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CONSTRAINT QUALIFICATIONS AND OPTIMALITY CONDITIONS IN NONSMOOTH LOCALLY STAR-SHAPED OPTIMIZATION USING CONVEXIFICATORS*

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Abstract: In this paper, KKT optimality conditions, for a nonsmooth optimization problem with a locally star-shaped feasible set and a semilocally convex objective function, are investigated. The appearing functions are not necessarily convex/locally Lipschitz/continuous. The optimality conditions are derived in terms of convexificators, after studying nonsmooth constraint qualification conditions. The results of the present paper extend some corresponding ones existing in the literature.

Key words: *nonsmooth optimization, generalized convexity, constraint qualification, convexificator, KKT conditions*

Mathematics Subject Classification: *90C26, 90C30, 90C46*

1 Introduction

In [16], Lasserre derived optimality conditions for an optimization problem with a convex differentiable objective function, differentiable constraint functions, and a convex feasible set. He assumed the feasible set to be convex, without imposing convexity on constraint functions. Lasserre proved the necessity and sufficiency of KKT conditions for optimality under Slater constraint qualification and a nondegeneracy assumption. Dutta and Lalitha [9] obtained corresponding results for nonsmooth problems with regular locally Lipschitz constraint functions in terms of Clarke generalized gradient [5]. Martínez-Legaz [21] extended two aforementioned works by studying a nonsmooth problem with a pseudoconvex objective function. He applied tangential subdifferential to derive KKT conditions.

In this study, we deal with a nonsmooth optimization problem whose objective function is semilocally convex and its feasible set is locally star-shaped (not necessarily convex). The constraint functions are not necessarily convex or locally Lipschitz or even continuous. The problem in question is nonsmooth and our main tool in getting KKT conditions is convexificator notion introduced by Jeyakumar and Luc [13]. Optimality conditions using convexificators have been studied by various scholars in recent years; see e.g. [6–8, 17–19, 25] and the references therein. For a locally Lipschitz function, most known subdifferentials, including Clarke generalized gradient, Michel-Penot, Ioffe, Mordukhovich, and Treiman subdifferential are convexificator [13]. Furthermore, tangential subdifferential (of a tangentially convex function) is a convexificator (see [21, Definition 5]).

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In the current work, after providing some preliminaries, a characterization of the normal cone of a locally star-shaped set is addressed which leads to a necessary and sufficient optimality condition. We show that the Slater constraint qualification and nondegeneracy condition considered in [9, 16, 21] imply Cottle constraint qualification. A new constraint qualification (CQ) is introduced and important connections between various CQs are proved. Necessary and sufficient KKT optimality conditions, in terms of convexificators, are discussed in the last part of the paper. The obtained outcomes generalize various results given in [9, 16, 21].

The rest of the paper is organized as follows. Section 2 contains preliminaries. Characterization of the normal cone of locally star-shaped sets and minimizers of semilocally convex functions are addressed in Section 3. CQs are discussed in Section 4; and KKT optimality conditions are investigated in Section 5. Section 6 concludes the paper.

2 Preliminaries

In this paper, we consider the following problem:

$$\min f(x) \quad s.t. \ \ g_i(x) \leq 0, \quad i = 1, \ldots, m, \tag{2.1}$$

where $f, g_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \ldots, m)$ are not necessarily locally Lipschitz or continuous or convex. Set

$$K := \{x : g_i(x) \leq 0, \quad i = 1, \ldots, m\}. \tag{2.2}$$

Assume $K \neq \emptyset$ and for $\bar{x} \in K$ set $I(\bar{x}) := \{i \in \{1, 2, \ldots, m\} : g_i(\bar{x}) = 0\}$.

For a set $S \subseteq \mathbb{R}^n$, we use the notations $conv\, S$, $int\, S$ and $cl\, S$ to denote the convex hull, the interior and the closure of S, respectively. Throughout the paper, the considered norm $\|\cdot\|$ is the Euclidean norm, i.e., $\|\cdot\| = \|\cdot\|_2$. The notation $\langle \cdot, \cdot \rangle$ is utilized to denote the inner product.

A nonempty set $C \subseteq \mathbb{R}^n$ is called a cone, if for any $x \in C$ and any scalar $\lambda \geq 0$, $\lambda x \in C$. For a nonempty set $S \subseteq \mathbb{R}^n$, the cone of feasible directions of, the tangent cone to, the adjacent cone to, and the normal cone to S at $\bar{x} \in cl\, S$, denoted by $D_S(\bar{x})$, $T_S(\bar{x})$, $A_S(\bar{x})$, and $N_S(\bar{x})$, respectively, are defined as

$$D_S(\bar{x}) := \{d \in \mathbb{R}^n : \exists \delta > 0 \ \ s.t. \ \ \forall \lambda \in (0, \delta), \quad \bar{x} + \lambda d \in S\},$$

$$T_S(\bar{x}) := \{d \in \mathbb{R}^n : \exists \{x_n\} \subseteq S, \ \exists \{\lambda_n\} \subseteq (0, +\infty), \ x_n \to \bar{x}, \ \lambda_n(x_n - \bar{x}) \to d\},$$

$$A_S(\bar{x}) := \{d \in \mathbb{R}^n : \forall t_n \downarrow 0, \ \exists \{d_n\} \subseteq \mathbb{R}^n, \ d_n \to d, \ \bar{x} + t_n d_n \in S\},$$

$$N_S(\bar{x}) := \{\zeta \in \mathbb{R}^n : \langle \zeta, d \rangle \leq 0, \forall d \in T_S(\bar{x})\}.$$

The polar cone and the strict polar cone of $S \subseteq \mathbb{R}^n$ are respectively defined by

$$S^\circ := \{d \in \mathbb{R}^n : \langle d, x \rangle \leq 0, \quad \forall x \in S\},$$

$$S^s := \{d \in \mathbb{R}^n : \langle d, x \rangle < 0, \quad \forall x \in S\}.$$

If $S^s \neq \emptyset$, then $cl\, S^s = S^\circ$. It is seen that $N_S(\bar{x}) = T_S^\circ(\bar{x})$.

The cone $D_S(\bar{x})$ is neither closed nor convex necessarily, while $A_S(\bar{x})$ and $T_S(\bar{x})$ are closed but not necessarily convex. In general, $D_S(\bar{x}) \subseteq A_S(\bar{x}) \subseteq T_S(\bar{x})$. The pseudotangent cone of $S \subseteq \mathbb{R}^n$ at $\bar{x}$, denoted by $PT_S(\bar{x})$, is defined as $PT_S(\bar{x}) := cl(conv(T_S(\bar{x})))$.

The cone and the convex cone generated by S, respectively, are defined as

$$cone(S) := \{\lambda y : \lambda \geq 0, \quad y \in S\},$$

$$pos(S) := \left\{ y \in \mathbb{R}^n \ : \exists \nu \in \mathbb{N}; \ \ y = \sum_{i=1}^{\nu} \lambda_i y_i, \ \ \lambda_i \geq 0, \ \ y_i \in S, \ i = 1, 2, \ldots, \nu \right\}.$$

Remark 2.1. If $S_1, \ldots, S_l \subseteq \mathbb{R}^n$ are convex sets, then

$$pos\left(\bigcup_{i=1}^{l} S_i \right) = \left\{ \sum_{i=1}^{l} \lambda_i d_i \ : \ d_i \in S_i, \ \ \lambda_i \geq 0, \ i = 1, 2, \ldots, l \right\}.$$

The lower and upper Dini directional derivatives play a vital role in defining convexificators. The effective domain of h is denoted by $dom\, h$.

Definition 2.1. The lower and upper Dini directional derivatives of $h : \mathbb{R}^n \to \mathbb{R}$ at $x \in dom\, h$ in direction $d \in \mathbb{R}^n$ are respectively defined by

$$h^-(x; d) := \liminf_{t \downarrow 0} \frac{h(x + td) - h(x)}{t}, \qquad h^+(x; d) := \limsup_{t \downarrow 0} \frac{h(x + td) - h(x)}{t}.$$

The directional derivative of h at $x \in dom\, h$ in direction $d \in \mathbb{R}^n$, denoted by $h'(x; d)$, is defined as

$$h'(x; d) := \lim_{t \downarrow 0} \frac{h(x + td) - h(x)}{t}.$$

Definition 2.2. [7,13] Let $h : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in dom\, h$ be given. The closed set $\partial^* h(\bar{x}) \subseteq \mathbb{R}^n$ is called

i) an upper convexificator (UC) of h at $\bar{x}$ if for each $d \in \mathbb{R}^n$,

$$h^-(\bar{x}; d) \leq \sup_{\zeta \in \partial^* h(\bar{x})} \langle \zeta, d \rangle.$$

ii) a convexificator of h at $\bar{x}$ if it is a UC of h at $\bar{x}$ and furthermore

$$h^+(\bar{x}; d) \geq \inf_{\zeta \in \partial_* h(\bar{x})} \langle \zeta, d \rangle.$$

iii) an upper semiregular convexificator (USRC) of h at $\bar{x}$ if for each $d \in \mathbb{R}^n$,

$$h^+(\bar{x}; d) \leq \sup_{\zeta \in \partial^* h(\bar{x})} \langle \zeta, d \rangle. \tag{2.3}$$

iv) an upper regular convexificator (URC) of h at $\bar{x}$ if (2.3) holds as equality for each $d \in \mathbb{R}^n$.

Gâteaux differentiable functions, regular functions in the sense of Clarke [5], and tangentially convex functions [21] are important functions which admit URC. If h is locally Lipschitz, then Clarke subdifferential [5] and Michel-Penot subdifferential [22] are USRCs [7].

Now we recall some constraint qualification conditions from the literature. Let K be as represented in (2.2). Hereafter assume that g_i functions, defining K, are not necessarily convex or differentiable.

Definition 2.3. We say that the Slater Constraint Qualification (SCQ) holds for Problem (2.1) if $int\, K \neq \emptyset$.

For a given $\bar{x} \in K$, define

$$\Gamma(\bar{x}) := \bigcup_{i \in I(\bar{x})} \partial^* g_i(\bar{x}).$$

Definition 2.4. Let $\bar{x} \in K$. Assume that for each $i \in I(\bar{x})$, g_i has a URC $\partial^* g_i(\bar{x})$ at $\bar{x}$. We say that

- Generalized Lasserre Constraint Qualification (GLCQ) holds at $\bar{x}$ if $0 \notin \partial^* g_i(\bar{x})$ for each $i \in I(\bar{x})$.
- Cottle Constraint Qualification (CCQ) holds at $\bar{x}$ if $\Gamma^s(\bar{x}) \neq \emptyset$.
- Guignard Constraint Qualification (GCQ) holds at $\bar{x}$ if $\Gamma^\circ(\bar{x}) \subseteq PT_K(\bar{x})$.
- Abadie Constraint Qualification (ACQ) holds at $\bar{x}$ if $\Gamma^\circ(\bar{x}) \subseteq T_K(\bar{x})$.

  Generalized Abadie Constraint Qualification (GACQ) holds at $\bar{x}$ if $\Gamma^\circ(\bar{x}) \subseteq A_K(\bar{x})$.
- The Zangwill Constraint Qualification (ZCQ) holds at $\bar{x}$ if $\Gamma^\circ(\bar{x}) \subseteq cl\, D_K(\bar{x})$.

It is clear that, ZCQ $\Rightarrow$ GACQ $\Rightarrow$ ACQ $\Rightarrow$ GCQ. See [2–4, 20, 24] for more details about CQs and their important role in optimization.

Definition 2.5. Let K be as represented in (2.2), and $\bar{x} \in K$. Assume that f and $g_i,\ i \in I(\bar{x})$ admit URCs $\partial^* f(\bar{x})$ and $\partial^* g_i(\bar{x})$ at $\bar{x}$, respectively. The vector $\bar{x}$ is called
(i) a Fritz John (FJ) point of (2.1) if

$$0 \in \lambda_0 conv(\partial^* f(\bar{x})) + \sum_{i \in I(\bar{x})} \lambda_i conv(\partial^* g_i(\bar{x}))$$

for some nonzero vector $(\lambda_0, \lambda_i,\ i \in I(\bar{x})) \geq 0$.
(ii) a KKT point of (2.1), if it is a FJ point with $\lambda_0 > 0$.

3 Locally Star-shaped Sets and Semilocally Convex Functions

It is known from the literature that KKT conditions are sufficient for optimality in Problem (2.1) if f, g_i functions are convex and differentiable. In the two last decades, several generalizations of convexity and differentiation have been introduced to extend the above result for wider classes of problems; see e.g. [1, 4, 5, 23] for some generalizations of convexity/differentiation and related discussions.

Since in our work the behavior of the objective function and the feasible set is important around the point under consideration in contrast to other areas, we focus on locally star-shaped sets and semilocally convex functions, defined and investigated by Ewing [10].

Definition 3.1. [10] A nonempty set $S \subseteq \mathbb{R}^n$ is said to be locally star-shaped at $\bar{x} \in S$, if corresponding to $\bar{x}$ and each $x \in S$, there exists $a(\bar{x}, x) \in (0, 1]$ such that

$$\bar{x} + \lambda(x - \bar{x}) \in S, \quad \forall \lambda \in (0, a(\bar{x}, x)). \tag{3.1}$$

If $a(\bar{x}, x) = 1$ for each $x \in S$, then S is said to be star-shaped at $\bar{x}$.

In the whole paper, $S \in \mathcal{L}(\bar{x})$ means S is locally star-shaped at $\bar{x}$. Open sets and convex sets are locally star-shaped at each of their elements, and cones are locally star-shaped at the origin. If S is closed and it is locally star-shaped at each $x \in S$, then S is convex [15].

There exist locally star-shaped sets (at some x) which are neither star-shaped nor locally convex (at x). For example, consider $S = \mathbb{R}^2 \backslash \{(x_1, x_2) \in \mathbb{R}^2 : \; x_2 = x_1^2, \; x_1 \neq 0\}$ at $\bar{x} = (0, 0)$.

Definition 3.2. [10] Let $S \in \mathcal{L}(\bar{x})$ and $h : S \to \mathbb{R}$ be a real-valued function. h is said to be semilocally convex (SLC) at $\bar{x}$ (denoted $h \in \mathcal{F}_S(\bar{x})$) if corresponding to $\bar{x}$ and each $x \in S$ there exists a positive number $d(\bar{x}, x) \leq a(\bar{x}, x)$ such that

$$h(\bar{x} + \lambda(x - \bar{x})) \leq h(\bar{x}) + \lambda(h(x) - h(\bar{x})), \quad \forall \lambda \in (0, d(\bar{x}, x)). \tag{3.2}$$

Consider $h(x) = [x]$, where $[\cdot]$ stands for the floor function. This function is not convex, while it is SLC at each point of $\mathbb{R}$. This example also shows that semilocally convexity does not imply continuity necessarily. If S is closed and convex and h is continuous and SLC, then by [14, Theorem 1.2] and [3, Theorem 3.2.2], h is convex.

If g_i functions defining K in (2.2) are SLC at $\bar{x} \in K$, then $K \in \mathcal{L}(\bar{x})$.

The SLC functions enjoy nice properties. If $S \in \mathcal{L}(\bar{x})$, $h \in \mathcal{F}_S(\bar{x})$, and $\bar{x}$ is a local minimizer of h over S, then $\bar{x}$ is a global minimizer of h over S. If, in addition, $\bar{x}$ is a strict local minimizer, then $\bar{x}$ is a strict global minimizer; see [10].

The following results address some important properties of locally star-shaped sets and SLC functions. These theorems generalize some popular results in classic convex analysis.

Theorem 3.1. *Let $S \in \mathcal{L}(\bar{x})$. Then*

(i) $T_S(\bar{x}) = cl(cone(S - \bar{x})) = cl(D_S(\bar{x}))$.

(ii) $N_S(\bar{x}) = (pos(S - \bar{x}))^\circ = \{d : \langle d, x - \bar{x}\rangle \leq 0, \; \forall x \in S\}$.

Proof. The inclusion $cl(cone(S - \bar{x})) \subseteq T_S(\bar{x})$ can be proved similar to [12, Theorem 4.8]. The proofs of other relations are not difficult and are hence omitted. □

Note that, unlike the convex sets, for a locally star-shaped set S, the normal cone $N_S^\circ(\bar{x})$ may not coincide with $T_S(\bar{x})$. For example, consider

$$S = \mathbb{R}^2 \setminus \Big\{(x_1, x_2) \in \mathbb{R}^2 \backslash \{(0, 0)\} : x_1, x_2 \leq 0\Big\}, \quad \bar{x} = (0, 0).$$

Theorem 3.2. *Let $S \in \mathcal{L}(\bar{x})$ and $h \in \mathcal{F}_S(\bar{x})$.*

i) *$\bar{x}$ is a minimizer of h on S if and only if $h^+(\bar{x}; x - \bar{x}) \geq 0$ for any $x \in S$.*

ii) *If h admits a URC $\partial^* h(\bar{x})$ at $\bar{x}$ and $0 \in conv\, \partial^* h(\bar{x}) + N_S(\bar{x})$, then $\bar{x}$ is a minimizer of h on S.*

Proof. (i) $h \in \mathcal{F}_S(\bar{x})$ implies the existence of $h'(\bar{x}; x - \bar{x})$ with $h'(\bar{x}; x - \bar{x}) = h^+(\bar{x}; x - \bar{x})$ and $h'(\bar{x}; x - \bar{x}) \leq h(x) - h(\bar{x})$ for any $x \in S$ (see [10]). Therefore, $h^+(\bar{x}; x - \bar{x}) \geq 0$ for any $x \in S$ implies $h(x) \geq h(\bar{x})$ for any $x \in S$. Conversely, if $h^+(\bar{x}; x - \bar{x}) = h'(\bar{x}; x - \bar{x}) < 0$ for some $x \in S$, then there exists some $\lambda > 0$ such that $x_\lambda := \bar{x} + \lambda(x - \bar{x}) \in S$ and $h(x_\lambda) < h(\bar{x})$. This contradicts the assumption.
(ii) If $0 \in conv\, \partial^* h(\bar{x}) + N_S(\bar{x})$, then $-\eta \in N_S(\bar{x})$ for some $\eta \in conv\, \partial^* h(\bar{x})$. Hence, $\langle \eta, d\rangle \geq 0$ for all $d \in T_S(\bar{x})$. Therefore, by Theorem 3.1(i), $\langle \eta, x - \bar{x}\rangle \geq 0$ for each $x \in S$. This leads to

$$\begin{aligned} h^+(\bar{x}; x - \bar{x}) &= \sup\{\langle \eta, x - \bar{x}\rangle : \eta \in \partial^* h(\bar{x})\} \\ &= \sup\{\langle \eta, x - \bar{x}\rangle : \eta \in conv\, \partial^* h(\bar{x})\} \geq 0, \end{aligned}$$

for any $x \in S$ and the proof is completed due to part (i). □

The converse of Theorem 3.2(ii) will be investigated in Theorem 5.3.

Martínez-Legaz [21] studied a generalization of convexity as follows.

Definition 3.3. [21] A function $h : \mathbb{R}^n \to \mathbb{R}$ is called tangentially convex at $x \in \mathbb{R}^n$ if for every $d \in \mathbb{R}^n$ the limit $h'(x; d) = \lim_{t \downarrow 0} \frac{h(x+td)-h(x)}{t}$ exists, is finite, and is a convex function of d.

Definition 3.4. [21] A function $h : \mathbb{R}^n \to \mathbb{R}$ which is tangentially convex at x is said to be pseudoconvex at x if $h(y) \geq h(x)$ for every $y \in \mathbb{R}^n$ with $h'(x; y - x) \geq 0$.

The following result shows that Theorem 3.2 is still valid if one replaces locally semiconvexity assumption with pseudoconvexity.

Theorem 3.3. *Let $S \in \mathcal{L}(\bar{x})$. Assume that h is pseudoconvex at $\bar{x}$ (in the sense of Definition 3.4). Then:*

(i) *The tangential subdifferential defined as*

$$\partial_T h(\bar{x}) = \{\eta \in \mathbb{R}^n | h'(\bar{x}; d) \geq \langle \eta, d \rangle, \quad \forall d \in \mathbb{R}^n\}$$

is a URC of h at $\bar{x}$.

(ii) *$\bar{x}$ is a minimizer of h over S if and only if*

$$h'(\bar{x}; x - \bar{x}) = \sup\{\langle \eta, x - \bar{x} \rangle : \eta \in \partial_T h(\bar{x})\} \geq 0, \qquad \forall x \in S.$$

(iii) *If $0 \in \partial_T h(\bar{x}) + N_S(\bar{x})$, then $\bar{x}$ is a minimizer of h over S.*

Proof. For part (i), see [21, p. 3]. The proof of parts (ii) and (iii) is similar to that of Theorem 3.2. □

In the next sections, we study optimality conditions and CQs for Problem (2.1) with a locally star-shaped feasible set K (as represented in (2.2)) and an SLC objective function f.

4 Constraint Qualifications

Constraint Qualifications (CQs) are some conditions which help us to derive optimality in optimization theory. Various CQs have been defined and investigated in the literature, see e.g. [2] and [3, Chapter 5] for some reviews. The main aim of this section is to introduce a new CQ and to establish the relationships between some CQs, including two CQs addressed in [9, 16, 21]. Lemma 4.1 is required in presenting these connections.

Lemma 4.1. *Assume that $K \in \mathcal{L}(\bar{x})$ is as represented in (2.2) and $g_i,\ i \in I(\bar{x})$ admits a URC $\partial^* g_i(\bar{x})$ at $\bar{x}$. Then $cl\, pos\big(\Gamma(\bar{x})\big) \subseteq N_K(\bar{x})$.*

Proof. As $N_K(\bar{x})$ is a closed convex cone, it is sufficient to show that $\partial^* g_i(\bar{x}) \subseteq N_K(\bar{x})$ for each $i \in I(\bar{x})$. Considering $i \in I(\bar{x})$ and $x \in K$ arbitrary, $K \in \mathcal{L}(\bar{x})$ implies $g_i^+(\bar{x}; x - \bar{x}) \leq 0$. Hence,

$$\langle \zeta, x - \bar{x} \rangle \leq 0, \quad \forall \Big(x \in K,\ i \in I(\bar{x}),\ \zeta \in \partial^* g_i(\bar{x})\Big).$$

Now the proof is completed by applying Theorem 3.1. □

Remark 4.1. Let $K \in \mathcal{L}(\bar{x})$ be as represented in (2.2). If $g_i,\ i \in I(\bar{x})$, admits a URC $\partial^* g_i(\bar{x})$ at $\bar{x}$, then, thanks to Theorem 3.1, $cl(D_K(\bar{x})) = A_K(\bar{x}) = T_K(\bar{x})$. Therefore,

$$\text{ZCQ} \Leftrightarrow \text{GACQ} \Leftrightarrow \text{ACQ}.$$

Now, we introduce a new CQ, as a modification of SCQ. Recall that $\Gamma(\bar{x}) = \bigcup_{i\in I(\bar{x})} \partial^* g_i(\bar{x})$.

Definition 4.1. Assume that $K \in \mathcal{L}(\bar{x})$ is as represented in (2.2), and $g_i,\ i \in I(\bar{x})$ admits a URC $\partial^* g_i(\bar{x})$ at $\bar{x}$. We say that directional constrain qualification (DCQ) holds at $\bar{x}$ if there exists $y \in K$ and $\varepsilon > 0$ such that $y + \varepsilon\frac{d}{\|d\|} \in K$, for each $d \in \Gamma(\bar{x})$.

Remark 4.2. Due to Definition 4.1, DCQ is well-defined when $d \neq 0$ for any $d \in \Gamma(\bar{x})$. Hence, in defining DCQ we have assumed GLCQ implicitly. Notice that GLCQ does not imply DCQ (See Example 4.1 in the current section).

SCQ+GLCQ implies DCQ while the converse does not hold (See Example 5.4 at the end of Section 5 of the present paper). If g_i is continuous for each i, then DCQ implies SCQ (see the proof of Theorem 4.4). Theorem 4.2 shows that DCQ implies CCQ.

Theorem 4.2. *Assume that $K \in \mathcal{L}(\bar{x})$ is as represented in (2.2), and $g_i,\ i \in I(\bar{x})$ admits a URC $\partial^* g_i(\bar{x})$ at $\bar{x}$. If DCQ holds at $\bar{x}$, then CCQ holds at $\bar{x}$.*

Proof. By DCQ there exist $y \in K$ and $\varepsilon > 0$ such that $y + \varepsilon\frac{d}{\|d\|} \in K$ for any $d \in \Gamma(\bar{x})$. Considering $d \in \Gamma(\bar{x})$ arbitrary, by Lemma 4.1, $d \in N_K(\bar{x})$, and hence by Theorem 3.1, $\langle d, z - \bar{x}\rangle \leq 0$ for each $z \in K$. Thus, we get

$$\langle \zeta, \varepsilon\frac{\zeta}{\|\zeta\|} + y - \bar{x}\rangle \leq 0, \quad \forall i \in I(\bar{x}),\ \forall \zeta \in \partial^* g_i(\bar{x}),$$

which implies

$$\langle \zeta, y - \bar{x}\rangle \leq -\varepsilon\|\zeta\|, \quad \forall i \in I(\bar{x}), \forall \zeta \in \partial^* g_i(\bar{x}).$$

Hence, taking $0 \notin \partial^* g_i(\bar{x})$ into account (Remark 4.2), we have

$$\langle \zeta, y - \bar{x}\rangle < 0, \quad \forall i \in I(\bar{x}), \forall \zeta \in \partial^* g_i(\bar{x}).$$

Therefore, CCQ holds at $\bar{x}$. □

The converse of the above theorem may not hold, i.e., CCQ does not imply DCQ necessarily.

Remark 4.3. Let $\bar{x}$ be a FJ point of (2.1). If DCQ holds, then according to Theorem 4.2, CCQ is fulfilled, and so

$$0 \notin conv(\cup_{i\in I(\bar{x})} conv(\partial^* g_i(\bar{x}))).$$

Therefore, a FJ point is a KKT point provided that DCQ holds.

In [9,16], the authors consider SCQ and GLCQ to obtain optimality conditions. On the other hand, some optimality conditions have been obtained in the literature under CCQ; see [3,17]. The following corollary, which is a direct consequence of Theorem 4.2, shows that SCQ+GLCQ implies CCQ. Therefore, some results presented in [9,16] can be derived from corresponding results in [17].

Corollary 4.3. *Under the assumptions of Theorem 4.2, if SCQ and GLCQ hold at $\bar{x}$, then CCQ holds at $\bar{x}$.*

Note that if $g_i,\ i = 1, \ldots, m$, are convex, then the subdifferential set in classic convex analysis can be considered as a URC, and so SCQ implies GLCQ (see [16]). Therefore, if $g_i,\ i = 1, \ldots, m$, are convex, then according to Corollary 4.3, SCQ implies CCQ.

The following theorem provides a connection between DCQ and ZCQ.

Theorem 4.4. *Let $K \in \mathcal{L}(\bar{x})$ be as represented in (2.2). Assume that g_i, $i \in I(\bar{x})$, admits a URC $\partial^* g_i(\bar{x})$ at $\bar{x}$; and g_i, $i \notin I(\bar{x})$, is upper semicontinuous at $\bar{x}$. If DCQ holds at $\bar{x}$, then ZCQ holds at $\bar{x}$.*

Proof. We choose $y \in K$ such that DCQ holds. Similar to the proof of Theorem 4.2, we get

$$\langle \zeta, y - \bar{x} \rangle \leq -\varepsilon \|\zeta\|, \quad \forall i \in I(\bar{x}), \forall \zeta \in \partial^* g_i(\bar{x}).$$

According to Remark 4.2, $0 \notin \partial^* g_i(\bar{x})$ for any $i \in I(\bar{x})$. Hence, due to the closedness of $\partial^* g_i(\bar{x})$, we have

$$g_i^+(\bar{x}; y - \bar{x}) = \sup_{\zeta \in \partial^* g_i(\bar{x})} \langle \zeta, y - \bar{x} \rangle \leq -\varepsilon \inf_{\zeta \in \partial^* g_i(\bar{x})} \|\zeta\| < 0, \quad \forall i \in I(\bar{x}).$$

So, for each $d \in \Gamma^\circ(\bar{x})$, we have

$$g_i^+(\bar{x}; d + t(y - \bar{x})) \leq g_i^+(\bar{x}; d) + t g_i^+(\bar{x}; y - \bar{x}) < 0, \quad \forall t > 0, \forall i \in I(\bar{x}).$$

Therefore, for sufficiently small λ values, $g_i\big(\bar{x} + \lambda(d + t(y - \bar{x}))\big) \leq 0$ for each $i \in I(\bar{x})$. Also, due to the upper semicontinuity assumption, for sufficiently small λ values we have $g_i\big(\bar{x} + \lambda(d + t(y - \bar{x}))\big) \leq 0$ for each $i \notin I(\bar{x})$. Hence $d + t(y - \bar{x}) \in D_K(\bar{x})$ for each $t > 0$, which leads to $d \in cl\, D_K(\bar{x})$. Therefore, $\Gamma^\circ(\bar{x}) \subseteq cl\, D_K(\bar{x})$ and the proof is completed. ☐

Li and Zhang [17] showed that in general, neither $\Gamma^\circ(\bar{x}) \subseteq T_K(\bar{x})$ nor $\Gamma^\circ(\bar{x}) \supseteq T_K(\bar{x})$ holds necessarily. Corollary 4.5 proves the equality of these two sets for locally star-shaped sets under appropriate assumptions.

Corollary 4.5. *Assume that $K \in \mathcal{L}(\bar{x})$ is as represented in (2.2), and g_i, $i \in I(\bar{x})$, admits a URC $\partial^* g_i(\bar{x})$ at $\bar{x}$.*

(i) *If g_i, $i \notin I(\bar{x})$, is upper semicontinuous at $\bar{x}$, then DCQ implies ACQ.*

(ii) *ACQ holds at $\bar{x}$ if and only if $\Gamma^\circ(\bar{x}) = T_K(\bar{x})$.*

Proof. (i) Apply Remark 4.1 and Theorem 4.4.
(ii) By Lemma 4.1, $T_K(\bar{x}) \subseteq T_K^{\circ\circ}(\bar{x}) = N_K^\circ(\bar{x}) \subseteq \Gamma^\circ(\bar{x})$. Hence, ACQ is equivalent to $\Gamma^\circ(\bar{x}) = T_K(\bar{x})$. ☐

Now, we continue with investigating a connection between ZCQ and CCQ. In [17], it has been proved that CCQ implies ZCQ provided that the constraint functions are locally Lipschitz at $\bar{x}$ and $\partial^* g_i(\bar{x})$, $i \in I(\bar{x})$, is a bounded USRC. In the following theorem, it is shown that the locally Lipschitz assumption is not required in the presence of a locally star-shaped feasible set and URCs.

Theorem 4.6. *Assume that $K \in \mathcal{L}(\bar{x})$ is as represented in (2.2), g_i, $i \in I(\bar{x})$ admits a bounded URC $\partial^* g_i(\bar{x})$ at $\bar{x}$, and g_i, $i \notin I(\bar{x})$ is upper semicontinuous at this point. If CCQ holds at $\bar{x}$, then ZCQ holds at $\bar{x}$.*

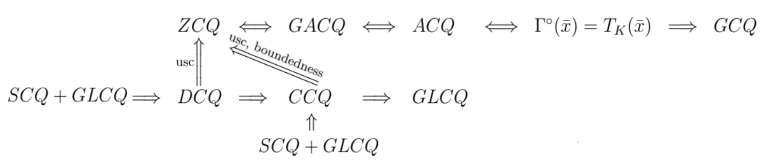
Proof. Let $d \in \Gamma^s(\bar{x})$ be arbitrary. Because of the boundedness of $\partial^* g_i(\bar{x})$ and $0 \notin \partial^* g_i(\bar{x})$,

$$g_i^+(\bar{x}; d) < 0, \quad \forall i \in I(\bar{x}).$$

Furthermore, due to the upper semicontinuity assumption, for sufficiently small λ values we have $g_i\big(\bar{x} + \lambda d\big) \leq 0$ for each $i \notin I(\bar{x})$. These imply $d \in D_K(\bar{x})$. Therefore, $\Gamma^\circ(\bar{x}) = cl \Gamma^s(\bar{x}) \subseteq cl D_K(\bar{x})$. ☐

Example 4.1 in the current section shows that CCQ does not imply ZCQ in the absence of the boundedness of URCs. Also, the converse does not hold; see Example 4.2 in the current section.

The following diagram summarizes the results of this section. In the whole diagram, we assume that $K \in \mathcal{L}(\bar{x})$ is as represented in (2.2) and $g_i,\ i \in I(\bar{x})$ admits a URC $\partial^* g_i(\bar{x})$ at $\bar{x}$. As can be seen from the results of the current section, for some of the presented implications, we need to assume upper semicontinuity for nonbinding constraints and/or boundedness for URCs of binding constraints. These two assumptions have been respectively indicated by "usc" and "boundedness" in the diagram.



The following two examples show that the converse of some implications presented in the above diagram may not hold.

Example 4.1. (CCQ $\nRightarrow$ DCQ), (GLCQ $\nRightarrow$ DCQ), and (CCQ $\nRightarrow$ ZCQ):
Consider $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g_1(x_1,x_2) = \begin{cases} \sqrt{x_1} + x_2, & x_1 \geq 0, \\ x_2, & x_1 < 0, \end{cases} \qquad g_2(x_1,x_2) = \begin{cases} -1, & x_1 = x_2 = 0, \\ x_1^2 + x_2^2, & o.w. \end{cases}$$

Then

$$K = \{(x_1,x_2) :\ g_1(x_1,x_2) \leq 0,\ g_2(x_1,x_2) \leq 0\} = \{(0,0)\}.$$

For $\bar{x} = (0,0)$, we have $I(\bar{x}) = \{1\}$ and $\partial^* g_1(0,0) = \{(\alpha, 1) : \alpha \geq 0\}$ is a URC of g_1 at $\bar{x}$. It can be seen that

$$(0,-1) \in (\Gamma(\bar{x}))^s, \quad (0,0) \notin \partial^* g_1(\bar{x}), \quad cl(D_K(\bar{x})) = \{(0,0)\}.$$

Hence, CCQ and GLCQ hold at $\bar{x}$ while DCQ and ZCQ do not hold at $\bar{x}$.

Example 4.2. (ZCQ $\nRightarrow$ CCQ): Consider

$$g(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

Let $\bar{x} = 0$. Then $[0, +\infty)$ is a URC of g at $\bar{x}$. Also, $K = \{x : g(x) \leq 0\} = \{x : x \leq 0\}$. Hence, CCQ does not hold at $\bar{x}$ while ZCQ holds at $\bar{x}$. Note that $\Gamma^\circ(\bar{x}) = (-\infty, 0] = cl(D_K(\bar{x}))$.

5 Optimality Conditions

In this section, some optimality conditions are derived using the characterization results established in previous sections. Results of this section generalize corresponding ones given in [9, 16, 21].

Lemma 5.1, which provides a full characterization of the normal cone, shows that the converse of the inclusion addressed in Lemma 4.1 holds under upper semicontinuity of nonbinding constraints and DCQ.

Lemma 5.1. *Assume that $K \in \mathcal{L}(\bar{x})$ is as represented in (2.2), $g_i,\ i \in I(\bar{x})$ admits a URC $\partial^* g_i(\bar{x})$ at $\bar{x}$, and $g_i,\ i \notin I(\bar{x})$ is upper semicontinuous at $\bar{x}$. If DCQ holds at $\bar{x}$, then*

$$N_K(\bar{x}) = cl\, pos\big(\Gamma(\bar{x})\big). \tag{5.1}$$

Proof. According to Corollary 4.5, $\big(pos(\Gamma(\bar{x}))\big)^\circ = \Gamma^\circ(\bar{x}) = T_K(\bar{x})$. Therefore,

$$N_K(\bar{x}) = T_K^\circ(\bar{x}) = \big(pos\Gamma(\bar{x})\big)^{\circ\circ} = cl\, pos\big(\Gamma(\bar{x})\big).$$

□

Theorem 5.2. *Assume that $K \in \mathcal{L}(\bar{x})$ is as represented in (2.2), $g_i,\ i \in I(\bar{x})$ admits a bounded URC $\partial^* g_i(\bar{x})$ at $\bar{x}$, and $g_i,\ i \notin I(\bar{x})$ is upper semicontinuous at $\bar{x}$. If DCQ holds at $\bar{x}$, then*

$$N_K(\bar{x}) = \Big\{ \sum_{i \in I(\bar{x})} \lambda_i conv(\partial^* g_i(\bar{x})) :\ \lambda_i \geq 0,\ i \in I(\bar{x}) \Big\}.$$

Proof. It is not difficult to see that,

$$pos\big(\cup_{i \in I(\bar{x})} \partial^* g_i(\bar{x})\big) = pos\big(\cup_{i \in I(\bar{x})} conv(\partial^* g_i(\bar{x})\big).$$

Since DCQ holds, CCQ is fulfilled and so

$$0 \notin conv(\cup_{i \in I(\bar{x})} conv(\partial^* g_i(\bar{x}))).$$

Therefore, $pos\big(\cup_{i \in I(\bar{x})} conv(\partial^* g_i(\bar{x}))\big)$ is a closed set and by Remark 2.1 and Lemma 5.1,

$$\begin{aligned} N_K(\bar{x}) &= cl\, pos\big(\cup_{i \in I(\bar{x})} (\partial^* g_i(\bar{x}))\big) \\ &= pos\big(\cup_{i \in I(\bar{x})} conv(\partial^* g_i(\bar{x}))\big) \\ &= \{\textstyle\sum_{i \in I(\bar{x})} \lambda_i conv(\partial^* g_i(\bar{x})) :\ \lambda_i \geq 0,\ i \in I(\bar{x})\}. \end{aligned}$$

□

In Theorem 5.2, the condition "$g_i,\ i \in I(\bar{x})$ admits a bounded URC" is essential and one cannot replace URC with USRC or UC. Example 5.1 clarifies it.

Example 5.1. Let

$$g(x) = \begin{cases} -x, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

and $K = \{x \in \mathbb{R} : g(x) \leq 0\} = \mathbb{R}$. For $\bar{x} = 0$, we have

$$g^+(\bar{x}; d) = g^-(\bar{x}; d) = \begin{cases} -d, & d \geq 0 \\ -\infty, & d < 0. \end{cases}$$

Therefore, $\partial^* g(\bar{x}) = \{-1\}$ is a USRC and a UC of g at $\bar{x}$. Furthermore, $K \in \mathcal{L}(\bar{x})$ and DCQ holds at $\bar{x}$ while

$$N_K(\bar{x}) = \{0\} \neq \{\lambda conv(\partial^* g(\bar{x})) :\ \lambda \geq 0\}.$$

Lasserre [16], Dutta and Lalitha [9], and Martínez-Legaz [21] showed that KKT conditions are necessary for optimality. To this end, they considered the FJ conditions at optimal solutions and then imposed some CQs to get KKT conditions. Their manner can be summarized as:

$$\text{Minimizer} \Longrightarrow \text{FJ} \overset{CQ}{\Longrightarrow} \text{KKT}$$

In Theorem 5.3 and Corollary 5.5, we use the characterization of $N_K(\bar{x})$, proved in Lemma 5.1 and Theorem 5.2, and we get KKT conditions without going through FJ conditions. The diagram below clarifies our manner:

$$\text{Minimizer} + (\text{DCQ}) + (N_K \text{ characterization}) \Rightarrow \text{KKT}$$

Theorem 5.3. *Let $K \in \mathcal{L}(\bar{x})$ and $f \in \mathcal{F}_K(\bar{x})$ has a bounded URC $\partial^* f(\bar{x})$ at $\bar{x}$. Assume that g_i $(i \in I(\bar{x}))$ admits a URC $\partial^* g_i(\bar{x})$ at $\bar{x}$, and g_i, $i \notin I(\bar{x})$ is upper semicontinuous at this point. If $\bar{x}$ is an optimal solution of Problem (2.1) which satisfies DCQ, then*

$$0 \in conv\, \partial^* f(\bar{x}) + cl\, pos(\Gamma(\bar{x})).$$

Proof. Because of the positive homogeneity, finiteness and convexity of the function $f^+(\bar{x};\cdot)$ (due to the existence of a bounded URC), by Theorems 3.1(i) and 3.2(i), we have

$$f^+(\bar{x}; d) \geq 0, \quad \forall d \in T_K(\bar{x}).$$

Therefore,

$$\sup\{\langle \eta, d\rangle : \eta \in conv\, \partial^* f(\bar{x})\} \geq 0, \quad \forall d \in T_K(\bar{x}). \tag{5.2}$$

By compactness of $conv\, \partial^* f(\bar{x})$ and due to (5.2), we have

$$\sup\{\langle \eta, d\rangle : \eta \in conv\, \partial^* f(\bar{x})\} + I_{T_K(\bar{x})}(d) \geq 0, \quad \forall d \in \mathbb{R}^n$$

where $I_{T_K(\bar{x})}(d)$ is equal to zero if $d \in T_K(\bar{x})$, and $+\infty$ otherwise. From Corollary 4.5, $T_K(\bar{x})$ is a closed convex cone, and then by [11, Example V.2.3.1], the support function of $N_K(\bar{x})$ is $I_{T_K(\bar{x})}$. Now by a manner similar to the proof of [11, Theorem VII.1.1.1], we have $0 \in conv(\partial^* f(\bar{x})) + N_K(\bar{x})$. The proof is completed because of Lemma 5.1. □

Remark 5.1. From the proof of Theorem 5.3, it can be seen that this result is still valid if one replaces bounded URC for f with bounded USRC/UC. Indeed, $f \in \mathcal{F}_K(\bar{x})$ implies the existence of $f'(\bar{x}; x-\bar{x})$ with $f'(\bar{x}; x-\bar{x}) = f^+(\bar{x}; x-\bar{x}) = f^-(\bar{x}; x-\bar{x})$ for any $x \in K$, and thanks to Theorem 3.2(i), $\bar{x}$ is a minimizer of f on K if and only if $f^{\pm}(\bar{x}; x-\bar{x}) \geq 0$ for any $x \in K$. Therefore, as $\partial^* f(\bar{x})$ is a bounded USRC/UC of f at $\bar{x}$, by Theorem 3.1(i),

$$\sup\{\langle \eta, d\rangle : \eta \in conv\, \partial^* f(\bar{x})\} \geq 0, \quad \forall d \in T_K(\bar{x}).$$

Now, the desired result can be derived by following the rest of the proof of Theorem 5.3.

Note that the above theorem may not hold if one assumes GLCQ instead of DCQ.

Example 5.2. Consider $\min f(x_1, x_2) = x_1$ subject to $(x_1, x_2) \in K$, where K is as described in Example 4.1. Let $\bar{x} = (0,0)$. Then $\partial^* f(0,0) = \{(1,0)\}$ is a URC of f at $\bar{x}$. It is clear that, $\bar{x}$ is a minimizer of f over K, and GLCQ holds at $\bar{x}$ while DCQ does not hold. Furthermore,

$$0 \notin conv\, \partial^* f(\bar{x}) + cl\, pos(\Gamma(\bar{x})).$$

Theorem 5.4 is a converse version of Theorem 5.3.

Theorem 5.4. *Let $K \in \mathcal{L}(\bar{x})$ and $f \in \mathcal{F}_K(\bar{x})$ has a URC $\partial^* f(\bar{x})$ at $\bar{x}$. Assume that g_i, $i \in I(\bar{x})$ admits a URC $\partial^* g_i(\bar{x})$ at $\bar{x}$. If $0 \in conv \partial^* f(\bar{x}) + cl\, pos(\Gamma(\bar{x}))$, then $\bar{x}$ is an optimal solution of Problem (2.1).*

Proof. Apply Theorem 3.2(ii) and Lemma 4.1. □

In Theorem 5.4, the condition "$g_i,\ i \in I(\bar{x})$ admits a URC" is essential and one can not replace URC with USRC or UC.

Example 5.3. Consider

$$\min f(x) = x, \quad s.t. \quad x \in K, \tag{5.3}$$

with K as described in Example 5.1. The set $\partial^* g(\bar{x}) = \{-1\}$ is a USRC and a UC of g at $\bar{x} = 0$. Also, $\partial^* f(\bar{x}) = \{1\}$ is a USRC and a UC of f at $\bar{x}$. Furthermore, $K \in \mathcal{L}(\bar{x})$, $f \in \mathcal{F}_K(\bar{x})$, and DCQ holds at $\bar{x}$. Although, $0 \in \partial^* f(\bar{x}) + cl\,pos(\Gamma(\bar{x}))$, here $\bar{x}$ is not an optimal solution.

The following corollary, which gives a full characterization of optimal solutions, is derived from Theorems 5.2-5.4.

Corollary 5.5. *Let $K \in \mathcal{L}(\bar{x})$ and $f \in \mathcal{F}_K(\bar{x})$ has a bounded URC $\partial^* f(\bar{x})$ at $\bar{x}$. Assume that $g_i,\ i \in I(\bar{x})$ admits a URC $\partial^* g_i(\bar{x})$ at $\bar{x}$, and $g_i,\ i \notin I(\bar{x})$ is upper semicontinuous at $\bar{x}$. Furthermore, assume that DCQ holds at $\bar{x}$ and $pos(\Gamma(\bar{x}))$ is closed (for instance, if $\partial^* g_i(\bar{x}),\ i \in I(\bar{x})$ is bounded). Then, $\bar{x}$ is an optimal solution of (2.1) if and only if there exist $\lambda_i \geq 0,\ i \in I(\bar{x})$ such that*

$$0 \in conv\,\partial^* f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i conv(\partial^* g_i(\bar{x})).$$

As mentioned before, in Theorem 5.3 and Corollary 5.5 the KKT conditions are gotten from optimality directly, in contrast to [9, Theorem 2.4], [16, Theorem 2.3] and [21, Theorem 9]. Furthermore, here we have used URC instead of usual gradient (used in [16]), Clarke generalized gradient (used in [9]) and tangential subdifferential (used in [21]). Moreover, here nonbinding g_i functions are assumed to be upper semicontinuous and not necessarily locally Lipschitz. It is worth mentioning that [9, Theorem 2.4], [16, Theorem 2.3] and [21, Theorem 9] result from Corollary 5.5.

We continue this section by an example in which the optimality of a point under consideration is concluded from the results of the present paper while it can not be derived from the results given in [9, 16, 21].

Example 5.4. Consider

$$\begin{aligned} &\min f(x_1, x_2) = \max\{x_1, x_2\} \\ &s.t. \quad g_i(x_1, x_2) \leq 0, \quad i = 1, 2, \end{aligned}$$

where

$$g_1(x_1, x_2) = -x_2, \qquad g_2(x_1, x_2) = \begin{cases} -1, & x_2 = |x_1| \text{ or } x_1 = 0 \\ x_1^2 + x_2^2, & o.w. \end{cases}$$

Let $\bar{x} = (0, 0)$. Then $I(\bar{x}) = \{1\}$. It is not difficult to see that $\partial^* g_1(0, 0) = \{(0, -1)\}$ is a URC of g_1 at $\bar{x}$, and $\partial^* f(0, 0) = \{(0, 1), (1, 0)\}$ is a URC for f at $\bar{x}$. Here, the set of feasible solutions is

$$K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 \geq 0\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = |x_1|\},$$

and it is locally star-shaped (not convex) at $\bar{x}$. We have,

$$(0, 0) = (0, 1) + (0, -1) \in \partial^* f(0, 0) + \partial^* g_1(0, 0).$$

Hence, $\bar{x} = (0, 0)$ is an optimal solution, because of Theorem 5.4.

The optimality of $\bar{x} = (0, 0)$ in this example cannot be derived from the results provided in [9, 16, 21], because K is not convex and also SCQ does not hold.

Example 5.5 presents a function which is neither locally Lipschitz nor convex nor differentiable, while admits a bounded URC.

Example 5.5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, & x = -\frac{1}{n},\ n \in \mathbb{N} \\ -1, & x < 0,\ x \neq -\frac{1}{n},\ n \in \mathbb{N} \\ x, & x \geq 0 \end{cases}$$

Let $\bar{x} = 0$. It is clear that f is neither locally Lipschitz nor convex nor differentiable at $\bar{x}$ while $\{0, 1\}$ is a bounded URC of f at $\bar{x}$.

6 Conclusions

In the current work, a nonsmooth optimization problem has been dealt with. Some CQs have been investigated and then KKT optimality conditions, in terms of convexificators, have been obtained.

The feasible set of the problem under consideration is assumed to be locally star-shaped. This assumption is less restrictive than convexity. The objective function is semilocally convex. The provided KKT conditions are gotten from optimality directly, in contrast to corresponding results existing in the literature. Furthermore, we have used upper regular convexificator instead of usual gradient, Clarke generalized gradient, and tangential subdifferential which have been used in the literature. Moreover, in the present paper nonbinding constraint functions are assumed to be upper semicontinuous and not necessarily locally Lipschitz. It is worth mentioning that optimality conditions given in the present paper are gotten under a CQ weaker than existing ones. Due to these, the results given in the present paper extend some important theorems existing in the literature. The problem for semi-infinite/multi-objective programming is worth studying.

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