Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 15, No. 3, pp. 441 – 456, 2019

PROPERTIES OF THE NONNEGATIVE SOLUTION SET OF MULTI-LINEAR EQUATIONS*

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Abstract: Multi-linear equations and tensor complementarity problems are two hot topics in recent years. It is known that the nonnegative solution set of multi-linear equations is a subset of the solution set of the corresponding tensor complementarity problem. In this paper, we first investigate the existence and uniqueness of nonnegative (positive) solution to multi-linear equations induced by some triangular tensors; and then, we discuss the non-existence of nonnegative solution to multi-linear equations induced by B (B_0) tensors or strictly diagonally dominant tensors. In addition, we also investigate the boundedness of the nonnegative solution set of multi-linear equations with some structured tensors. The obtained properties of the nonnegative solution set of multi-linear equations give some characteristics on the solution set of tensor complementarity problems.

Key words: *structured tensors, multi-linear equations, existence of solution, boundedness of solution set*

Mathematics Subject Classification: *15A48, 15A69, 65F10, 65H10, 65N22*

1 Introduction

Suppose that m and n are two positive integers. Denote $[n] := \{1, 2, \ldots, n\}$. A real mth order n-dimensional tensor $\mathcal{A} = (a_{i_1\cdots i_m})$ is a multi-array of real entries $a_{i_1\cdots i_m}$, where $i_j \in [n]$ for any $j \in [m]$. Denote the set of all the real mth order n-dimensional tensors by $\mathbb{T}_{m,n}$. Our main consideration is the following system of multi-linear equations:

$$\mathcal{A}x^{m-1} = b, \tag{1.1}$$

where $\mathcal{A} \in \mathbb{T}_{m,n}$ and $b \in \mathbb{R}^n := \{b = (b_1, b_2, ..., b_n)^\top : b_i \in \mathbb{R}\}$ are given, $x \in \mathbb{R}^n$ is the vector of variables, and $\mathcal{A}x^{m-1} \in \mathbb{R}^n$ is defined by

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}, \quad \forall i \in [n].$$

It is obvious that the system of multi-linear equations (1.1) is an extension of the system of linear equations, and a special system of general nonlinear equations.

*This author's work was supported by the National Natural Science Foundation of China (Grant Nos. 11431002 and 19871051).

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The system of multi-linear equations is related to the tensor complementarity problem denoted by TCP, which is to find a vector $x \in \mathbb{R}^n$ such that

$$x \geq 0,\ \mathcal{A}x^{m-1} - b \geq 0,\ x^\top(\mathcal{A}x^{m-1} - b) = 0, \tag{1.2}$$

where $\mathcal{A} \in \mathbb{T}_{m,n}$ and $b \in \mathbb{R}^n$ are given. The TCP has been studied extensively in recent years [1–3, 6–8, 12, 13, 18, 19, 22–26, 29, 30]. It is obvious that every nonnegative solution of multi-linear equations (1.1) is a solution of the TCP (1.2).

Recently, the system of multi-linear equations has been attracting a lot of attention and has been a new topic emerged from the tensor community. Li and Ng [15] proposed some iterative methods for solving a set of sparse nonnegative multi-linear equations arising from data mining applications and obtained the linear convergence of the Jacobi and Gauss-Seidel methods under suitable conditions. Ding and Wei [5] proposed several iterative algorithms for solving multi-linear nonsingular $\mathcal{M}$-equations. Han [9] proposed a homotopy method for solving multi-linear equations with $\mathcal{M}$-tensors. Xie, Jin and Wei [27] defined the generalized circulant tensors and considered solving multi-linear equations with a circulant tensor by a fast algorithm based on the fast Fourier transform. Xie, Jin and Wei [28] proposed a new tensor method for multi-linear equations based on the rank-1 approximation of the coefficient tensor which is a strong $\mathcal{M}$-tensor. Li, Xie and Xu [14] proposed a Newton-Gauss-Seidel method for multi-linear equations, and the proposed method can be extended to solve a general system of symmetric tensor equations. Liu, Li, and Vong [17] proposed some tensor splitting algorithms for multi-linear equations. More recently, He, Ling, Qi and Zhou [10] proved that solving multilinear systems with M-tensors is equivalent to solving systems of nonlinear equations where the involving functions are P-functions, by which the authors proposed a globally and quadratically convergent algorithm for solving multilinear systems with M-tensors.

As previously mentioned, the research on the system of multi-linear equations mainly focuses on how to design effective methods to solve it. There are also two papers to study properties of the solution set of multi-linear equations. Ding and Wei [5] proved that the system of nonsingular $\mathcal{M}$-equations has a unique positive solution for any positive right-hand-side vector b; and Liu, Li, and Vong [17] discussed the existence and uniqueness conditions of solution to multi-linear equations. Motivated by the papers mentioned above, we investigate properties of the nonnegative solution set of multi-linear equations. More specifically, we investigate the existence and uniqueness, and the non-existence of nonnegative solution of multi-linear equations and the boundedness of the nonnegative solution set of multi-linear equations, where the associated tensors are some triangular tensors or B (B_0) tensors or strictly diagonally dominant tensors.

The rest of this paper is organized as follows. In Section 2, we recall some basic definitions and results which will be used in the sequel. In Section 3, we discuss the existence and uniqueness of nonnegative (positive) solution to multi-linear equations with some triangular tensors and extend the obtained results to the case of a more general form of multi-linear equations. Several non-existence results for multi-linear equations with some structured tensors are also included in this section. In Section 4, we discuss the boundedness of the solution set of multi-linear equations with some structured tensors. The final conclusions are given in Section 5.

2 Preliminaries

Given $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{T}_{m,n}$. $\mathcal{A}$ is called a *nonnegative tensor* if all of its entries are nonnegative. The entries of $\mathcal{A}$ are said to be *diagonal entries* if and only if $i_1 = i_2 = \cdots = i_m$;

and $\mathcal{A}$ is said to be a *unit tensor* if its diagonal entries $a_{ii\cdots i} = 1$ for all $i \in [n]$ and other entries are all zeros. We use the definition of triangular part of a tensor given in [5], i.e., the *lower triangular part* of $\mathcal{A}$ contains the entries $a_{i_1 i_2 \cdots i_m}$ with $i_1 \in [n]$ and $i_2, i_3, \ldots, i_m \leq i_1$; and other entries are said to be the *off-lower triangular entries*. The *strictly lower triangular part* consists of the entries $a_{i_1 i_2 \cdots i_m}$ with $i_1 \in [n] \backslash \{1\}$ and $i_2, i_3, \ldots, i_m < i_1$. Similarly, the *upper triangular part* of $\mathcal{A}$ contains the entries $a_{i_1 i_2 \cdots i_m}$ with $i_1 \in [n]$ and $i_2, i_3, \ldots, i_m \geq i_1$; and other entries are said to be *off-upper triangular entries*. The *strictly upper triangular part* consists of the entries $a_{i_1 i_2 \cdots i_m}$ with $i_1 \in [n-1]$ and $i_2, i_3, \ldots, i_m > i_1$.

Definition 2.1. A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{T}_{m,n}$ is called a lower (an upper) triangular tensor if its entries in the off-lower(off-upper) triangular part are zeros.

In the last few years, many classes of structured tensors have been introduced and studied, see the excellent monograph by Qi and Luo [21]. In the following, we recall some structured tensors and related results which will be used in this paper.

The eigenvalues of tensors are initially introduced and studied by Qi [20] and Lim [16]. If a scalar λ and a nonzero vector $x \in \mathbb{R}^n$ satisfy

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

where $\mathcal{A} \in \mathbb{T}_{m,n}$ and $x^{[m-1]} := (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})^\top$, then λ is said to be an H-eigenvalue of $\mathcal{A}$ and x is said to be a corresponding H-eigenvector. The spectral radius of $\mathcal{A}$ is defined by

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an } H\text{-eigenvalue of } \mathcal{A}\}.$$

A tensor which can be expressed by $\mathcal{A} = s\mathcal{I} - \mathcal{B}$ is called an $\mathcal{M}$-tensor, where $\mathcal{B}$ is a nonnegative tensor and $s \geq \rho(\mathcal{B})$. Furthermore, if $s > \rho(\mathcal{B})$, $\mathcal{A}$ is said to be a nonsingular $\mathcal{M}$-tensor.

The following definition and result about strong P tensors were given by Bai, Huang and Wang [1].

Definition 2.2. $\mathcal{A} \in \mathbb{T}_{m,n}$ is a strong P tensor if and only if

$$\max_{i \in [n]} \{(x_i - y_i)[(\mathcal{A}x^{m-1})_i - (\mathcal{A}y^{m-1})_i]\} > 0 \quad \text{for any } x, y \in \mathbb{R}^n \text{ with } x \neq y.$$

Theorem 2.1. *If $\mathcal{A} \in \mathbb{T}_{m,n}$ is a strong P tensor, then the TCP (1.2) has a unique solution for any $b \in \mathbb{R}^n$.*

The following definition and result about B (B_0) tensors were given by Song and Qi [23].

Definition 2.3. $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{T}_{m,n}$ is said to be a B tensor if and only if

$$\begin{cases} \displaystyle\sum_{i_2, i_3, \ldots, i_m = 1}^{n} a_{i i_2 \cdots i_m} > 0, \;\; \forall i \in [n], \\ \displaystyle\frac{1}{n^{m-1}} \left(\sum_{i_2, i_3, \ldots, i_m = 1}^{n} a_{i i_2 \cdots i_m} \right) > a_{i j_2 \cdots j_m}, \;\; \forall (j_2, j_3, \ldots, j_m) \neq (i, i, \ldots, i); \end{cases}$$

and a B_0 tensor if and only if

$$\begin{cases} \displaystyle\sum_{i_2, i_3, \ldots, i_m = 1}^{n} a_{i i_2 \cdots i_m} \geq 0, \;\; \forall i \in [n], \\ \displaystyle\frac{1}{n^{m-1}} \left(\sum_{i_2, i_3, \ldots, i_m = 1}^{n} a_{i i_2 \ldots i_m} \right) \geq a_{i j_2 \cdots j_m}, \;\; \forall (j_2, j_3, \ldots, j_m) \neq (i, i, \ldots, i). \end{cases}$$

Theorem 2.2. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{T}_{m,n}$. If $\mathcal{A}$ is a B tensor, then*

$$a_{ii\cdots i} > \sum_{a_{ii_2\cdots i_m}<0} |a_{ii_2\cdots i_m}|, \ \ \forall i \in [n];$$

and if $\mathcal{A}$ is a B_0 tensor, then

$$a_{ii\cdots i} \geq \sum_{a_{ii_2\cdots i_m}<0} |a_{ii_2\cdots i_m}|, \ \ \forall i \in [n].$$

The following definition about (strictly) diagonally dominant tensors were given by Ding, Luo and Qi [3].

Definition 2.4. $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{T}_{m,n}$ is called a diagonally dominant tensor if and only if

$$|a_{ii\cdots i}| - \sum_{(i_2,i_3,\ldots,i_m)\neq(i,i,\ldots,i)} |a_{ii_2\cdots i_m}| \geq 0, \ \ \forall i \in [n];$$

and a strictly diagonally dominant tensor if and only if

$$|a_{ii\cdots i}| - \sum_{(i_2,i_3,\ldots,i_m)\neq(i,i,\ldots,i)} |a_{ii_2\cdots i_m}| > 0, \ \ \forall i \in [n].$$

In the following two sections, we also use the following notations: $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i \in [n]\}$ and $\mathbb{R}^n_{++} := \{x \in \mathbb{R}^n : x_i > 0 \text{ for all } i \in [n]\}$.

$\boxed{3}$ Existence, Uniqueness and Non-existence of Nonnegative Solution to Multi-linear Equations

In this section, we main study the existence and uniqueness of nonnegative (positive) solution to multi-linear equations (1.1) with some triangular tensors, and extend the obtained results to a class of multi-linear equations which contains (1.1) as a special case. In addition, we also give several results on the non-existence of nonnegative solution to multi-linear equations (1.1) with a B tensor or a diagonally dominant tensor.

Theorem 3.1. *Let $b \in \mathbb{R}^n$ and $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{T}_{m,n}$ be a lower triangular tensor with positive diagonal entries. If*

(C1) *all entries in the strictly lower triangular part are nonpositive,*

then, $\mathcal{A}x^{m-1} = b$ has at least one nonnegative solution if $b \in \mathbb{R}^n_+$; and $\mathcal{A}x^{m-1} = b$ has at least one positive solution if $b \in \mathbb{R}^n_{++}$.

Furthermore, if one of the following conditions is met:

(C2) *all entries in the lower triangular part, which are neither in the strictly lower triangular part nor in the diagonal part, are nonnegative;*

(C3) *$\mathcal{A}$ is a strong P tensor,*

then, $\mathcal{A}x^{m-1} = b$ has a unique nonnegative solution if $b \in \mathbb{R}^n_+$; and $\mathcal{A}x^{m-1} = b$ has a unique positive solution if $b \in \mathbb{R}^n_{++}$.

Proof. Denote $F(x) := \mathcal{A}x^{m-1} - b$ with $F_i(x) := (\mathcal{A}x^{m-1})_i - b_i$ for all $i \in [n]$. We divide the proof into the following four parts.

(i) We assume that $b \in \mathbb{R}^n_+$ and Condition (C1) holds. Since $\mathcal{A}$ is a lower triangular tensor, it follows that

$$\begin{cases} F_1(x) = a_{11\cdots 1}x_1^{m-1} - b_1, \\ F_2(x) = \sum\limits_{i_2,\ldots,i_m=1}^{2} a_{2i_2\cdots i_m} x_{i_2}\cdots x_{i_m} - b_2, \\ \cdots\cdots \\ F_i(x) = \sum\limits_{i_2,\ldots,i_m=1}^{i} a_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m} - b_i, \\ \cdots\cdots \\ F_n(x) = \sum\limits_{i_2,\ldots,i_m=1}^{n} a_{ni_2\cdots i_m} x_{i_2}\cdots x_{i_m} - b_n. \end{cases} \tag{3.1}$$

It is easy to see that $F_i(x)$ is a function of variables $x_1, \ldots, x_i$ for any $i \in [n]$. Thus, we can consider successively $F_i(x) = 0$ for $i \in [n]$ to construct a nonnegative (positive) solution of $F(x) = 0$.

For $F_1(x) = 0$, we define a real-valued function $f_1 : \mathbb{R} \to \mathbb{R}$ by

$$f_1(x_1) := a_{11\cdots 1}x_1^{m-1} - b_1. \tag{3.2}$$

Then, $f_1(x_1) = 0$ has a nonnegative solution $\bar{x}_1 = \left(\frac{b_1}{a_{11\cdots 1}}\right)^{\frac{1}{m-1}}$ since $a_{11\cdots 1} > 0$ and $b_1 \geq 0$.

For $F_2(x) = 0$, we define a real-valued function $f_2 : \mathbb{R} \to \mathbb{R}$ by

$$f_2(x_2) := \sum_{i_2,\ldots,i_m=1}^{2} a_{2i_2\cdots i_m} z_{i_2}\cdots z_{i_m} - b_2 \tag{3.3}$$

where $z_{i_j} (j = 2, \ldots, m)$ are defined by

$$z_{i_j} := \begin{cases} x_2, & i_j = 2, \\ \bar{x}_1, & i_j = 1. \end{cases}$$

Clearly, $f_2(0) = a_{211\cdots 1}\bar{x}_1^{m-1} - b_2 \leq 0$ since $a_{211\cdots 1} \leq 0$ and $b_2 \geq 0$. If $f_2(0) = 0$, we take $\bar{x}_2 = 0$. If $f_2(0) < 0$, there exists $\hat{x}_2 > 0$ such that $f_2(\hat{x}_2) > 0$ because of $a_{22\cdots 2} > 0$. Then there exists $\bar{x}_2 \in (0, \hat{x}_2)$ such that $f_2(\bar{x}_2) = 0$ from zero point theorem.

Now, we show the result by mathematical induction. For any $k \in \{3, \ldots, n\}$, we assume that, for any $i \in \{3, \ldots, k-1\}$, by $F_i(x) = 0$ we define $f_i : \mathbb{R} \to \mathbb{R}$ (similar to (3.3)) and obtain that there exists $\bar{x}_i \in \mathbb{R}^n_+$ such that $f_i(\bar{x}_i) = 0$. We show that the result holds when $i = k$. In this case, we define real-valued functions $f_k : \mathbb{R} \to \mathbb{R}$ by

$$f_k(x_k) := \sum_{i_2,\ldots,i_m=1}^{k} a_{ki_2\cdots i_m} z_{i_2}\cdots z_{i_m} - b_k \tag{3.4}$$

where $z_{i_j} (j = 2, \ldots, m)$ are defined by

$$z_{i_j} := \begin{cases} x_k, & i_j = k, \\ \bar{x}_i, & i_j = i \ \text{ with } i \in \{1, 2, 3, \ldots, k-1\}. \end{cases}$$

Since $b_k \geq 0$ and $a_{ki_2\cdots i_m} \leq 0$ for all $i_2, \ldots, i_m \in [k-1]$, it follows that

$$f_k(0) = \sum_{i_2,\ldots,i_m=1}^{k-1} a_{ki_2\cdots i_m} \bar{x}_{i_2} \cdots \bar{x}_{i_m} - b_k \leq 0.$$

If $f_k(0) = 0$, we take $\bar{x}_k = 0$. If $f_k(0) < 0$, there exists $\hat{x}_k > 0$ such that $f_k(\hat{x}_k) > 0$ because of $a_{ki\cdots i} > 0$. Then there exists $\bar{x}_k \in (0, \hat{x}_k)$ such that $f_k(\bar{x}_k) = 0$ from zero point theorem. Thus, we can obtain recursively nonnegative real numbers $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ with $f_i(\bar{x}_i) = 0$ for all $i \in [n]$. Denote $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)^\top$. Then, by definitions of $F_i(x)$ and $f_i(x_i)$ (i.e., (3.1), (3.2) and (3.4)), it is easy to see that $\bar{x} \in \mathbb{R}^n_+$ and $F_i(\bar{x}) = 0$ for all $i \in [n]$. Thus, $\mathcal{A}x^{m-1} = b$ has at least one nonnegative solution if $b \in \mathbb{R}^n_+$.

(ii) We assume that $b \in \mathbb{R}^n_{++}$ and Condition (C1) holds. In this case, it is easy to see that $f_i(0) < 0$ for any $i \in \{2, 3, \ldots, n\}$ where f_i is defined by (3.2) and (3.4); and hence, similar to the proof given in (i), we can obtain that $\mathcal{A}x^{m-1} = b$ has at least one positive solution if $b \in \mathbb{R}^n_{++}$.

(iii) We assume that Conditions (C1) and (C2) hold. Based on the proof given in (i), we further show that each real-valued function f_i defined by (3.2) and (3.4) is strictly monotonically increasing on $\mathbb{R}_+$.

First, it is easy to show that the function f_1 defined by (3.2) is strictly monotonically increasing on $\mathbb{R}_+$ since $a_{11\cdots 1} > 0$.

Second, for any fixed $i \in \{2, 3, \ldots, n\}$ and $k \in \{0, 1, \ldots, m-1\}$, we define

$$\Omega_k^{(i)} := \left\{ (i_2, \ldots, i_m) : \begin{array}{l} \text{there are } k \text{ indices } i_j \text{ satisfying } i_j = i; \text{and} \\ \text{other indices } i_j \text{ belong to } \{1, 2, \ldots, i-1\} \end{array} \right\}$$

and

$$p_k^{(i)} := \sum_{(i_2,\ldots,i_m)\in\Omega_k^{(i)}} a_{ii_2\cdots i_m} z_{i_2} \cdots z_{i_m}$$

where $z_{i_j} (j = 2, \ldots, m)$ are defined by

$$z_{i_j} := \left\{ \begin{array}{ll} 1, & i_j = i, \\ \bar{x}_l, & i_j = l \text{ with } l \in \{1, 2, 3, \ldots, i-1\}. \end{array} \right.$$

Then, the real-valued function f_i defined by (3.4) can be re-written as

$$f_i(x_i) = p_{m-1}^{(i)} x_i^{m-1} + p_{m-2}^{(i)} x_i^{m-2} + \cdots + p_2^{(i)} x_i^2 + p_1^{(i)} x_i + p_0^{(i)} - b_i,$$

and hence,

$$f_i'(x_i) = (m-1) p_{m-1}^{(i)} x_i^{m-2} + (m-2) p_{m-2}^{(i)} x_i^{m-3} + \cdots + 2 p_2^{(i)} x_i + p_1^{(i)}.$$

It is easy to see that $p_{m-1}^{(i)} = a_{ii\cdots i} > 0$. In addition, by Condition (C2), it is easy to see that $a_{ii_2\cdots i_m} \geq 0$ for all $(i_2, \ldots, i_m) \in \Omega_k^{(i)}$ with $k \neq 0$. This, together with $(\bar{x}_1, \ldots, \bar{x}_{i-1})^\top \in \mathbb{R}^{i-1}_+$ and the definition of $p_k^{(i)}$, implies that $p_k^{(i)} \geq 0$ for all $k \in \{1, \ldots, m-2\}$. Thus,

$$f_i'(x_i) > 0 \quad \forall x_i \in (0, +\infty),$$

which implies that for any $i \in \{2, 3, \ldots, n\}$, the function f_i is strictly monotonically increasing on $\mathbb{R}_+$.

Therefore, for any $i \in \{1, 2, \ldots, n\}$, the function f_i is strictly monotonically increasing on $\mathbb{R}_+$. This, together with (i) and (ii), implies that $\mathcal{A}x^{m-1} = b$ has a unique nonnegative solution for any $b \in \mathbb{R}^n_+$ and $\mathcal{A}x^{m-1} = b$ has a unique positive solution if $b \in \mathbb{R}^n_{++}$.

(iv) We assume that Conditions (C1) and (C3) hold. Since every nonnegative solution of $\mathcal{A}x^{m-1} = b$ must be a solution of the TCP (1.2), and by Theorem 2.1, the TCP has a unique solution when Condition (C3) holds, it follows from the results obtained in (i) and (ii) that $\mathcal{A}x^{m-1} = b$ has a unique nonnegative solution for any $b \in \mathbb{R}^n_+$ and $\mathcal{A}x^{m-1} = b$ has a unique positive solution if $b \in \mathbb{R}^n_{++}$.

Combining (i)-(iv), we complete the proof. $\square$

In [5], Ding and Wei obtained the following results:

- Let $\mathcal{A} \in \mathbb{T}_{m,n}$ be a lower triangular nonsingular $\mathcal{M}$-tensor. Then, $\mathcal{A}x^{m-1} = b$ has at least one nonnegative solution if b is a nonnegative vector, and $\mathcal{A}x^{m-1} = b$ has a unique positive solution if b is a positive vector.

It is known that for any lower triangular nonsingular $\mathcal{M}$-tensor, its diagonal entries are positive [4], and its off-diagonal entries in the lower triangular part are nonpositive [4, 5]. It is easy to see that it is possible that a lower triangular tensor, which has positive entries and satisfies condition (C1), is not a nonsingular $\mathcal{M}$-tensor. In particular, if this tensor additionally satisfies condition (C2), then it must not be a nonsingular $\mathcal{M}$-tensor.

In addition, for any lower triangular tensor $\mathcal{A} \in \mathbb{T}_{m,n}$ with positive diagonal entries, a natural question is *whether condition (C1) and condition (C3) are compatible or not.* The following example can answer this question.

Example 3.2. Let $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{T}_{4,3}$ with $a_{1111} = a_{2222} = a_{3333} = a_{3311} = 1$, $a_{3131} = -1$, and others being zeros; and $b = (1, 2, 3)^T$.

Obviously, $\mathcal{A}$ is a lower triangular tensor with positive diagonal entries and satisfies Condition (C1), and it is easy to see that

$$\max_{i \in \{1,2,3\}} (x_i - y_i)(\mathcal{A}x^3 - \mathcal{A}y^3)_i = \max_{i \in \{1,2,3\}} (x_i - y_i)(x_i^3 - y_i^3) > 0$$

for any $x, y \in \mathbb{R}^3$ with $x \neq y$, which implies that $\mathcal{A}$ is a strong P tensor, i.e., Condition (C3) holds. Thus, the concerned tensor satisfies conditions given in Theorem 3.1. Since the given b is a positive vector, by Theorem 3.1, $\mathcal{A}x^{m-1} = b$ has a unique positive solution. Indeed, it is easy to check that $(1, \sqrt[3]{2}, \sqrt[3]{3})^T$ is the unique positive solution of $\mathcal{A}x^{m-1} = b$. In addition, it should be pointed that the tensor concerned in this example is not a nonsingular $\mathcal{M}$-tensor since the off-diagonal entry a_{3311} is positive.

Therefore, the results given in Theorem 3.1 can be viewed as some supplements of the above results by Ding and Wei.

In the same way as the proof of Theorem 3.1, we can obtain the following result.

Corollary 3.3. *Let $\mathcal{A} \in \mathbb{T}_{m,n}$ be a lower triangular tensor with positive diagonal entries and Condition (C1) in Theorem 3.1 be satisfied. Suppose that, for any $k \in \{2, \ldots, m-1\}$, $\mathcal{B}_k \in \mathbb{T}_{k,n}$ are given lower triangular tensors whose entries in strictly lower triangular part are nonpositive. Then, the system*

$$\mathcal{A}x^{m-1} + \mathcal{B}_{m-1}x^{m-2} + \cdots + \mathcal{B}_2 x = b$$

has at least one nonnegative (positive) solution if $b \in \mathbb{R}^n_+$ ($b \in \mathbb{R}^n_{++}$).

Furthermore, if each $\mathcal{B}_k$ ($k \in \{2,3,\ldots,m-1\}$) has positive diagonal entries; and for $\mathcal{A}$ and each $\mathcal{B}_k$ ($k \in \{2,3,\ldots,m-1\}$), if all entries in the lower triangular part, which are neither in the strictly lower triangular part nor in the diagonal part, are nonnegative, then the above system has a unique nonnegative (positive) solution if $b \in \mathbb{R}^n_+$ ($b \in \mathbb{R}^n_{++}$).

In Theorem 3.1 and Corollary 3.3, we considered the existence of nonnegative (positive) solution to multi-linear equations (1.1) with a lower triangular tensor. If the lower triangular tensor is replaced by an upper triangular tensor, then similar results can be obtained, which are given in the following.

Theorem 3.4. *Let $\mathcal{A} \in \mathbb{T}_{m,n}$ be an upper triangular tensor with positive diagonal entries. If all entries in the strictly upper triangular part are nonpositive, then $\mathcal{A}x^{m-1} = b$ has at least one nonnegative solution for any $b \in \mathbb{R}^n_+$; and $\mathcal{A}x^{m-1} = b$ has at least one positive solution if $b \in \mathbb{R}^n_{++}$.*

Furthermore, if one of the following conditions is met:

- *all entries in the upper triangular part, which are neither in the strictly upper triangular part nor in the diagonal part, are nonnegative;*
- *$\mathcal{A}$ is a strong P tensor,*

then, $\mathcal{A}x^{m-1} = b$ has a unique nonnegative solution if $b \in \mathbb{R}^n_+$; and $\mathcal{A}x^{m-1} = b$ has a unique positive solution if $b \in \mathbb{R}^n_{++}$.

Corollary 3.5. *Let $\mathcal{A} \in \mathbb{T}_{m,n}$ be an upper triangular tensor with positive diagonal entries, and all entries in the strictly upper triangular part be nonpositive. Suppose that, for any $k \in \{2,\ldots,m-1\}$, $\mathcal{B}_k \in \mathbb{T}_{k,n}$ are given upper triangular tensors whose entries in strictly upper triangular part are nonpositive. Then, the system*

$$\mathcal{A}x^{m-1} + \mathcal{B}_{m-1}x^{m-2} + \cdots + \mathcal{B}_2 x = b$$

has at least one nonnegative (positive) solution if $b \in \mathbb{R}^n_+$ ($b \in \mathbb{R}^n_{++}$).

Furthermore, if each $\mathcal{B}_k$ ($k \in \{2,3,\ldots,m-1\}$) has positive diagonal entries; and for $\mathcal{A}$ and each $\mathcal{B}_k$ ($k \in \{2,3,\ldots,m-1\}$), if all entries in the upper triangular part, which are neither in the strictly upper triangular part nor in the diagonal part, are nonnegative, then the above system has a unique nonnegative (positive) solution if $b \in \mathbb{R}^n_+$ ($b \in \mathbb{R}^n_{++}$).

Remark 3.1. As a byproduct, every condition on the existence of nonnegative (positive) solution to multi-linear equations (1.1) is also the condition on the existence of solution to the corresponding TCP (1.2).

In the following, we consider the non-existence of nonnegative solution to multi-linear equations (1.1) which is induced by B (B_0) tensors or diagonally dominant tensors.

Theorem 3.6. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{T}_{m,n}$ be a B tensor and $b \in \mathbb{R}^n$ satisfies $b \le 0$ and $b \neq 0$. Then, $\mathcal{A}x^{m-1} = b$ has no nonnegative solution.*

Proof. Suppose that there exists $\bar{x} \ge 0$ satisfying $\mathcal{A}\bar{x}^{m-1} = b$, then $\bar{x} \neq 0$ since $b \neq 0$. Thus, if we denote $|\bar{x}_{i_0}| := \|\bar{x}\|_\infty$, then $\bar{x}_{i_0} = |\bar{x}_{i_0}| > 0$. Furthermore, we have

$$\frac{(\mathcal{A}\bar{x}^{m-1})_{i_0}}{\bar{x}_{i_0}^{m-1}} = \sum_{i_2,\ldots,i_m=1}^{n} a_{i_0 i_2 \cdots i_m} \frac{\bar{x}_{i_2} \cdots \bar{x}_{i_m}}{\bar{x}_{i_0}^{m-1}}$$

$$
\begin{aligned}
&= a_{i_0 i_0 \cdots i_0} + \sum_{(i_2,\dots,i_m)\neq(i_0,\dots,i_0)} a_{i_0 i_2 \cdots i_m} \frac{\bar{x}_{i_2}\cdots\bar{x}_{i_m}}{\|\bar{x}\|_\infty^{m-1}} \\
&\geq a_{i_0 i_0 \cdots i_0} - \sum_{a_{i_1 i_2 \cdots i_m}<0} |a_{i_1 i_2 \cdots i_m}| \\
&> 0 \\
&\geq \frac{b_{i_0}}{\bar{x}_{i_0}^{m-1}},
\end{aligned}
$$

where the second inequality holds by Theorem 2.2. This contradicts the assumption that $\bar{x}$ solves $\mathcal{A}x^{m-1} = b$. So, $\mathcal{A}x^{m-1} = b$ has no nonnegative solution. $\square$

In the same way as the proof of Theorem 3.6, we can show the following result.

Corollary 3.7. *If $\mathcal{A}$ is a B_0 tensor and $b \in \mathbb{R}^n$ satisfies $b < 0$, then $\mathcal{A}x^{m-1} = b$ has no nonnegative solution.*

Theorem 3.8. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{T}_{m,n}$ be a strictly diagonally dominant tensor with $a_{ii\cdots i} > 0$ for all $i \in [n]$ and $b \in \mathbb{R}^n$ satisfies $b \leq 0$ and $b \neq 0$. Then, $\mathcal{A}x^{m-1} = b$ has no nonnegative solution. In addition, if m is odd, then $\mathcal{A}x^{m-1} = b$ has no real solution.*

Proof. Suppose that there exists $\hat{x} \geq 0$ satisfying $\mathcal{A}\hat{x}^{m-1} = b$, then $\hat{x} \neq 0$ since $b \neq 0$. Thus, if we denote $|\hat{x}_{i_0}| := \|\hat{x}\|_\infty$, then $\hat{x}_{i_0} = |\hat{x}_{i_0}| > 0$. Furthermore, we have

$$
\begin{aligned}
\frac{(\mathcal{A}\hat{x}^{m-1})_{i_0}}{\|\hat{x}\|_\infty^{m-1}} &= \mathcal{A}\left(\frac{\hat{x}}{\|\hat{x}\|_\infty}\right)^{m-1}_{i_0} \\
&= \sum_{i_2,i_3,\dots,i_m=1}^{n} a_{i_0 i_2 \cdots i_m} \frac{\hat{x}_{i_2}\cdots\hat{x}_{i_m}}{\|\hat{x}\|_\infty^{m-1}} \\
&= a_{i_0 i_0 \cdots i_0} \frac{\hat{x}_{i_0}^{m-1}}{\|\hat{x}\|_\infty^{m-1}} + \sum_{(i_2,\dots,i_m)\neq(i_0,\dots,i_0)} a_{i_0 i_2 \cdots i_m} \frac{\hat{x}_{i_2}\cdots\hat{x}_{i_m}}{\|\hat{x}\|_\infty^{m-1}} \\
&= a_{i_0 i_0 \cdots i_0} + \sum_{(i_2,\dots,i_m)\neq(i_0,\dots,i_0)} a_{i_0 i_2 \cdots i_m} \frac{\hat{x}_{i_2}\cdots\hat{x}_{i_m}}{\|\hat{x}\|_\infty^{m-1}} \\
&\geq a_{i_0 i_0 \cdots i_0} - \sum_{(i_2,\dots,i_m)\neq(i_0,\dots,i_0)} |a_{i_0 i_2 \cdots i_m}| \\
&> 0 \\
&\geq \frac{b_{i_0}}{\|\hat{x}\|_\infty^{m-1}},
\end{aligned}
$$

where the first inequality holds by the triangle inequality and $\hat{x}_{i_0}^{m-1} = \|\hat{x}\|_\infty^{m-1}$, the second inequality holds by Definition 2.4. This contradicts the assumption that $\hat{x}$ solves $\mathcal{A}x^{m-1} = b$. So, $\mathcal{A}x^{m-1} = b$ has no nonnegative solution.

Since m is odd, it follows that $x_i^{m-1} = |x_i|^{m-1}$ for all $i \in [n]$. Thus, the second result of this theorem can be proved in a similar way as the proof above. $\square$

Similar to Theorem 3.6, the following results can be obtained easily.

Theorem 3.9. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{T}_{m,n}$ be a diagonally dominant tensor with $a_{ii\cdots i} \geq 0$ for all $i \in [n]$ and $b \in \mathbb{R}^n$ satisfies $b < 0$. Then, $\mathcal{A}x^{m-1} = b$ has no nonnegative solution. In addition, if m is odd, then $\mathcal{A}x^{m-1} = b$ has no real solution.*

4 Boundedness of the Solution Set of Multi-linear Equations

In this section, we discuss the boundedness of the nonnegative solution set of $\mathcal{A}x^{m-1} = b$ where $\mathcal{A}$ is nonnegative tensor or B tensor. We also show that an arbitrary real solution of $\mathcal{A}x^{m-1} = b$ with $\mathcal{A}$ being a strictly diagonally tensor can be bounded. All bounds we obtained in this section only depend on entries of $\mathcal{A}$ and components of b.

The class of nonnegative tensors is an important class of tensors. If a nonnegative tensor has all positive diagonal entries, then it is possible that the nonnegative solution set of the corresponding $\mathcal{A}x^{m-1} = b$ is nonempty. For example, suppose that $A = (a_{i_1i_2i_3i_4}) \in \mathbb{T}_{4,3}$ with $a_{1111} = a_{1112} = a_{2222} = a_{3333} = 1$ and others being zeros, and $b = (2, 1, 1)^\top$, then it is easy to show that $\bar{x} = (1, 1, 1)^\top$ is a solution of $\mathcal{A}x^{m-1} = b$. The following theorem demonstrates that the nonnegative solution set of $\mathcal{A}x^{m-1} = b$ is bounded if $\mathcal{A}$ lies in this class of tensors.

Theorem 4.1. *Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{T}_{m,n}$ be a nonnegative tensor with positive diagonal entries and $b \in \mathbb{R}^n_+$. If $\bar{x}$ is a nonnegative solution of $\mathcal{A}x^{m-1} = b$, then*

$$\|\bar{x}\|_\infty \le \left(\frac{\|b\|_\infty}{\min\{a_{ii\cdots i} : i \in [n]\}}\right)^{\frac{1}{m-1}}.$$

Proof. Clearly, $\bar{x} = 0$ is a unique nonnegative solution of $\mathcal{A}x^{m-1} = b$ if and only if $b = 0$, since $\mathcal{A}$ is a nonnegative tensor and $a_{ii\cdots i} > 0$ for all $i \in [n]$. This implies the result holds if $b = 0$. While $b \neq 0$, there must be $\bar{x} \neq 0$. Denote $\bar{x}_{i_0} := \|\bar{x}\|_\infty$, then $\bar{x}_{i_0} > 0$. Thus,

$$\begin{aligned}
\frac{\|b\|_\infty}{\|\bar{x}\|_\infty^{m-1}} &\ge \frac{b_{i_0}}{\|\bar{x}\|_\infty^{m-1}} = \frac{(\mathcal{A}\bar{x}^{m-1})_{i_0}}{\|\bar{x}\|_\infty^{m-1}} \\
&= \sum_{i_2,\ldots,i_m=1}^{n} a_{i_0i_2\cdots i_m} \frac{\bar{x}_{i_2}\cdots\bar{x}_{i_m}}{\|\bar{x}\|_\infty^{m-1}} \\
&= a_{i_0i_0\cdots i_0} + \sum_{(i_2,i_3,\ldots,i_m)\neq(i_0,i_0,\ldots,i_0)} a_{i_0i_2\cdots i_m} \frac{\bar{x}_{i_2}\bar{x}_{i_3}\cdots\bar{x}_{i_m}}{\|\bar{x}\|_\infty^{m-1}} \\
&\ge a_{i_0i_0\cdots i_0} \\
&> 0.
\end{aligned}$$

So, the desired result holds. □

Remark 4.1. In Theorem 4.1, the condition "positive diagonal entries" cannot be weakened. Recall that a nonnegative tensor $\mathcal{A} \in \mathbb{T}_{m,n}$ is *strictly nonnegative* if $\mathcal{A}x^{m-1} > 0$ for any $x \in \mathbb{R}^n_{++}$ [11]. It is obvious that a nonnegative tensor with positive diagonal entries must be a strictly nonnegative tensor, but the inverse is not true. If the nonnegative tensor $\mathcal{A}$ has at least a diagonal entry being zero, then it is possible that the nonnegative solution set of $\mathcal{A}x^{m-1} = b$ is unbounded even if $\mathcal{A}$ is strictly nonnegative. This can be seen in the following example.

Example 4.2. Let $\mathcal{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{T}_{4,3}$, where $a_{1111} = a_{1222} = a_{2222} = a_{3122} = 1$ and others are zeros, and $b = (1, 1, 0)^\top$.

Obviously,

$$\mathcal{A}x^3 = \begin{pmatrix} x_1^3 + x_2^3 \\ x_2^3 \\ x_1x_2^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = b. \tag{4.1}$$

It is easy to see that the concerned tensor $\mathcal{A}$ is a strictly nonnegative tensor. However, it is also easy to show that $(0,1,k)^\top$ is a solution of (4.1) for any $k \in \mathbb{R}$, which implies that the nonnegative set of this system of multi-linear equations is unbounded.

B tensors have been studied extensively in the literature. If $\mathcal{A}$ is a B tensor, then it is possible that $\mathcal{A}x^{m-1} = b$ has one nonnegative solution. For example, let $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{T}_{4,3}$ with $a_{1111} = 28$, $a_{1333} = 1$, $a_{2222} = 2$, $a_{2333} = -1$, $a_{3333} = 1$, and others being zeros, and $b = (56, -25, 27)^\top$. Then, it is easy to see that $(1,1,3)^\top$ is a solution of $\mathcal{A}x^{m-1} = b$. In the following, we show that the nonnegative solution set of $\mathcal{A}x^{m-1} = b$ is bounded when $\mathcal{A}$ is a B tensor.

Lemma 4.3. *If $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{T}_{m,n}$ is a B tensor, then $\bar{x} = 0$ is the unique nonnegative solution of $\mathcal{A}x^{m-1} = 0$.*

Proof. Obviously, 0 is a solution of $\mathcal{A}x^{m-1} = 0$. We only need to show that $\mathcal{A}x^{m-1} = 0$ has no nonzero nonnegative solution. Suppose that there exists $\bar{x} \in \mathbb{R}^n_+$ with $\bar{x} \neq 0$ such that $\mathcal{A}\bar{x}^{m-1} = 0$, then $\bar{x}_{i_0} := \|\bar{x}\|_\infty > 0$. Thus, we have

$$\begin{aligned}
\frac{(\mathcal{A}\bar{x}^{m-1})_{i_0}}{\bar{x}_{i_0}^{m-1}} &= \mathcal{A}\left(\frac{\bar{x}}{\|\bar{x}\|_\infty}\right)^{m-1}_{i_0} = \sum_{i_2,\ldots,i_m=1}^{n} a_{i_0 i_2 \cdots i_m} \frac{\bar{x}_{i_2} \cdots \bar{x}_{i_m}}{\|\bar{x}\|_\infty^{m-1}} \\
&= \sum_{a_{i_0 i_2 \cdots i_m} \geq 0} a_{i_0 i_2 \cdots i_m} \frac{\bar{x}_{i_2} \cdots \bar{x}_{i_m}}{\|\bar{x}\|_\infty^{m-1}} + \sum_{a_{i_0 i_2 \cdots i_m} < 0} a_{i_0 i_2 \cdots i_m} \frac{\bar{x}_{i_2} \cdots \bar{x}_{i_m}}{\|\bar{x}\|_\infty^{m-1}} \\
&\geq a_{i_0 i_0 \cdots i_0} + \sum_{a_{i_0 i_2 \cdots i_m} < 0} a_{i_0 i_2 \cdots i_m} \\
&> 0,
\end{aligned}$$

where the last inequality holds since $\mathcal{A}$ is a B tensor. This contracts that $\bar{x}$ solves $\mathcal{A}x^{m-1} = 0$. So, the desired result holds. □

Theorem 4.4. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{T}_{m,n}$ be a B tensor and $b \in \mathbb{R}^n$. If $\bar{x}$ is a nonnegative solution of $\mathcal{A}x^{m-1} = b$, then*

$$\|\bar{x}\|_\infty \leq \left(\frac{\|b\|_\infty}{\min\{a_{ii,\ldots,i} + \sum\limits_{a_{ii_2 \cdots i_m} < 0} a_{ii_2 \cdots i_m} : i \in [n]\}} \right)^{\frac{1}{m-1}}.$$

Proof. If $b = 0$, it follows from Lemma 4.3 that $\bar{x} = 0$ is the unique nonnegative solution of (1.1), which implies that the desired result holds trivially. In the following, we assume that $b \neq 0$. Then, $\bar{x} \neq 0$, and hence, $\bar{x}_{i_0} := \|\bar{x}\|_\infty > 0$. So, we have

$$\begin{aligned}
\frac{\|b\|_\infty}{\|\bar{x}\|_\infty^{m-1}} &\geq \frac{|b_{i_0}|}{\|\bar{x}\|_\infty^{m-1}} = \frac{|(\mathcal{A}\bar{x}^{m-1})_{i_0}|}{\|\bar{x}\|_\infty^{m-1}} \\
&= \left| \sum_{i_2,\ldots,i_m=1}^{n} a_{i_0 i_2 \ldots i_m} \frac{\bar{x}_{i_2} \cdots \bar{x}_{i_m}}{\|\bar{x}\|_\infty^{m-1}} \right| \\
&\geq \sum_{a_{i_0 i_2 \cdots i_m} \geq 0} a_{i_0 i_2 \cdots i_m} \frac{\bar{x}_{i_2} \cdots \bar{x}_{i_m}}{\|\bar{x}\|_\infty^{m-1}} + \sum_{a_{i_0 i_2 \cdots i_m} < 0} a_{i_0 i_2 \cdots i_m} \frac{\bar{x}_{i_2} \cdots \bar{x}_{i_m}}{\|\bar{x}\|_\infty^{m-1}} \\
&\geq a_{i_0 i_0 \cdots i_0} + \sum_{a_{i_0 i_2 \cdots i_m} < 0} a_{i_0 i_2 \cdots i_m}
\end{aligned}$$

$$> \quad 0.$$

The desired result holds. □

Remark 4.2. In Theorem 4.4, if the condition "B tensor" is replaced by "B_0 tensor", then it is possible that the nonnegative solution set of $\mathcal{A}x^{m-1} = b$ is unbounded. This can be seen in the following example.

Example 4.5. Let $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{T}_{4,4}$, where $a_{1111} = a_{2222} = a_{3333} = 1$, $a_{1222} = a_{2333} = a_{3444} = -1$, and others are zeros, and $b = (1, 1, 1, 0)^\top$.

It is easy to show that $\mathcal{A}$ is a B_0 tensor; and $(\sqrt[3]{k^3+3}, \sqrt[3]{k^3+2}, \sqrt[3]{k^3+1}, k)^\top$ is a solution of the system of equations

$$\mathcal{A}x^{m-1} = \begin{pmatrix} x_1^3 - x_2^3 \\ x_2^3 - x_3^3 \\ x_3^3 - x_4^3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = b$$

for any $k \in \mathbb{R}$.

Remark 4.3. In [22], the authors discussed the boundedness of the solution set of the TCP with a B tensor in terms of two operator norms. Generally, it is difficult to compute these two norms. In Theorem 4.4, we obtained a bound of the nonnegative solution set of $\mathcal{A}x^{m-1} = b$ with a B tensor, which is also a bound of a subset set of the solution set of the corresponding TCP. However, the bound we obtained in Theorem 4.4 only depends on entries of $\mathcal{A}$ and right-hand-side vector b, which is easy to be computed.

In the following, we discuss the boundedness of the solution set of multi-linear equations (1.1).

Theorem 4.6. *Given nonzero tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{T}_{m,n}$ and $b \in \mathbb{R}^n$. If $\bar{x}$ is an arbitrarily real solution of $\mathcal{A}x^{m-1} = b$, then*

$$\|\bar{x}\|_\infty \geq \left(\frac{\|b\|_\infty}{\sum_{i_2,\ldots,i_m=1}^n |a_{i_0 i_2 \cdots i_m}|} \right)^{\frac{1}{m-1}},$$

where $i_0 := \arg\max\limits_{i \in [n]} |b_i|$.

Proof. If $b = 0$, then it is clear that the desired result holds. In the following, we assume that $b \neq 0$. Thus, it follows that $\bar{x} \neq 0$, which implies that $\|\bar{x}\|_\infty > 0$. Since $|b_{i_0}| = \|b\|_\infty$ and $b \neq 0$, it follows that $|b_{i_0}| > 0$. Thus,

$$\begin{aligned} 0 \quad &< \quad \frac{\|b\|_\infty}{\|\bar{x}\|_\infty^{m-1}} = \frac{|b_{i_0}|}{\|\bar{x}\|_\infty^{m-1}} = \frac{|(\mathcal{A}\bar{x}^{m-1})_{i_0}|}{\|\bar{x}\|_\infty^{m-1}} = \left| \mathcal{A} \left(\frac{\bar{x}}{\|\bar{x}\|_\infty} \right)^{m-1}_{i_0} \right| \\ &= \quad \left| \sum_{i_2,\ldots,i_m=1}^n a_{i_0 i_2 \cdots i_m} \frac{\bar{x}_{i_2} \cdots \bar{x}_{i_m}}{\|\bar{x}\|_\infty^{m-1}} \right| \\ &\leq \quad \sum_{i_2,\ldots,i_m=1}^n \left| a_{i_0 i_2 \cdots i_m} \frac{\bar{x}_{i_2} \cdots \bar{x}_{i_m}}{\|\bar{x}\|_\infty^{m-1}} \right| \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i_2,\ldots,i_m=1}^{n} |a_{i_0 i_2\cdots i_m}| \left| \frac{\bar{x}_{i_2}\cdots\bar{x}_{i_m}}{\|\bar{x}\|_\infty^{m-1}} \right| \\
&\leq \sum_{i_2,\ldots,i_m=1}^{n} |a_{i_0 i_2\cdots i_m}|,
\end{aligned}
$$

which implies that the desired result holds. ☐

Now, we discuss the boundedness of the solution set of multi-linear equations where the associated tensor is a strictly diagonally dominant tensor.

Lemma 4.7. *If $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{T}_{m,n}$ is a strictly diagonally dominant tensor, then $\bar{x} = 0$ is a unique real solution of $\mathcal{A}x^{m-1} = 0$.*

Proof. Obviously, 0 is a solution of $\mathcal{A}x^{m-1} = 0$. In the following, we show that $\mathcal{A}x^{m-1} = 0$ has no nonzero real solution. Suppose that $\hat{x} \neq 0$ is a real solution of $\mathcal{A}x^{m-1} = 0$. Then, $|\hat{x}_{i_0}| := \|\hat{x}\|_\infty > 0$. Thus, we have

$$
\begin{aligned}
\left| \frac{(\mathcal{A}\hat{x}^{m-1})_{i_0}}{\|\hat{x}\|_\infty^{m-1}} \right| &= \left| \sum_{i_2,i_3,\ldots,i_m=1}^{n} a_{i_0 i_2\cdots i_m} \frac{\hat{x}_{i_2}\cdots\hat{x}_{i_m}}{\|\hat{x}\|_\infty^{m-1}} \right| \\
&= \left| a_{i_0 i_0\cdots i_0} \frac{\hat{x}_{i_0}^{m-1}}{\|\hat{x}\|_\infty^{m-1}} + \sum_{(i_2,i_3,\ldots,i_m)\neq(i_0,i_0,\ldots,i_0)} a_{i_0 i_2\cdots i_m} \frac{\hat{x}_{i_2}\cdots\hat{x}_{i_m}}{\|\hat{x}\|_\infty^{m-1}} \right| \\
&\geq \left| a_{i_0 i_0\cdots i_0} \frac{\hat{x}_{i_0}^{m-1}}{\|\hat{x}\|_\infty^{m-1}} \right| - \left| \sum_{(i_2,i_3,\ldots,i_m)\neq(i_0,i_0,\ldots,i_0)} a_{i_0 i_2\cdots i_m} \frac{\hat{x}_{i_2}\cdots\hat{x}_{i_m}}{\|\hat{x}\|_\infty^{m-1}} \right| \\
&\geq |a_{i_0 i_0\cdots i_0}| - \sum_{(i_2,i_3,\ldots,i_m)\neq(i_0,i_0,\ldots,i_0)} |a_{i_0 i_2\cdots i_m}| \\
&> 0,
\end{aligned}
$$

which contradicts that $\hat{x}$ solves $\mathcal{A}x^{m-1} = 0$. So, the desired result holds. ☐

Theorem 4.8. *Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{T}_{m,n}$ be a strictly diagonally dominant tensor and $b \in \mathbb{R}^n$. If $\bar{x}$ is a real solution of $\mathcal{A}x^{m-1} = b$, then*

$$
\|\bar{x}\|_\infty \leq \left(\frac{\|b\|_\infty}{\min\{|a_{ii\cdots i}| - \sum_{(i_2,i_3,\ldots,i_m)\neq(i,i,\ldots,i)} |a_{i i_2\cdots i_m}| : i \in [n]\}} \right)^{\frac{1}{m-1}}.
$$

Proof. When $b = 0$, the desired result holds by Lemma 4.7. In the following, we assume that $b \neq 0$. Then, $\bar{x} \neq 0$, which implies that $|\bar{x}_{i_0}| := \|\bar{x}\|_\infty > 0$. So, we have

$$
\begin{aligned}
\frac{\|b\|_\infty}{\|\bar{x}\|_\infty^{m-1}} &\geq \frac{|b_{i_0}|}{\|\bar{x}\|_\infty^{m-1}} = \frac{|(\mathcal{A}\bar{x}^{m-1})_{i_0}|}{\|\bar{x}\|_\infty^{m-1}} = \left| \mathcal{A}\left(\frac{\bar{x}}{\|\bar{x}\|_\infty^{m-1}} \right)_{i_0} \right| \\
&= \left| \sum_{i_2,i_3,\ldots,i_{m=1}}^{n} a_{i_0 i_2\cdots i_m} \frac{\bar{x}_{i_2}\cdots\bar{x}_{i_m}}{\|\bar{x}\|_\infty^{m-1}} \right|
\end{aligned}
$$

$$
\begin{aligned}
&\geq \left| a_{i_0 i_0 \cdots i_0} \frac{\bar{x}_{i_0}^{m-1}}{\|\bar{x}\|_\infty^{m-1}} \right| - \left| \sum_{(i_2,i_3,\ldots,i_m)\neq(i_0,i_0,\ldots,i_0)} a_{i_0 i_2 \cdots i_m} \frac{\bar{x}_{i_2} \cdots \bar{x}_{i_m}}{\|\bar{x}\|_\infty^{m-1}} \right| \\
&= |a_{i_0 i_0 \cdots i_0}| - \left| \sum_{(i_2,i_3,\ldots,i_m)\neq(i_0,i_0,\ldots,i_0)} a_{i_0 i_2 \cdots i_m} \frac{\bar{x}_{i_2} \cdots \bar{x}_{i_m}}{\|\bar{x}\|_\infty^{m-1}} \right| \\
&\geq |a_{i_0 i_0 \cdots i_0}| - \sum_{(i_2,i_3,\ldots,i_m)\neq(i_0,i_0,\ldots,i_0)} |a_{i_0 i_2 \cdots i_m}| \left| \frac{\bar{x}_{i_2} \cdots \bar{x}_{i_m}}{\|\bar{x}\|_\infty^{m-1}} \right| \\
&\geq |a_{i_0 i_0 \cdots i_0}| - \sum_{(i_2,i_3,\ldots,i_m)\neq(i_0,i_0,\ldots,i_0)} |a_{i_0 i_2 \cdots i_m}|.
\end{aligned}
$$

Thus, the desired result holds. □

Remark 4.4. In Theorem 4.8, if "strictly diagonally dominant tensor" is replaced by "diagonally dominant tensor", then it is possible that the result of Theorem 4.8 does not hold. This can be seen in the following example.

Example 4.9. Let $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{T}_{4,4}$, where $a_{1111} = 2$, $a_{1333} = -1$, $a_{2222} = 3$, $a_{2333} = -1$, $a_{3333} = a_{4444} = 1$, $a_{3444} = a_{4333} = -1$, and others are zeros. Let $b = (1, 3, 0, 0)^\top$.

Obviously, the concerned tensor is a diagonally dominant tensor, but not a strictly diagonally dominant tensor. It is easy to show that for any $k \in \mathbb{R}$, $(\sqrt[3]{\frac{1+k^3}{2}}, \sqrt[3]{\frac{3+k^3}{3}}, k, k)^\top$ is a solution of the system of equations

$$
\mathcal{A}x^{m-1} = \begin{pmatrix} 2x_1^3 - x_3^3 \\ 3x_2^3 - x_3^3 \\ x_3^3 - x_4^3 \\ x_4^3 - x_3^3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix} = b, \tag{4.2}
$$

which implies that the set of real solutions to (4.2) is unbounded.

5 Conclusions

In this paper, we proved that the nonnegative (positive) solution set of multi-linear equations (1.1) which induced by a lower (an upper) triangular tensor with positive diagonal entries and non-positive strictly lower(upper) part is nonempty for any nonnegative (positive) right-hand-side vector. We also investigated the uniqueness of nonnegative (positive) solution to these multi-linear equations if the associated tensors satisfy some additional assumptions. Meanwhile, we investigated the non-existence of nonnegative solution to multi-linear equations induced by B (B_0) tensors or strictly diagonally dominant tensors. In addition, we also discussed the boundedness of the nonnegative solution set of multi-linear equations with some structured tensors including nonnegative tensors and B tensors.

In recent years, various kinds of structured tensors have been studied well. In this paper, we just discussed the existence and uniqueness of nonnegative (positive) solution to multi-linear equations with associated tensors being some triangular tensors. It is worthy of investigating the existence and uniqueness of solution to multi-linear equations with associated tensors being other structured tensors.

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Manuscript received 29 January 2018
revised 28 July 2018
accepted for publication 8 August 2018

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