mu}_0^T \nabla \mathbf{g}(\mathbf{x}^*) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

showing that non-degeneracy fails to hold at $\mathbf{x}^*$.

(ii) In addition to the Q-convexity of $\mathbf{g}$ at a given point $\mathbf{x}^*$, if Slater's condition holds, then non-degeneracy condition is satisfied at $\mathbf{x}^*$. To see this, suppose now by contradiction that there exists $\boldsymbol{\mu}_0 \in Q^* \backslash \{\mathbf{0}\}$ satisfying $\boldsymbol{\mu}_0^T \mathbf{g}(\mathbf{x}^*) = 0$ and $\boldsymbol{\mu}_0^T \nabla \mathbf{g}(\mathbf{x}^*) = \mathbf{0}$. It then follows from Q-convexity of $\mathbf{g}$ at $\mathbf{x}^*$ that $\boldsymbol{\mu}_0^T \mathbf{g}(\hat{\mathbf{x}}) - \boldsymbol{\mu}_0^T \mathbf{g}(\mathbf{x}^*) = \boldsymbol{\mu}_0^T \mathbf{g}(\hat{\mathbf{x}}) - \boldsymbol{\mu}_0^T \mathbf{g}(\mathbf{x}^*) - \boldsymbol{\mu}_0^T \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) \geq 0$ for a Slater's point $\hat{\mathbf{x}}$. This contradicts to the fact that $\boldsymbol{\mu}_0^T \mathbf{g}(\hat{\mathbf{x}}) < 0 = \boldsymbol{\mu}_0^T \mathbf{g}(\mathbf{x}^*)$.

Remark 3.5. In the case of $Q = \mathbb{R}^m_+ := \{(x_1, x_2, \ldots, x_m)^T \in \mathbb{R}^m : x_i \geq 0,\ \forall i = 1, \ldots, m\}$, non-degeneracy conditions at $\mathbf{x}^*$ can be view as the *Mangasarian-Fromovitz constraint qualification* at $\mathbf{x}^*$ and non-degeneracy conditions at $\mathbf{x}^*$ in [8, 12] as well. Indeed,

$$\begin{aligned}
&\exists \mathbf{v} \in \mathbb{R}^n \text{ such that } \nabla g_i(\mathbf{x}^*)^T \mathbf{v} < 0,\ \forall i \in I(\mathbf{x}^*)\\
\Leftrightarrow &\mathbf{0} \notin \mathrm{co}\{\nabla g_i(\mathbf{x}^*) : i \in I(\mathbf{x}^*)\}\\
\Leftrightarrow &\forall \boldsymbol{\mu} := (\mu_1, \mu_2, \ldots, \mu_m)^T \in \mathbb{R}^m_+ \backslash \{\mathbf{0}\} \text{ s.t. } \mu_i g_i(\mathbf{x}^*) = 0,\ i = 1, 2, \ldots, m,\\
&\sum_{i=1}^m \mu_i \nabla g_i(\mathbf{x}^*) \neq \mathbf{0},
\end{aligned}$$

and for each $i \in \{1, 2, \ldots, m\}$, by taking $\boldsymbol{\mu} := \boldsymbol{e}_i$, where $\boldsymbol{e}_i$ is the unit vector in $\mathbb{R}^m$ with the ith component is 1 and the others are 0, one has $\nabla g_i(\mathbf{x}^*) \neq \mathbf{0}$ whenever $i \in I(\mathbf{x}^*)$. Note that Slater-type cone constraint qualification at $\mathbf{x}^*$ is also equivalent to the *Robinson constraint qualification* at $\mathbf{x}^*$ [4, Lemma 2.99, p. 69]. Then, as the considered set $\{\mathbf{x} \in \mathbb{R}^n,\ g_i(\mathbf{x}) \leq 0,\ i = 1, 2, \ldots, m\}$ is not necessarily convex, it seems that Theorem 3.1 extends [5, Theorem 2.1] to non-convex setting on the set $\{\mathbf{x} \in \mathbb{R}^n,\ g_i(\mathbf{x}) \leq 0,\ i = 1, 2, \ldots, m\}$.

Now, we are in the position to give necessary and sufficient KKT optimality conditions for a weak Pareto minimum of (MOP).

Theorem 3.2. *Consider the problem* (MOP) *and let both Assumption 1 and the condition* (1.2) *be satisfied at a feasible point* $\boldsymbol{x}^*$.

(i) *If* $\boldsymbol{x}^*$ *is a weak Pareto minimum of* (MOP)*, then* $\boldsymbol{x}^*$ *is a KKT point.*

(ii) *Conversely, if* $\boldsymbol{x}^*$ *is a non-trivial KKT point with multipliers* $\boldsymbol{\lambda}$ *and* $\boldsymbol{\mu}$*, and* $L^<_{\boldsymbol{\lambda}^T \boldsymbol{f}}(\boldsymbol{x}^*)$ *is convex then* $\boldsymbol{x}^*$ *is a weak Pareto minimum of* (MOP).

Proof. (i) Let $\mathbf{x}^* \in \mathcal{X}$ be a weak Pareto minimum of (MOP). By Lemma 2.2, there exist $\boldsymbol{\lambda} \in K^*$ and $\boldsymbol{\mu} \in Q^*$ not both zero such that $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0$ and

$$\Big(\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)\Big)(\mathbf{x} - \mathbf{x}^*) \geq 0,\ \forall \mathbf{x} \in \mathbb{R}^n. \tag{3.4}$$

As the inequality (3.4) holds for every $\mathbf{x} \in \mathbb{R}^n$, we conclude that

$$\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*) = \mathbf{0} \text{ and } \boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0.$$

Moreover, we assert that $\boldsymbol{\lambda} = \mathbf{0}$. Otherwise, it follows in turn that $\boldsymbol{\mu} \neq \mathbf{0}$, which stands in a contradiction to Assumption 1, and therefore, $\boldsymbol{\lambda} \neq \mathbf{0}$.

(ii) Let $\mathbf{x}^* \in \mathcal{X}$ be an arbitrary non-trivial KKT point, i.e.,

$$\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*) = \mathbf{0};\ \boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0,$$

for some non-zero vectors $\boldsymbol{\lambda} \in \mathbb{R}^p$ and $\boldsymbol{\mu} \in \mathbb{R}^m$. This together with Assumption 1 implies that $\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*)$ must ultimately be non-zero vector. It can be seen that if the set $L^<_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*)$ is empty, then $\mathbf{x}^*$ actually is a weak Pareto minimum of (MOP). In fact, if $\mathbf{x}^*$ is not

The set $\{\mathbf{x} \in \mathbb{R}^n,\ g_i(\mathbf{x}) \leq 0,\ i = 1, 2, \ldots, m\}$ is said to satisfy the *Mangasarian-Fromovitz constraint qualification* [4] at $\mathbf{x}^*$ if there exists $\mathbf{v} \in \mathbb{R}^n$ s.t. $\nabla g_i(\mathbf{x}^*)^T \mathbf{v} < 0$ for each $i \in I(\mathbf{x}^*) := \{i \in \{1, 2, \ldots, m\} : g_i(\mathbf{x}^*) = 0\}$.

One says that the set $\{\mathbf{x} \in \mathbb{R}^n,\ g_i(\mathbf{x}) \leq 0,\ i = 1, 2, \ldots, m\}$ satisfies the *Robinson constraint qualification* at $\mathbf{x}^*$ if $\mathbf{0} \in \mathrm{int}\{\mathbf{g}(\mathbf{x}^*) + \nabla \mathbf{g}(\mathbf{x}^*)(\mathbb{R}^n - \mathbf{x}^*) + \mathbb{R}^m_+\}$ where $\mathbf{g}(\mathbf{x}) := (g_1(\mathbf{x}), g_2(\mathbf{x}), \ldots, g_m(\mathbf{x}))^T$.

a weak Pareto minimum of (MOP), there exists $\mathbf{x} \in \mathcal{X}$ such that $\mathbf{f}(\mathbf{x}^*) - \mathbf{f}(\mathbf{x}) \in \text{int}K$. So, by the virtue of Lemma 2.1, $\boldsymbol{\lambda}^T\mathbf{f}(\mathbf{x}^*) > \boldsymbol{\lambda}^T\mathbf{f}(\mathbf{x})$, which contradicts to the fact that $L^{<}_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*) = \emptyset$. Let us consider in the case that $L^{<}_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*) \neq \emptyset$. Applying Proposition 2.3(i) with $f(\mathbf{x}) := \boldsymbol{\lambda}^T\mathbf{f}(\mathbf{x})$, we obtain that

$$\boldsymbol{\lambda}^T\nabla\mathbf{f}(\mathbf{x}^*)(\mathbf{u} - \mathbf{x}^*) \leq 0, \ \forall \mathbf{u} \in L^{<}_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*). \tag{3.5}$$

Therefore, by Lemma 2.4,

$$\boldsymbol{\lambda}^T\nabla\mathbf{f}(\mathbf{x}^*)(\mathbf{v} - \mathbf{x}^*) = -\boldsymbol{\mu}^T\nabla\mathbf{g}(\mathbf{x}^*)(\mathbf{v} - \mathbf{x}^*) \geq 0, \ \forall \mathbf{v} \in \mathcal{X}. \tag{3.6}$$

Note that,

$$\{\mathbf{y} \in \mathbb{R}^n : \mathbf{f}(\mathbf{x}^*) - \mathbf{f}(\mathbf{y}) \in \text{int}K\} \subseteq L^{<}_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*).$$

Thus, in order to obtain $\mathbf{x}^*$ to be a weak Pareto minimum of (MOP), it suffices to show that $\mathcal{X} \subseteq \mathbb{R}^n \backslash L^{<}_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*)$ equivalently, $L^{<}_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*) \cap \mathcal{X} = \emptyset$. Suppose, ad absurdum, $L^{<}_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*) \cap \mathcal{X} \neq \emptyset$. Thus, from (3.5) and (3.6) we get the assertion that $\boldsymbol{\lambda}^T\nabla\mathbf{f}(\mathbf{x}^*)(\mathbf{w} - \mathbf{x}^*) = \mathbf{0}$ for any $\mathbf{w} \in L^{<}_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*) \cap \mathcal{X}$. Furthermore, as the set $L^{<}_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*)$ being open, for each $\mathbf{d} \in \mathbb{R}^n$ we can find $t > 0$ small enough such that $\mathbf{w} + t\mathbf{d} \in L^{<}_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*)$. Hence,

$$t\boldsymbol{\lambda}^T\nabla\mathbf{f}(\mathbf{x}^*)\mathbf{d} = \boldsymbol{\lambda}^T\nabla\mathbf{f}(\mathbf{x}^*)(\mathbf{w} + t\mathbf{d} - \mathbf{x}^*) - \boldsymbol{\lambda}^T\nabla\mathbf{f}(\mathbf{x}^*)(\mathbf{w} - \mathbf{x}^*) \leq 0,$$

and consequently, $\boldsymbol{\lambda}^T\nabla\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$, a contradiction. Thus, $L^{<}_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*) \cap \mathcal{X} = \emptyset$, and $\mathbf{x}^*$ is a weak Pareto minimum of (MOP) as desired. □

Remark 3.6. It is worth mentioning here that Proposition 2.3 plays a significant role in Theorem 3.2(ii) for ensuring a feasible point $\mathbf{x}^*$ to be a weak Pareto minimum of (MOP). Beside, non-degeneracy condition (Assumption 1) at $\mathbf{x}^*$ need to be assumed for guaranteeing $\boldsymbol{\lambda}^T\nabla\mathbf{f}(\mathbf{x}^*) \neq \mathbf{0}$ with correspond to multiplier vector $\boldsymbol{\lambda} \in K^*\backslash\{\mathbf{0}\}$. In contrast, it generally does not need constraint qualification to establish the sufficient optimality conditions. Therefore, it might be reasonably assumed the assertion $\boldsymbol{\lambda}^T\nabla\mathbf{f}(\mathbf{x}^*) \neq \mathbf{0}$ instead of assuming the non-degeneracy condition at $\mathbf{x}^*$. However, keeping in mind the fact that we need to justify the convexity of $L^{<}_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*)$ with the same choice $\boldsymbol{\lambda}$, and so in this case the multiplier vector $\boldsymbol{\lambda}$ turn out to be difficult to determine for satisfying all conditions in Theorem 3.2(ii) simultaneously. This being a reason why non-degeneracy condition make used in Theorem 3.2(ii). Another reason is that justifying the non-degeneracy condition is done before verifying sufficient optimality conditions.

We now demonstrate with the following example to indicate that Theorem 3.2 may be conveniently applied in some cases however Theorem 3.1 and Theorem 3.2 of [21] cannot be used even when the feasible set $\mathcal{X}$ is convex.

Example 3.1. Consider the following muti-objective optimization problem (MOP) over cone constraint:

$$\begin{array}{ll} K- & \text{Minimize } \mathbf{f}(x) := (x+1, x^3 - 5x^2 + 8x - 3)^T \\ & \text{subject to } x \in \mathcal{X} := \{x \in \mathbb{R} : -\mathbf{g}(x) \in Q\}, \end{array}$$

where $\mathbf{g}(x) := (x - 1, x^2 - x - 1)^T$, $K := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq 0, \ x_2 \geq 0\}$ and $Q := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \leq 0, x_2 \leq x_1\}$. A straightforward calculation shows that:

- $\mathcal{X} = [2, +\infty)$,

- $K^* = K$,
- $Q^* = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leq 0,\ x_2 \leq -x_1\}$,
- $x^* := 2$ satisfies the non-trivial KKT conditions by taking $\boldsymbol{\lambda} := (2, 0)^T$ and $\boldsymbol{\mu} := (1, -1)^T$,
- $L^{<}_{\boldsymbol{\lambda}^T \mathbf{f}}(x^*) = (-\infty, 2)$ is convex, and
- it is easily to seen that Assumption 1 and the condition (1.2) are satisfied.

Applying Theorem 3.2 (ii), we can conclude that x^* is a weak Pareto minimum of (MOP). However, it can be checked that $\mathbf{g}$ is not Q-convex, i.e.,

$$\mathbf{g}(1) - \mathbf{g}(2) - \nabla \mathbf{g}(2)(1 - 2) = (0, 1)^T \notin Q,$$

but the feasible set $\mathcal{X}$ is convex. Furthermore, the function $\mathbf{f}$ is not K-pseudoconvex at $x^* := 2$, because if we take $x = 0$ then

$$-\nabla \mathbf{f}(x^*)(x - x^*) = (2, 0)^T \notin \text{int}K, \text{ whereas } \mathbf{f}(x^*) - \mathbf{f}(x) = (2, 4)^T \in \text{int}K.$$

Hence, the corresponding results [21] is not applicable. □

Note that the multiplier vector $\boldsymbol{\mu}$ is assumed to be non-zero vector (the non-triviality of the KKT conditions) in order to ensure that $\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) \neq \mathbf{0}$ in Theorem 3.2(ii). The following example demonstrates that this assumption cannot be dropped.

Example 3.2. Let $\mathbf{f}(x) := (x+1, -(x-2)^3)^T$, $\mathbf{g}(x) := (x^2 - 1, 2x - 1)^T$, $K := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \geq -x_1,\ x_1 \geq 0\}$ and $Q := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq x_2,\ x_1 \geq 0\}$. It is not hard to check that $\mathcal{X} = [1, 2]$, $x^* := 2$ is a KKT point with $\boldsymbol{\lambda} := (0, -1)^T$ and $\boldsymbol{\mu} := (0, 0)^T$, and all the conditions in Theorem 3.2 (ii) are fullfilled. However x^* is not even a weak Pareto minimum, i.e., if we take $x := \frac{3}{2}$ then $f(x^*) - f(x) = (3, 0)^T - (\frac{5}{2}, \frac{1}{8})^T = (\frac{1}{2}, -\frac{1}{8})^T \in \text{int}K$. The main reason is that x^* is not a non-trivial KKT point. □

To appreciate Theorem 3.2 we present an example that is applicable while the aforementioned result in [21] is not.

Example 3.3. Consider the following multi-objective optimization problem (MOP) over cone constraint:

$$K- \quad \begin{array}{l} \text{Minimize } \mathbf{f}(x) := (x^2 - 1, -x^3 + 5x^2 - 8x + 5)^T \\ \text{subject to } x \in \mathcal{X} := \{x \in \mathbb{R} : -\mathbf{g}(x) \in Q\}, \end{array}$$

where $\mathbf{g}(x) := (x^3 + x^2 + x, x^3 + 2x^2 - 5x + 8)^T$, $K := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 \leq x_1\}$ and $Q := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \leq 0, x_2 \leq x_1\}$. Evidently, $\mathbf{f}$, and $\mathbf{g}$ are not K-convex, and Q-convex, respectively. Indeed, $\mathbf{f}(1) - \mathbf{f}(0) - \nabla \mathbf{f}(0)(1 - 0) = (1, 4)^T \notin K$, and $\mathbf{g}(1) - \mathbf{g}(0) - \nabla \mathbf{g}(0)(1 - 0) = (2, 3)^T \notin Q$. It is easy to verify that $\mathcal{X} = [0, 2] \cup [4, +\infty)$. Then we have already seen that the feasible set $\mathcal{X}$ is not convex. Therefore, the results in [21] cannot be applicable. cone constraint: it is not hard to verify that

- $K^* = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leq 0,\ x_2 \geq -x_1\}$,
- $Q^* = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leq 0,\ x_2 \leq -x_1\}$,
- $x^* := 0$ satisfies the non-trivial KKT conditions by taking $\boldsymbol{\lambda} := (1, -1)^T$ and $\boldsymbol{\mu} := (-8, 0)^T$,

- Assumption 1 and the condition (1.2) are satisfied, and
- $L^{<}_{\boldsymbol{\lambda}^T\mathbf{f}}(x^*) = (-\infty, 0)$, which is convex.

Hence, Theorem 3.2(ii) indicates that x^* is a weak Pareto minimum of (MOP). □

Next, we will see now how the convexity of $L_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*)$ together with the strict convexity of $L_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*)$ at a non-trivial KKT point $\mathbf{x}^*$ possess $\mathbf{x}^*$ to be a Pareto minimum of (MOP).

Theorem 3.3. *Consider the problem* (MOP) *and let both Assumption 1 and the condition* (1.2) *be satisfied at a feasible point* $\boldsymbol{x}^*$. *If* $\boldsymbol{x}^*$ *is a non-trivial KKT point with multipliers* $\boldsymbol{\lambda}$ *and* $\boldsymbol{\mu}$, $L_{\boldsymbol{\lambda}^T\boldsymbol{f}}(\boldsymbol{x}^*)$ *is convex, and additionally* $L_{\boldsymbol{\lambda}^T\boldsymbol{f}}(\boldsymbol{x}^*)$ *is strictly convex at* $\boldsymbol{x}^*$, *then* $\boldsymbol{x}^*$ *is a Pareto minimum of* (MOP).

Proof. In a similar manner of the second argument as the proof of Theorem 3.2, by the KKT conditions and Proposition 2.3(ii), we arrive at the following assertion

$$\boldsymbol{\lambda}^T\nabla\mathbf{f}(\mathbf{x}^*)(\mathbf{v}-\mathbf{x}^*) \geq 0 \geq \boldsymbol{\lambda}^T\nabla\mathbf{f}(\mathbf{x}^*)(\mathbf{u}-\mathbf{x}^*),\ \forall\mathbf{v}\in\mathcal{X}, \forall\mathbf{u}\in L_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*), \tag{3.7}$$

and $\boldsymbol{\lambda}^T\nabla\mathbf{f}(\mathbf{x}^*) \neq \mathbf{0}$. To establish the desired results, we argue first by using Lemma 2.1 that

$$\{\mathbf{y}\in\mathbb{R}^n : \mathbf{f}(\mathbf{x}^*) - \mathbf{f}(\mathbf{y}) \in K\backslash\{\mathbf{0}\}\} \subseteq L_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*)\backslash\{\mathbf{x}^*\}.$$

Thus, we only need to justify this containment

$$\mathcal{X} \subseteq \mathbb{R}^n\backslash(L_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*)\backslash\{\mathbf{x}^*\}).$$

We argue by contradiction that there exists some $\mathbf{w}\in\mathcal{X}$ such that $\mathbf{w}\neq\mathbf{x}^*$ and $\mathbf{w}\in L_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*)$. Taking (3.7) into account we actually have

$$\boldsymbol{\lambda}^T\nabla\mathbf{f}(\mathbf{x}^*)(\mathbf{w}-\mathbf{x}^*) = 0.$$

Furthermore, as $\boldsymbol{\lambda}^T\nabla\mathbf{f}(\mathbf{x}^*) \in N(L_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*), \mathbf{x}^*)\backslash\{\mathbf{0}\}$ (by the second inequality in (3.7)) and $L_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*)$ is strictly convex, then $\boldsymbol{\lambda}^T\nabla\mathbf{f}(\mathbf{x}^*)(\mathbf{w}-\mathbf{x}^*) < 0$. This is a contradiction, and thereby implying that $\mathbf{x}^*$ is a Pareto minimum of (MOP). □

Remark 3.7. In Example 3.3 with $\boldsymbol{\lambda} := (1, -1)^T$, it is evident that $L_{\boldsymbol{\lambda}^T\mathbf{f}}(x^*)$ is strictly convex at $x^* := 0$, and hence, by Theorem 3.3, x^* is a Pareto minimum of (MOP) (see, Figure 1).

Remark 3.8. It should be noted that to obtain a Pareto minimum in the literature (see [9, 21] and other references therein), the multiplier vector $\boldsymbol{\lambda}$ in KKT conditions need to be taken from the strict positive dual cone of K, K^{s^*}, which defined as

$$K^{s^*} := \{\mathbf{z}\in\mathbb{R}^n : \mathbf{x}^T\mathbf{z} > 0 \text{ for all } \mathbf{x}\in K\backslash\{\mathbf{0}\}\}.$$

However, in this case study the multiplier vector $\boldsymbol{\lambda}$ is not necessarily to take from the strict positive dual cone. In fact, as K defined in Example 3.3 and $\boldsymbol{\lambda} := (1, -1)^T$, Then elementary calculations give us

$$K^{s^*} = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 > 0,\ x_2 > -x_1\}$$

and so, $\boldsymbol{\lambda} \notin K^{s^*}$.

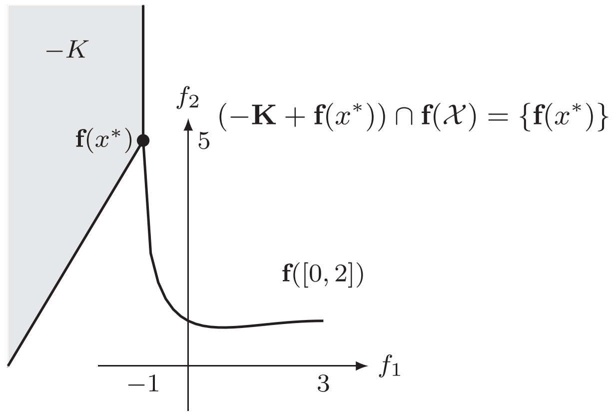


Figure 1: In Example 3.3, $x^* := 0$ is a Pareto minimum of (MOP).

To this end, we now give an example showing that the strict convexity of $L_{\boldsymbol{\lambda}^T\mathbf{f}}(x^*)$ with corresponding multiplier $\boldsymbol{\lambda}$ is essential for $\mathbf{x}^*$ under the question to be a Pareto minimum of (MOP) in Theorem 3.3.

Example 3.4. Let $\mathbf{x} := (x_1, x_2)^T \in \mathbb{R}^2$, $\mathbf{f}(\mathbf{x}) := (x_1^2, x_2 - x_1)^T$, $\mathbf{g}(\mathbf{x}) := (-x_1^3 + 3x_1 + x_2, x_1 - x_2)^T$ and $K = Q := \{\mathbf{x} \in \mathbb{R}^2 : x_1 \le 0,\ x_2 \ge 0\}$. It is easy to check that the feasible set $\mathcal{X}$ is not convex and the condition (1.2) is valid at $\mathbf{x}^* := (1,1)^T \in \mathcal{X}$. Then elementary calculations give us

- $K^* = Q^* = K$,
- $\mathbf{g}(\mathbf{x}^*) = (3,0)^T$, $\nabla\mathbf{g}(\mathbf{x}^*) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, $\mathbf{f}(\mathbf{x}^*) = (1,0)^T$, $\nabla\mathbf{f}(\mathbf{x}^*) = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$,
- $\mathbf{x}^*$ satisfies Assumption 1 and the non-trivial KKT conditions by taking $\boldsymbol{\lambda} = \boldsymbol{\mu} := (0,1)^T$,
- $L^<_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*) = \{(x_1,x_2)^T \in \mathbb{R}^2 : x_2 < x_1\}$ and $L_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*) = \{(x_1,x_2)^T \in \mathbb{R}^2 : x_2 \le x_1\}$, which are convex sets.

By Theorem 3.2 (ii), we can conclude that $\mathbf{x}^*$ is a weak Pareto minimum of (MOP). However, the set $L_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*)$ is not a strictly convex set at $\mathbf{x}^*$, i.e., it is clear that $N(L_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*), \mathbf{x}^*) = \{(-r,r)^T \in \mathbb{R}^2 : r \ge 0\}$. So, by taking $\mathbf{u} := (-1,1)^T \in N(L_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*), \mathbf{x}^*) \backslash \{(0,0)^T\}$ and $\mathbf{y} := (2,2)^T \in L_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*) \backslash \{(0,0)^T\}$, one has $\mathbf{u}^T(\mathbf{y} - \mathbf{x}^*) = 0$. Actually, a point $\mathbf{x}^*$ is not even a Pareto minimum, i.e., if we take $\bar{\mathbf{x}} := (-2,-2)^T \in \mathcal{X}$, one has

$$\mathbf{f}(\mathbf{x}^*) - \mathbf{f}(\bar{\mathbf{x}}) = (-3,0)^T \in K \backslash \{(0,0)^T\}.$$

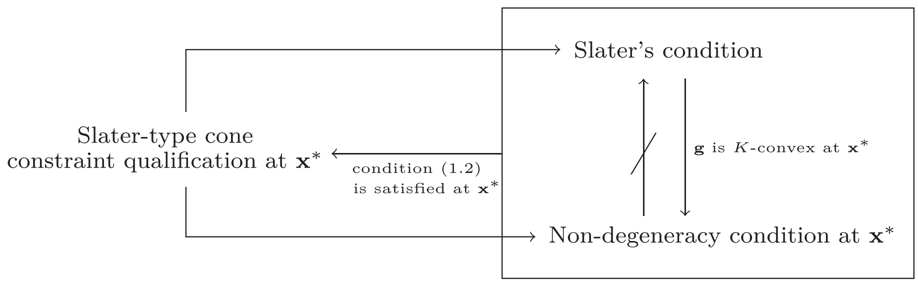
Remark 3.9. It is worth noting that the convexity of $L^<_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*)$ (resp. $L_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*)$) in Theorem 3.2 (resp. in Theorem 3.3) can be viewed as a generalized quasiconvexity of $\mathbf{f}$ at $\mathbf{x}^*$ due to the notion of $*$-*quasiconvexity* [10] in the sense that for each $\boldsymbol{\lambda} \in K^*$ the function $\boldsymbol{\lambda}^T\mathbf{f} : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex. It is quite clear from the definition that $*$-quasiconvexity of $\mathbf{f}$ guarantees

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *quasiconvex* if its sublevel set $L_f(\mathbf{x})$ at $\mathbf{x}$ is convex for all $\mathbf{x} \in \mathbb{R}^n$ or, equivalently, the strict sublevel set $L^<_f(\mathbf{x})$ at $\mathbf{x}$ is convex for all $\mathbf{x} \in \mathbb{R}^n$.

the convexity of the level set $L^{<}_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*)$ or of $L_{\boldsymbol{\lambda}^T\mathbf{f}}(\mathbf{x}^*)$. In fact, the function $\mathbf{f}$ in Example 3.4 is not $*$-quasiconvex, i.e., by taking $\boldsymbol{\lambda} := (-1,1)^T \in K^*$ and $\mathbf{x} := (1,1)^T$, the sublevel set $L_f(\mathbf{x})$ is not convex. For related conditions for cone quasiconvex mappings we refer the reader to [3, 13, 16].

4 Conclusions

In this paper, we have established necessary and sufficient the Karush-Kuhn-Tucker optimality conditions for a weak Pareto minimum as well as a Pareto minimum of a differentiable multi-objective optimization problem (MOP) over cone constraint without the convexity of the feasible set, and the cone-convexity of objective and constraint functions. We also have proposed constraint qualifications, and have discussed the relationship among them which can be summarized in following diagram whenever $\mathbf{x}^* \in \mathcal{X}$:



References

[1] S. Aggarwal, Optimality and Duality in Mathematical Programming Involving Generalized Convex Functions, Ph.D. Thesis, University of Delhi, Delhi, 1998.

[2] Q.H. Ansari and J.C. Yao, *Recent Developments in Vector Optimization*, Springer-Verlag, Berlin, Heidelberg, 2012.

[3] J. Benoist, J.M. Borwein and N. Popovici, A characterization of quasiconvex vector-valued functions, *Proc. Am. Math. Soc.* 131 (2003) 1109–1113.

[4] J.F. Bonnans, A. Shapiro and N. York, *Perturbation Analysis of Optimization Problems*, Springer, 2000.

[5] N.H. Chieu, V. Jeyakumar, G. Li, H. Mohebi, Constraint qualifications for convex optimization without convexity of constraints : New connections and applications to best approximation, *Eur. J. Oper. Res.* 265 (2018) 19–25.

[6] J. Dutta and C.S. Lalitha, Optimality conditions in convex optimization revisited, *Optim. Lett.* 7 (2013) 221–229.

[7] M. Ehrgott, *Multicriteria Optimization*, second ed., Springer, Berlin, 2005

[8] Q. Ho, Necessary and sufficient KKT optimality conditions in non-convex optimization, *Optim. Lett.* 11 (2017) 41–46.

[9] J. Jahn, *Vector Optimization-theory, Applications, and Extensions*, Springer, Berlin Heidelberg New York, 2003

[10] V. Jeyakumar, W. Oettli and M. Natividad, A solvability theorem for a class of quasiconvex mappings with applications to optimization, *J. Math. Anal. Appl.* 179 (1993) 537–546.

[11] P.Q. Khanh, H.T. Quyen and J.C. Yao, Optimality conditions under relaxed quasiconvexity assumptions using star and adjusted subdifferentials, *Eur. J. Oper. Res.* 212 (2011) 235–241.

[12] J.B. Lasserre, On representations of the feasible set in convex optimization, *Optim. Lett.* 4 (2010) 1–5.

[13] D.T. Luc, *Theory of Vector Optimization*, Lecture Notes in Economics and Mathematical Systems, vol. 319. Springer, Berlin Heidelberg New York, 1989

[14] J.E. Martinez-Legaz, Optimality conditions for pseudoconvex minimization over convex sets defined by tangentially convex constraints, *Optim. Lett.* 9 (2015) 1017–1023.

[15] V. Pareto, *Course d'economie politique*, Rouge, Lausanne, 1896

[16] P. H. Sach, Characterization of scalar quasiconvexity and convexity of locally Lipschitz vector-valued maps, *Optimization* 46 (1999) 283-310.

[17] Y. Sawaragi, H. Nakayama and T. Tanino, *Theory of Multiobjective Optimization*, Mathematics in Science and Engineering, vol. 176. Academic Press, Inc., Orlando, 1985.

[18] W. Stadler, Multicriteria optimization in mechanics: a survey, *Appl. Mech. Rev.* 37 (1984) 277–286.

[19] W. Stadler, *Multicriteria Optimization in Engineering and in the Sciences*, Plenum Press, New York, 1988.

[20] S.K. Suneja, S. Aggarwal and S. Davar, Multiobjective symmetric duality involving cones, *Eur. J. Oper. Res.*141(2002) 471–479.

[21] S.K. Suneja, S. Sharma, M.B. Grover and M. Kapoor, A different approach to cone-convex optimization, *American J. Oper. Res.* 3 (2013) 536–541.

[22] T. Weir, B. Mond and B. D. Craven, Weak minimization and duality, *Numer. Func. Anal. Optim.* 9 (1987) 181–192.

[23] S. Yamamoto, D. Kuroiwa, Constraint qualifications for KKT optimality condition in convex optimization with locally Lipschitz inequality constraints, *Linear and Nonlinear Anal.* 2 (2016) 101–111.

[24] P.L. Yu, *Multiple-Criteria Decision Making: Concepts*, Techniques and Extensions, Plenum Press, New York, 1985.

Manuscript received 20 May 2018
revised 25 September 2018
accepted for publication 30 September 2018

Nithirat Sisarat
Department of Mathematics, Faculty of Science
Naresuan University
Phitsanulok 65000, Thailand
E-mail address: nithirats@hotmail.com

Rabian Wangkeeree
Naresuan University
Phitsanulok 65000, Thailand
Research center for Academic Excellence in Mathematics
Naresuan University
Phitsanulok 65000, Thailand
E-mail address: rabianw@nu.ac.th