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STABLE FARKAS LEMMAS AND DUALITY FOR NONCONVEX COMPOSITE SEMI-INFINITE PROGRAMMING PROBLEMS

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Abstract: The purpose of this paper is to study a nonconvex composite semi-infinite programming problem. By using the properties of the epigraph of conjugate function, we introduce a new data qualification condition and give its equivalent characterizations. Based on the new data qualification condition, we completely characterize the stable Farkas lemma in dual form and the non-asymptotic stable Farkas lemma for composite semi-infinite programming problems without any convexity and lower semicontinuity assumptions on involved functions. Moreover, we also completely characterize the stable Fenchel-Lagrange duality and the stable Lagrange duality. Our results improve the corresponding results obtained by Fang, Li and Ng and by Sun. Some examples are given to illustrate our results.

Key words: *composite semi-infinite programming, data qualification condition, nonconvex optimization, stable duality, stable Farkas lemma*

Mathematics Subject Classification: *90C34, 90C26, 49N15*

1 Introduction

The Farkas lemma [15] plays an important role in the development of linear programming and optimization theory. During the last three decades, several versions of Farkas-type results have been given in the literature for convex inequality systems [2, 6], composite convex inequality systems [5, 31], cone convex systems [24] and semi-infinite or infinite systems [11, 12, 19, 21, 27, 34]. Overviews on the development and applications of Farkas-type

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results can be found in [13,22]. Recently, there has been an increasing interest in the study of stable Farkas lemmas and stable duality for optimization problems. Jeyakumar and Lee [25] obtained some characterizations of stable Farkas lemmas and the stable Lagrange duality for cone convex optimization problems with the objective being lower semicontinuous functions and the constraint being continuous functions. Fang et al. [16] provided necessary and sufficient conditions of stable Farkas lemmas and the stable Lagrange duality for semi-infinite convex optimization problems in locally convex spaces without any lower semicontinuity assumptions. Sun [39] obtained stable Farkas lemmas in dual form and the stable Fenchel-Lagrange duality for semi-infinite convex optimization problems in locally convex spaces without lower semicontinuity assumptions.

In recent years, composite convex programming has received much attention because it offers a unified framework for treating different kinds of optimization problems. Many optimization problems from engineering, economics and finance involve composite convex functions. Thus, many works on composite convex programming problems have been published; see, e.g., [3–5, 10, 28, 31, 33, 41, 42] and the references therein.

In the above references, the convexity and the semicontinuity of the involved functions play an important role in deriving the characterization of Farkas lemmas and the duality theory for optimization problems. However, many optimization problems naturally involve nonconvex and non-continuous functions. For example, in DC programming, the functions are neither convex nor lower semicontinuous; see, e.g., [14, 17, 23, 39, 40]. Recently, Boncea and Grad [1] obtained some characterizations of ε-duality theorems for nonconvex composed optimization problems without functional constraints.

In this paper, without any convexity and lower semicontinuity assumptions, we establish some stable Farkas lemmas and the stable duality for the following composite semi-infinite programming problem:

$$\begin{array}{lll}(\mathrm{P}) & \text{Minimize} & f(h(x)), \\ & \text{subject to} & f_t(x) \leq 0,\ t \in T, \\ & & x \in C,\end{array}$$

where T is an arbitrary index set, X and Y are locally convex spaces, $C \subset X$ is a nonempty set, $K \subset Y$ is a closed convex cone, $f : Y \to \mathbb{R} \cup \{+\infty\}$ is a proper K-increasing function, $h : X \to Y^\bullet$ is a proper function, and $f_t : X \to \mathbb{R} \cup \{+\infty\}$, $t \in T$, is proper functions. Composite semi-infinite programming problems provide a unified mathematical model for a wide range of practical problems, which includes as special cases semi-infinite programming problems, conic programming problems, and composite convex inequality systems. However, to the best of our knowledge, there are no stable Farkas lemma type results and stable duality results on composite semi-infinite programming problems due to some theoretical and technical difficulties.

The rest of the paper is organized as follows. Section 2 contains some basic definitions and preliminary results. In Section 3, a new data qualification condition is provided and its equivalent characterizations are established. Based on the new data qualification condition, the stable Farkas lemma in dual form and the non-asymptotic stable Farkas lemma for composite semi-infinite programming problems are obtained. Using the results obtained in Section 3, the stable Fenchel-Lagrange duality and the stable Lagrange duality are given in Section 4. It is worth mentioning that these results are obtained without any convexity and lower semicontinuity assumptions. Our results improve the corresponding results obtained by Fang, Li and Ng [16] and by Sun [39].

2 Preliminaries

Throughout this paper, we assume that X and Y are locally convex spaces with X^* and Y^* being their dual spaces endowed with the weak*-topologies $w(X^*, X)$ and $w(Y^*, Y)$, respectively. For a subset $D \subseteq X^*$, we use cl$D$ for the weak*-closure of D, and cone$D$ for the convex cone generated by $D \cup \{0\}$. Given a nonempty set $B \subseteq X$, we denote by δ_B the indicator function of B, i.e., $\delta_B(x) = 0$ if $x \in B$ and $\delta_B(x) = +\infty$ if $x \notin B$.

Let $K \subseteq Y$ be a closed convex cone. Denote by $K^\oplus$ the dual cone of K, i.e.,

$$K^\oplus := \{y^* \in Y^* : \langle y^*, y\rangle \geq 0, \ \forall\, y \in K\},$$

where we denote by $\langle y^*, y\rangle = y^*(y)$ the value at y of the continuous linear functional y^*. Consider the ordering $\leq_K$ in Y induced by K as

$$y_1 \leq_K y_2 \Leftrightarrow y_2 - y_1 \in K, \ \forall\, y_1, y_2 \in Y.$$

We add to Y a greatest element with respect to $\leq_K$ denoted by ∞_K which does not belong to Y and $Y^\bullet := Y \cup \{\infty_K\}$. Then for any $y \in Y^\bullet$ one has $y \leq_K \infty_K$ and we consider the following operations on $Y^\bullet$: $y + \infty_K = \infty_K + y = \infty_K$ and $t\infty_K = \infty_K$ for all $t \geq 0$. For the problem (P), we set $f(\infty_K) = +\infty$.

Let I be an arbitrary index set, $\{X_i : \ i \in I\}$ be a family of subset of X, and let $\wp$ be the collection of all the nonempty finite subsets of I. Then

$$\text{cone}\left(\bigcup_{i\in I} X_i\right) = \bigcup_{J\in\wp} \text{cone}\left(\bigcup_{j\in J} X_j\right) = \bigcup_{J\in\wp}\left(\sum_{j\in J} \text{cone}X_j\right).$$

Let us denote by $\mathbb{R}^{(T)}$ the following linear vector space [20]:

$$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t\in T} : \lambda_t = 0 \text{ for all } t \in T \text{ except for finitely many } \lambda_t \neq 0\}.$$

The nonnegative cone of $\mathbb{R}^{(T)}$ is denoted by

$$\mathbb{R}^{(T)}_+ := \{\lambda = (\lambda_t)_{t\in T} \in \mathbb{R}^{(T)} : \lambda_t \geq 0, \ t \in T\}.$$

It is easy to see that $\mathbb{R}^{(T)}_+$ is a convex cone of $\mathbb{R}^{(T)}$. For $\lambda \in \mathbb{R}^{(T)}_+$, the supporting set corresponding to λ is defined by $T(\lambda) := \{t \in T : \lambda_t > 0\}$, which is a finite subset of T. Let Z be a linear vector space. For $\lambda \in \mathbb{R}^{(T)}$ and $\{z_t\}_{t\in T} \subseteq Z$, we set

$$\sum_{t\in T} \lambda_t z_t := \begin{cases} \sum_{t\in T(\lambda)} \lambda_t z_t, & \text{if } T(\lambda) \neq \emptyset, \\ 0, & \text{if } T(\lambda) = \emptyset. \end{cases}$$

Let $g : X \to \mathbb{R} \cup \{+\infty\}$. We denote by $\text{dom}(g) := \{x \in X : g(x) < +\infty\}$ its effective domain and by

$$\text{epi}g := \{(x, r) \in X \times \mathbb{R} : g(x) \leq r\}$$

its epigraph, respectively. The function g is said to be proper if $\text{dom}(g) \neq \emptyset$. The conjugate function of g, $g^* : X^* \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$g^*(x^*) := \sup\{\langle x^*, x\rangle - g(x) : x \in \text{dom}(g)\}.$$

The following relation is the well-known Fenchel-Young inequality

$$g^*(x^*) + g(x) \geq \langle x^*, x\rangle, \ \forall\, x \in X, \ \forall\, x^* \in X^*.$$

Definition 2.1. A function $f : Y \to \mathbb{R} \cup \{+\infty\}$ is said to be K-increasing on Y if, for $y_1, y_2 \in Y$ with $y_1 \leq_K y_2$, we have $f(y_1) \leq f(y_2)$.

Remark 2.2. It is easy to see that if f is K-increasing on Y, then $\mathrm{dom} f^* \subset K^{\oplus}$.

Definition 2.3. A function $h : X \to Y^\bullet$ is said to be proper if its domain $\mathrm{dom}(h) = \{x \in X : h(x) \in Y\}$ is nonempty.

Definition 2.4 ([26])**.** A function $h : X \to Y^\bullet$ is said to be star K-lower semicontinuous if the function $\lambda h := \lambda \circ h$ is lower semicontinuous for all $\lambda \in K^{\oplus}$.

The following lemmas will be used in the sequel.

Lemma 2.5 ([16, p. 1315])**.** *Let $\varphi, \psi : X \to \mathbb{R} \cup \{+\infty\}$ be two proper functions. Then*

(i) $epi\varphi^* + epi\psi^* \subseteq epi(\varphi + \psi)^*$.

(ii) $\varphi \leq \psi \Rightarrow \varphi^* \geq \psi^* \Rightarrow epi\varphi^* \subseteq epi\psi^*$.

Lemma 2.6 ([7, Theorem 2.1])**.** *Let $\varphi, \psi : X \to \mathbb{R} \cup \{+\infty\}$ be two proper convex lower semicontinuous functions such that $dom\varphi \cap dom\psi \neq \emptyset$. Then*

$$epi(\varphi + \psi)^* = cl(epi\varphi^* + epi\psi^*).$$

Lemma 2.7. *Let $K \subseteq Y$ be a closed convex cone. Let $f : Y \to \mathbb{R} \cup \{+\infty\}$ be a proper K-increasing function and $h : X \to Y^\bullet$ be a proper function. Assume that $h^{-1}(dom(f)) \neq \emptyset$. For any $x^* \in X^*$ and $\xi \in K^{\oplus}$, we have*

$$(f \circ h)^*(x^*) \leq f^*(\xi) + (\xi h)^*(x^*).$$

Proof. By the Fenchel-Young inequality, for any $x^* \in X^*$, $\xi \in K^{\oplus}$ and $x \in h^{-1}(\mathrm{dom}(f))$, $(\xi h)^*(x^*) + (\xi h)(x) \geq \langle x^*, x \rangle$ and $(f \circ h)(x) + f^*(\xi) \geq \langle \xi, h(x) \rangle$. It follows that

$$(\xi h)^*(x^*) + f^*(\xi) \geq \langle x^*, x \rangle - (f \circ h)(x).$$

Therefore, the conclusion holds. $\square$

Throughout this paper, following [43], we adapt the convention that

$$(+\infty) - (+\infty) = (+\infty) + (-\infty) = (-\infty) - (-\infty) = (-\infty) + (+\infty) = +\infty,$$

$$0 \cdot (+\infty) = +\infty \text{ and } 0 \cdot (-\infty) = 0.$$

3 Stable Farkas-Type Results

In this section, we present some characterizations of stable Farkas lemmas for composite semi-infinite programming problems, where the functions involved are not necessarily convex nor lower semicontinuous.

Let A be the feasible set of the problem (P), i.e.,

$$A := \{x \in C : f_t(x) \leq 0, \ \forall\, t \in T\}.$$

In the rest of this paper, without other specifications, we always assume that $h^{-1}(\mathrm{dom}(f)) \cap A \neq \emptyset$. Consider the cone

$$K_1 := \mathrm{cone}\left\{ \bigcup_{t\in T} \mathrm{epi} f_t^* \right\} + \mathrm{epi}\delta_C^*, \tag{3.1}$$

which was introduced by Dinh et al. [12]. In general, $K_1 \subseteq \mathrm{epi}\delta_A^*$ (see [16, (3.6)]). Dinh et al. [12] also obtained that if each f_t, $t \in T$ is a proper lower semicontinuous convex function and C is a closed convex set, then $\mathrm{epi}\delta_A^* = \mathrm{cl}K_1$. For more results on this topic, we refer the reader to [3, 4, 8, 9, 12] and the references therein.

3.1 A New Data Qualification Condition

In this subsection, we introduce a new data qualification condition for the problem (P). The condition not only involves the constraint set and the constraint functions; but also the objective function. We will give its equivalent characterizations. Let us first prove the following proposition.

Proposition 3.1. *Let the functions $F, G, H : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be defined by $F(x, y) = f(y)$, $G(x, y) = \delta_A(x)$ and $H(x, y) = \delta_{\{(x,y)\in X\times Y : y-h(x)\in K\}}(x, y)$, for any $(x, y) \in X \times Y$. Then*

(i) *F, G and H are proper functions and*

$$dom(F) \cap dom(G) \cap dom(H) \neq \emptyset.$$

(ii) *For $(x^*, r) \in X^* \times \mathbb{R}$,*

$$(x^*, r) \in epi(f \circ h + \delta_A)^* \Leftrightarrow (x^*, 0, r) \in epi(F + G + H)^*.$$

(iii)

$$epiF^* = \{0\} \times epif^*,$$

$$epiG^* = \{(p, 0, r) : (p, r) \in epi\delta_A^*\}$$

and

$$epiH^* = \bigcup_{\lambda\in K^\oplus} \{(p, -\lambda, r) : (p, r) \in epi(\lambda h)^*\}.$$

Proof. (i) By the assumptions, it is easy to see that F, G and H are proper functions. Since $h^{-1}(\mathrm{dom}(f)) \cap A \neq \emptyset$, there exists $y \in \mathrm{dom}(f)$ such that $h^{-1}(y) \cap A \neq \emptyset$. It follows that there exists $x \in A$ such that $h(x) = y$ and so $y - h(x) = 0 \in K$. Thus, $(x, y) \in \mathrm{dom}(H)$. Note that $\mathrm{dom}(F) = X \times \mathrm{dom}(f)$ and $\mathrm{dom}(G) = A \times Y$. Therefore, $(x, y) \in \mathrm{dom}(F) \cap \mathrm{dom}(G) \cap \mathrm{dom}(H)$.

(ii) As f is K-increasing, for any $x^* \in X^*$, we can conclude that

$$\inf_{x\in X}[(f \circ h + \delta_A)(x) - \langle x^*, x\rangle] = \inf_{\substack{x\in X, y\in Y \\ y-h(x)\in K}} [(f(y) + \delta_A(x) - \langle x^*, x\rangle].$$

It follows that

$$-(f \circ h + \delta_A)^*(x^*) = \inf_{x\in X}\{(f \circ h)(x) - \langle x^*, x\rangle + \delta_A(x)\}$$

$$= \inf_{\substack{x\in X, y\in Y\\ y-h(x)\in K}} \{f(y) - \langle x^*, x\rangle + \delta_A(x)\}$$
$$= \inf_{x\in X, y\in Y} \{f(y) - \langle x^*, x\rangle + \delta_A(x) + \delta_{\{(x,y)\in X\times Y: y-h(x)\in K\}}(x,y)\}$$
$$= -(F+G+H)^*(x^*,0).$$

This implies that (ii) holds.

(iii) By the definition of conjugate functions, we have

$$F^*(x^*,y^*) = \sup_{x\in X, y\in Y}\{\langle x^*, x\rangle + \langle y^*, y\rangle - f(y)\} = \sup_{x\in X}\langle x^*, x\rangle + \sup_{y\in Y}\{\langle y^*, y\rangle - f(y)\},$$

$$G^*(x^*,y^*) = \sup_{x\in X, y\in Y}\{\langle x^*, x\rangle + \langle y^*, y\rangle - \delta_A(x)\} = \sup_{x\in X}\{\langle x^*, x\rangle - \delta_A(x)\} + \sup_{y\in Y}\langle y^*, y\rangle$$

and

$$H^*(x^*,y^*) = \sup_{x\in X, y\in Y}\{\langle x^*, x\rangle + \langle y^*, y\rangle - \delta_{\{(x,y)\in X\times Y: y-h(x)\in K\}}(x,y)\}$$
$$= \sup_{\substack{x\in X, y\in Y\\ y-h(x)\in K}}\{\langle x^*, x\rangle + \langle y^*, y\rangle\}.$$

Let $k = y - h(x)$. Then

$$H^*(x^*,y^*) = \sup_{x\in X, k\in K}\{\langle x^*, x\rangle + \langle y^*, k + h(x)\rangle\}$$
$$= \sup_{x\in X}\{\langle x^*, x\rangle + \langle y^*, h(x)\rangle\} + \sup_{k\in K}\langle y^*, k\rangle.$$

It follows that

$$F^*(x^*,y^*) = \begin{cases} f^*(y^*), & \text{if } x^* = 0,\\ +\infty, & \text{otherwise,}\end{cases} \quad G^*(x^*,y^*) = \begin{cases} \delta_A^*(x^*), & \text{if } y^* = 0,\\ +\infty, & \text{otherwise}\end{cases}$$

and

$$H^*(x^*,y^*) = \begin{cases} (-y^*h)^*(x^*), & \text{if } y^* \in -K^\oplus,\\ +\infty, & \text{otherwise.}\end{cases}$$

It is easy to see that $\text{epi}F^* = \{0\}\times \text{epi}f^*$. Now let $(p,b,r)\in \text{epi}G^*$. Then $G^*(p,b) \le r$. This is equivalent to $b = 0$ and $\delta_A^*(p) \le r$. So, $(p,r)\in \text{epi}\delta_A^*$. Thus, $\text{epi}G^* = \{(p,0,r) : (p,r)\in \text{epi}\delta_A^*\}$. Similarly, we have

$$\text{epi}H^* = \bigcup_{\lambda\in K^\oplus}\{(p,-\lambda,r) : (p,r)\in \text{epi}(\lambda h)^*\}.$$

The proof is complete. $\square$

If $Y = X$, $h(x) = x$ for any $x\in X$ and $K = \{0\}$, then we have the following corollary.

Corollary 3.2. *Let $dom(f)\cap A \neq \emptyset$. Let the functions $F, G, H : X\times X \to \mathbb{R}\cup\{+\infty\}$ be defined by $F(x,y) = f(y)$, $G(x,y) = \delta_A(x)$ and $H(x,y) = \delta_{\{(x,y)\in X\times X: x=y\}}(x,y)$, for any $(x,y)\in X\times X$. Then (i) F, G and H are proper functions and $dom(F)\cap dom(G)\cap dom(H) \neq \emptyset$; (ii) for $(x^*,r)\in X^*\times\mathbb{R}$, $(x^*,r)\in epi(f+\delta_A)^* \Leftrightarrow (x^*,0,r)\in epi(F+G+H)^*$; (iii) $epiF^* = \{(0,p,r) : (p,r)\in epif^*\}$, $epiG^* = \{(p,0,r) : (p,r)\in epi\delta_A^*\}$ and $epiH^* = \cup_{x^*\in X^*}\{(x^*,-x^*,r) : r\ge 0\}$.*

Remark 3.3. If $Y = X$, $h(x) = x$ for any $x \in X$ and $K = \{0\}$, then for $\lambda \in K^{\oplus}$,

$$(\lambda h)^*(x^*) = \begin{cases} 0, & \text{if } x^* = \lambda; \\ +\infty, & \text{otherwise.} \end{cases}$$

In this paper, we consider the following set

$$M := \{0\} \times \mathrm{epi} f^* + \{(p, 0, r) : (p, r) \in K_1\} + \bigcup_{\lambda \in K^{\oplus}} \{(p, -\lambda, r) : (p, r) \in \mathrm{epi}((\lambda h)^*)\},$$

where K_1 is defined in (3.1). We have the following proposition holds.

Proposition 3.4. *The following relation holds:*

$$M \subseteq epi(F + G + H)^*.$$

Proof. From Proposition 3.1 and Lemma 2.5, we have

$$\begin{aligned} M =& \{0\} \times \mathrm{epi} f^* + \{(p, 0, r) : (p, r) \in K_1\} + \bigcup_{\lambda \in K^{\oplus}} \{(p, -\lambda, r) : (p, r) \in \mathrm{epi}((\lambda h)^*)\} \\ \subseteq& \{0\} \times \mathrm{epi} f^* + \{(p, 0, r) : (p, r) \in \mathrm{epi}\delta_A^*\} + \bigcup_{\lambda \in K^{\oplus}} \{(p, -\lambda, r) : (p, r) \in \mathrm{epi}((\lambda h)^*)\} \\ =& \mathrm{epi} F^* + \mathrm{epi} G^* + \mathrm{epi} H^* \\ \subseteq& \mathrm{epi}(F + G)^* + \mathrm{epi} H^* \\ \subseteq& \mathrm{epi}(F + G + H)^*. \end{aligned}$$

The proof is complete. $\square$

The following example shows that the converse inclusion does not hold in general.

Example 3.5. Let $X = Y = \mathbb{R}$, $K = [0, +\infty)$, $T = [0, +\infty)$ and $C = \mathbb{R}$. Define $h, f_t, f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ respectively by $h(x) = 1 - x^2$, $f_t(x) = tx$, $t \in T$ and

$$f(y) = \begin{cases} 0, & \text{if } y < 0; \\ 1, & \text{if } y = 0; \\ +\infty, & \text{if } y > 0. \end{cases}$$

Then, $X^* = Y^* = \mathbb{R}$, $K^{\oplus} = [0, +\infty)$ and $A = (-\infty, 0]$. Moreover, f is not lower semicontinuous, $f^* = \delta_{[0,+\infty)}$, $f_t^* = \delta_{\{t\}}$ and $\delta_C^* = \delta_{\{0\}}$. If $\lambda = 0$, then $(\lambda h)^* = \delta_{\{0\}}$. If $\lambda > 0$, then $(\lambda h)^*(x^*) = +\infty$ for any $x^* \in \mathbb{R}$. Note that for any $x, y \in \mathbb{R}$,

$$(F + G + H)(x, y) = \begin{cases} 0, & \text{if } y < 0, x \le -\sqrt{1 - y}; \\ 1, & \text{if } y = 0, x \le -1; \\ +\infty, & \text{otherwise} \end{cases}$$

and for any $x^*, y^* \in \mathbb{R}$,

$$(F + G + H)^*(x^*, y^*) = \begin{cases} -x^*, & \text{if } x^* \ge 0, y^* \ge 0; \\ +\infty, & \text{otherwise.} \end{cases}$$

Hence,

$$\mathrm{epi} f^* = [0, +\infty) \times [0, +\infty),$$

$$\begin{aligned}
&\mathrm{epi}\delta_C^* = \{0\} \times [0, +\infty),\\
&\mathrm{cone}\left\{\bigcup_{t\in T} \mathrm{epi} f_t^*\right\} = [0, +\infty) \times [0, +\infty),\\
&\mathrm{epi}(F + G + H)^* = \{(x, y, z) : x \geq 0, y \geq 0, z \geq -x\}
\end{aligned}$$

and

$$\begin{aligned}
M &= \{0\} \times \mathrm{epi} f^* + \{(p, 0, r) : (p, r) \in K_1\} + \bigcup_{\lambda\in K^\oplus} \{(p, -\lambda, r) : (p, r) \in \mathrm{epi}((\lambda h)^*)\}\\
&= \{0\} \times [0, +\infty) \times [0, +\infty) + [0, +\infty) \times \{0\} \times [0, +\infty) + \{0\} \times \{0\} \times [0, +\infty)\\
&= [0, +\infty) \times [0, +\infty) \times [0, +\infty).
\end{aligned}$$

Obviously,

$$\mathrm{epi}(F + G + H)^* \nsubseteq M.$$

Considering the possible relationships between M and $\mathrm{epi}(F + G + H)^*$, we introduce the following data qualification condition.

Definition 3.6. We say that the problem (P) satisfies the data qualification condition (DQC, in brief) if

$$\mathrm{epi}(F + G + H)^* \cap (X^* \times \{0\} \times \mathbb{R}) = M \cap (X^* \times \{0\} \times \mathbb{R}).$$

Remark 3.7. By Proposition 3.4, (DQC) holds if and only if

$$\mathrm{epi}(F + G + H)^* \cap (X^* \times \{0\} \times \mathbb{R}) \subseteq M \cap (X^* \times \{0\} \times \mathbb{R}).$$

The following proposition gives a sufficient condition ensuring that (DQC) holds.

Proposition 3.8. *Assume that f is a proper convex, lower semicontinuous K-increasing function, h is a proper K-convex and star K-lower semicontinuous function, f_t, $t \in T$ is proper lower semicontinuous convex functions and C is a closed convex set. If M is weak*-closed, then (DQC) holds.*

Proof. By Lemma 2.6,

$$\begin{aligned}
\mathrm{epi}(F + G + H)^* =& \mathrm{cl}(\mathrm{epi}(F + G)^* + \mathrm{epi}H^*)\\
=& \mathrm{cl}(\mathrm{cl}(\mathrm{epi}F^* + \mathrm{epi}G^*) + \mathrm{epi}H^*)\\
=& \mathrm{cl}(\mathrm{epi}F^* + \mathrm{epi}G^* + \mathrm{epi}H^*).
\end{aligned}$$

On the other hand, it is easy to see that $\mathrm{cl}\{(p, 0, r) : (p, r) \in K_1\} = \{(p, 0, r) : (p, r) \in \mathrm{cl}K_1\} = \{(p, 0, r) : (p, r) \in \mathrm{epi}\delta_A^*\}$. Note that M is weak*-closed. Then

$$\begin{aligned}
M =& \mathrm{cl}M\\
=& \mathrm{cl}\left\{\{0\} \times \mathrm{epi} f^* + \mathrm{cl}\{(p, 0, r) : (p, r) \in K_1\} + \bigcup_{\lambda\in K^\oplus} \{(p, -\lambda, r) : (p, r) \in \mathrm{epi}((\lambda h)^*)\}\right\}\\
=& \mathrm{cl}\left\{\{0\} \times \mathrm{epi} f^* + \{(p, 0, r) : (p, r) \in \mathrm{epi}\delta_A^*\} + \bigcup_{\lambda\in K^\oplus} \{(p, -\lambda, r) : (p, r) \in \mathrm{epi}((\lambda h)^*)\}\right\}.
\end{aligned}$$

It follows from this and Proposition 3.1 that (DQC) holds. The proof is complete. □

Remark 3.9. It is worth mentioning that M is weak*-closed can be replaced by the weaker assumption that

$$\mathrm{cl}M \cap (X^* \times \{0\} \times \mathbb{R}) = M \cap (X^* \times \{0\} \times \mathbb{R}),$$

i.e., that M is closed regarding the set $X^* \times \{0\} \times \mathbb{R}$. The concept of closedness regarding a set was introduced by Pomerol in his PhD thesis [35], [36]. Recently, Boţ has used this concept systematically in the context of data qualifications on conjugate duality [2, Chapter II, Section 9].

We now give an equivalent characterization of (DQC).

Theorem 3.10. *The following statements are equivalent:*

(i) *(DQC) holds.*

(ii) *For any $x^* \in X^*$, we have*

$$\begin{aligned}&(f \circ h + \delta_A)^*(x^*)\\ &= \min_{\substack{\lambda \in \mathbb{R}^{(T)}_+, \xi \in K^{\oplus} \\ u, v_t \in X^*, t \in T(\lambda)}} \left[f^*(\xi) + (\xi h)^*(u) + \sum_{t \in T(\lambda)} \lambda_t f_t^*(v_t) + \delta_C^*\left(x^* - u - \sum_{t \in T(\lambda)} \lambda_t v_t \right) \right].\end{aligned}$$

Proof. (i) $\Rightarrow$ (ii): Suppose that (DQC) holds. For any $x^* \in X^*$, by Lemma 2.7 and the definition of conjugate function,

$$\begin{aligned}&f^*(\xi) + (\xi h)^*(u) \geq (f \circ h)^*(u) \geq \langle u, x \rangle - (f \circ h)(x),\\ &f_t^*(v_t) \geq \langle v_t, x \rangle - f_t(x) \geq \langle v_t, x \rangle,\\ &\delta_C^*\left(x^* - u - \sum_{t \in T(\lambda)} \lambda_t v_t \right) \geq \langle x^* - u - \sum_{t \in T(\lambda)} \lambda_t v_t, x \rangle - \delta_C(x)\end{aligned}$$

for each $\lambda \in \mathbb{R}^{(T)}_+$, $\xi \in K^{\oplus}$, $t \in T(\lambda)$, $u, v_t \in X^*$ and $x \in h^{-1}(\mathrm{dom}(f)) \cap A \neq \emptyset$. It follows that

$$f^*(\xi) + (\xi h)^*(u) + \sum_{t \in T(\lambda)} \lambda_t f_t^*(v_t) + \delta_C^*\left(x^* - u - \sum_{t \in T(\lambda)} \lambda_t v_t \right) \geq \langle x^*, x \rangle - (f \circ h)(x).$$

This implies that

$$f^*(\xi) + (\xi h)^*(u) + \sum_{t \in T(\lambda)} \lambda_t f_t^*(v_t) + \delta_C^*\left(x^* - u - \sum_{t \in T(\lambda)} \lambda_t v_t \right) \geq (f \circ h + \delta_A)^*(x^*). \quad (3.2)$$

If $(f \circ h + \delta_A)^*(x^*) = +\infty$, the conclusion holds trivially by (3.2). Now let $(f \circ h + \delta_A)^*(x^*) \in \mathbb{R}$. Note that $(x^*, (f \circ h + \delta_A)^*(x^*)) \in \mathrm{epi}(f \circ h + \delta_A)^*$. By Proposition 3.1(ii),

$$(x^*, 0, (f \circ h + \delta_A)^*(x^*)) \in \mathrm{epi}(F + G + H)^*.$$

This together with (DQC) yields

$$(x^*, 0, (f \circ h + \delta_A)^*(x^*)) \in M \cap (X^* \times \{0\} \times \mathbb{R}).$$

Then there exist $\xi' \in K^{\oplus}$, $\lambda' \in \mathbb{R}_+^T$, $(\xi', f^*(\xi')+\eta') \in \mathrm{epi} f^*$ with $\eta' \geq 0$, $(u', \alpha') \in \mathrm{epi}(\xi' h)^*$, $(w', \gamma') \in \mathrm{epi}\delta_C^*$ and $(v_t', \beta_t') \in \mathrm{epi} f_t^*$ with $t \in T(\lambda')$ such that

$$(x^*, 0, (f \circ h + \delta_A)^*(x^*)) = (0, \xi', f^*(\xi')+\eta') + (u', -\xi', \alpha') + \sum_{t \in T(\lambda')} \lambda_t'(v_t', 0, \beta_t') + (w', 0, \gamma'),$$

which gives $x^* = u' + \sum_{t \in T(\lambda')} \lambda_t' v_t' + w'$ and

$$\begin{aligned}(f \circ h + \delta_A)^*(x^*) &= f^*(\xi') + \eta' + \alpha' + \sum_{t \in T(\lambda')} \lambda_t' \beta_t' + \gamma' \\ &\geq f^*(\xi') + (\xi' h)^*(u') + \sum_{t \in T(\lambda')} \lambda_t' f_t^*(v_t') + \delta_C^*(w') \\ &= f^*(\xi') + (\xi' h)^*(u') + \sum_{t \in T(\lambda')} \lambda_t' f_t^*(v_t') + \delta_C^*\Bigg(x^* - u' - \sum_{t \in T(\lambda')} \lambda_t' v_t'\Bigg).\end{aligned}$$

Combining this with (3.2), we have

$$\begin{aligned}&(f \circ h + \delta_A)^*(x^*) \\ =&f^*(\xi') + (\xi' h)^*(u') + \sum_{t \in T(\lambda')} \lambda_t' f_t^*(v_t') + \delta_C^*\Bigg(x^* - u' - \sum_{t \in T(\lambda')} \lambda_t' v_t'\Bigg) \\ =&\min_{\substack{\lambda \in \mathbb{R}_+^{(T)}, \xi \in K^{\oplus} \\ u, v_t \in X^*, t \in T(\lambda)}} \left[f^*(\xi) + (\xi h)^*(u) + \sum_{t \in T(\lambda)} \lambda_t f_t^*(v_t) + \delta_C^*\Bigg(x^* - u - \sum_{t \in T(\lambda)} \lambda_t v_t\Bigg) \right].\end{aligned}$$

Hence, (ii) holds.

(ii) $\Rightarrow$ (i): As $M \subseteq \mathrm{epi}(F+G+H)^*$, we only need to prove that $\mathrm{epi}(F+G+H)^* \cap (X^* \times \{0\} \times \mathbb{R}) \subseteq M \cap (X^* \times \{0\} \times \mathbb{R})$. Let $(x^*, 0, r) \in \mathrm{epi}(F+G+H)^* \cap (X^* \times \{0\} \times \mathbb{R})$. By Proposition 3.1(ii), $(x^*, r) \in \mathrm{epi}(f \circ h + \delta_A)^*$, i.e., $(f \circ h + \delta_A)^*(x^*) \leq r$. From the hypothesis we know that there exist $\xi' \in K^{\oplus}$, $\lambda' \in \mathbb{R}_+^T$ and $u', v_t' \in X^*$ with $t \in T(\lambda')$ such that

$$(f \circ h + \delta_A)^*(x^*) = f^*(\xi') + (\xi' h)^*(u') + \sum_{t \in T(\lambda')} \lambda_t' f_t^*(v_t') + \delta_C^*\Bigg(x^* - u' - \sum_{t \in T(\lambda')} \lambda_t' v_t'\Bigg).$$

This implies

$$f^*(\xi') + (\xi' h)^*(u') + \sum_{t \in T(\lambda')} \lambda_t' f_t^*(v_t') + \delta_C^*\Bigg(x^* - u' - \sum_{t \in T(\lambda')} \lambda_t' v_t'\Bigg) \leq r,$$

and so

$$\delta_C^*\Bigg(x^* - u' - \sum_{t \in T(\lambda')} \lambda_t' v_t'\Bigg) \leq r - f^*(\xi') - (\xi' h)^*(u') - \sum_{t \in T(\lambda')} \lambda_t' f_t^*(v_t'),$$

i.e.,

$$\Bigg(x^* - u' - \sum_{t \in T(\lambda')} \lambda_t' v_t', r - f^*(\xi') - (\xi' h)^*(u') - \sum_{t \in T(\lambda')} \lambda_t' f_t^*(v_t')\Bigg) \in \mathrm{epi}\delta_C^*.$$

It follows that

$$\begin{aligned}
(x^*, 0, r) =&(0, \xi', f^*(\xi')) + \sum_{t\in T(\lambda')} \lambda'_t(v'_t, 0, f^*_t(v'_t)) + (u', -\xi', (\xi' h)^*(u')) \\
&+ (x^* - u' - \sum_{t\in T(\lambda')} \lambda'_t v'_t, 0, r - f^*(\xi') - (\xi' h)^*(u') - \sum_{t\in T(\lambda')} \lambda'_t f^*_t(v'_t)) \\
\in&\{0\} \times \text{epi}(f)^* + \{(p, 0, s) : (p, s) \in \text{cone}\left\{\bigcup_{t\in T} \text{epi} f^*_t\right\} \\
&+ \bigcup_{\lambda\in K^\oplus} \{(p, -\lambda, s) : (p, s) \in \text{epi}((\lambda h)^*)\} + \{(p, 0, s) : (p, s) \in \text{epi}\delta^*_C\} \\
=&M,
\end{aligned}$$

which implies that $\text{epi}(F + G + H)^* \cap (X^* \times \{0\} \times \mathbb{R}) \subseteq M \cap (X^* \times \{0\} \times \mathbb{R})$. The proof is complete. □

The following example illustrates Theorem 3.10, where the function involved are possibly neither convex nor lower semicontinuous.

Example 3.11. Let $\mathbb{Q}$ and $\mathbb{Q}_+$ denote the set of rational numbers and the set of nonnegative rational numbers respectively. Let $X = Y = \mathbb{R}$, $K = (-\infty, 0]$, $T = (-\infty, 0]$ and $C = \mathbb{Q}$. Define $f, h, f_t : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ respectively by $f = \delta_{[0,+\infty)}$, $h = \delta_{\mathbb{Q}_+}$ and $f_t(x) = tx$, $t \in T$. Obviously, C is not a convex set and h is neither convex nor lower semicontinuous.. Moreover, we have $X^* = Y^* = \mathbb{R}$, $K^\oplus = (-\infty, 0]$, $A = \mathbb{Q}_+$ and $f \circ h + \delta_A = \delta_{\mathbb{Q}_+}$. By a simple computation, $f^* = \delta_{(-\infty,0]}$, $(f \circ h + \delta_A)^* = \delta_{(-\infty,0]}$, $f^*_t = \delta_{\{t\}}$ and $\delta^*_C = \delta_{\{0\}}$. If $\lambda = 0$, then $(\lambda h)^* = \delta_{\{0\}}$. If $\lambda < 0$, then $(\lambda h)^*(x^*) = +\infty$ for any $x^* \in \mathbb{R}$. It is easy to see that for any $x, y \in \mathbb{R}$,

$$(F + G + H)(x, y) = \begin{cases} 0, & \text{if } x \in \mathbb{Q}_+, y = 0; \\ +\infty, & \text{otherwise} \end{cases}$$

and for any $x^*, y^* \in \mathbb{R}$,

$$(F + G + H)^*(x^*, y^*) = \begin{cases} 0, & \text{if } x^* \le 0, y^* \in \mathbb{R}; \\ +\infty, & \text{otherwise.} \end{cases}$$

It follows that

$$\begin{aligned}
&\text{epi} f^* = (-\infty, 0] \times [0, +\infty), \\
&\text{epi}\delta^*_C = \{0\} \times [0, +\infty), \\
&\text{epi}(f \circ h + \delta_A)^* = (-\infty, 0] \times [0, +\infty), \\
&\text{cone}\left\{\bigcup_{t\in T} \text{epi} f^*_t\right\} = (-\infty, 0] \times [0, +\infty),
\end{aligned}$$

and

$$\text{epi}(F + G + H)^* = (-\infty, 0] \times (-\infty, +\infty) \times [0, +\infty).$$

Note that

$$\begin{aligned} M &= \{0\} \times \mathrm{epi} f^* + \{(p,0,r) : (p,r) \in K_1\} + \bigcup_{\lambda \in K^\oplus} \{(p,-\lambda,r) : (p,r) \in \mathrm{epi}((\lambda h)^*)\} \\ &= \{0\} \times (-\infty,0] \times [0,+\infty) + (-\infty,0] \times \{0\} \times [0,+\infty) + \{0\} \times \{0\} \times [0,+\infty) \\ &= (-\infty,0] \times (-\infty,0] \times [0,+\infty). \end{aligned}$$

Therefore,

$$\mathrm{epi}(F+G+H)^* \cap (X^* \times \{0\} \times \mathbb{R}) = M \cap (X^* \times \{0\} \times \mathbb{R})$$

and (ii) of Theorem 3.10 holds.

$\boxed{3.2}$ Stable Farkas Lemmas

In this subsection, we obtain some stable Farkas lemmas for the problem (P). In the following theorem we give a stable Farkas lemma in dual form for the problem (P). Stable Farkas lemmas in dual form for convex and/or lower semicontinuous functions have been investigated in [5, 6, 13, 14, 17, 31, 34, 39, 40].

Theorem 3.12. *For the problem (P), (DQC) holds if and only if the following statements are equivalent:*

(i) *For every $x^* \in X^*$ and every $\alpha \in \mathbb{R}$,*

$$x \in C,\ f_t(x) \leq 0,\ t \in T \Rightarrow f(h(x)) \geq \langle x^*, x\rangle + \alpha.$$

(ii) *For every $x^* \in X^*$ and every $\alpha \in \mathbb{R}$, there exist $\xi \in K^\oplus$, $\lambda \in \mathbb{R}_+^T$ and $u, v_t \in X^*$ with $t \in T(\lambda)$ such that*

$$f^*(\xi) + (\xi h)^*(u) + \sum_{t \in T(\lambda)} \lambda_t f_t^*(v_t) + \delta_C^*\left(x^* - u - \sum_{t \in T(\lambda)} \lambda_t v_t\right) \leq -\alpha.$$

Proof. ($\Rightarrow$) Suppose that (DQC) holds. We will prove (i)$\Leftrightarrow$(ii). If (i) holds, then

$$(f \circ h + \delta_A)(x) \geq \langle x^*, x\rangle + \alpha,\ \forall\, x \in X,$$

or equivalently,

$$\langle x^*, x\rangle - (f \circ h + \delta_A)(x) \leq -\alpha,\ \forall\, x \in X.$$

This implies that

$$(f \circ h + \delta_A)^*(x^*) \leq -\alpha. \tag{3.3}$$

Since (DQC) holds, by Theorem 3.10(ii), there exist $\xi \in K^\oplus$, $\lambda \in \mathbb{R}_+^T$ and $u, v_t \in X^*$ with $t \in T(\lambda)$ such that

$$(f \circ h + \delta_A)^*(x^*) = f^*(\xi) + (\xi h)^*(u) + \sum_{t \in T(\lambda)} \lambda_t f_t^*(v_t) + \delta_C^*\left(x^* - u - \sum_{t \in T(\lambda)} \lambda_t v_t\right).$$

This equation, together with (3.3), yields (ii).

Conversely, assume that (ii) holds. Then for any $x^* \in X^*$ and any $\alpha \in \mathbb{R}$, there exist $\xi \in K^{\oplus}$, $\lambda \in \mathbb{R}^T_+$ and $u, v_t \in X^*$ with $t \in T(\lambda)$ such that

$$f^*(\xi) + (\xi h)^*(u) + \sum_{t\in T(\lambda)} \lambda_t f_t^*(v_t) + \delta_C^*\left(x^* - u - \sum_{t\in T(\lambda)} \lambda_t v_t\right) \le -\alpha.$$

By the definition of conjugate function, for any $x \in X$ and $y \in Y$, we have

$$\begin{aligned}
-\alpha \ge & f^*(\xi) + (\xi h)^*(u) + \sum_{t\in T(\lambda)} \lambda_t f_t^*(v_t) + \delta_C^*\left(x^* - u - \sum_{t\in T(\lambda)} \lambda_t v_t\right) \\
\ge & \langle \xi, y\rangle - f(y) + \langle u, x\rangle - (\xi h)(x) + \sum_{t\in T(\lambda)} \lambda_t \left(\langle v_t, x\rangle - f_t(x)\right) \\
& + \langle x^* - u - \sum_{t\in T(\lambda)} \lambda_t v_t, x\rangle - \delta_C(x) \\
= & \langle \xi, y\rangle - f(y) - (\xi h)(x) - \sum_{t\in T(\lambda)} \lambda_t f_t(x) + \langle x^*, x\rangle - \delta_C(x).
\end{aligned}$$

Taking $y = h(x)$ in above inequality, we have

$$f(h(x)) + \sum_{t\in T(\lambda)} \lambda_t f_t(x) + \delta_C(x) \ge \langle x^*, x\rangle + \alpha,\ \forall\, x \in X. \tag{3.4}$$

From (3.4), one gets

$$f(h(x)) \ge \langle x^*, x\rangle + \alpha,\ \forall\, x \in A.$$

($\Leftarrow$) Assume that (i)$\Leftrightarrow$(ii) holds. We now prove that (DQC) holds. As $M \subseteq \operatorname{epi}(F+G+H)^*$, we only need to prove that $\operatorname{epi}(F+G+H)^* \cap (X^* \times \{0\} \times \mathbb{R}) \subseteq M \cap (X^* \times \{0\} \times \mathbb{R})$. Let $(x^*, 0, r) \in \operatorname{epi}(F+G+H)^* \cap (X^* \times \{0\} \times \mathbb{R})$. By Proposition 3.1(ii), $(x^*, r) \in \operatorname{epi}(f \circ h + \delta_A)^*$, i.e., $(f \circ h + \delta_A)^*(x^*) \le r$. It follows that

$$\langle x^*, x\rangle - (f \circ h + \delta_A)(x) \le r,\ \forall\, x \in X.$$

This implies

$$f(h(x)) \ge \langle x^*, x\rangle - r,\ \forall\, x \in A.$$

Thus, there exist $\xi \in K^{\oplus}$, $\lambda \in \mathbb{R}^T_+$ and $u, v_t \in X^*$ with $t \in T(\lambda)$ such that

$$f^*(\xi) + (\xi h)^*(u) + \sum_{t\in T(\lambda)} \lambda_t f_t^*(v_t) + \delta_C^*\left(x^* - u - \sum_{t\in T(\lambda)} \lambda_t v_t\right) \le r.$$

That is

$$\delta_C^*\left(x^* - u - \sum_{t\in T(\lambda)} \lambda_t v_t\right) \le r - f^*(\xi) - (\xi h)^*(u) - \sum_{t\in T(\lambda)} \lambda_t f_t^*(v_t),$$

or equivalently,

$$\left(x^* - u - \sum_{t\in T(\lambda)} \lambda_t v_t, r - f^*(\xi) - (\xi h)^*(u) - \sum_{t\in T(\lambda)} \lambda_t f_t^*(v_t)\right) \in \operatorname{epi}\delta_C^*.$$

It follows that

$$\begin{aligned}(x^*,0,r) =&(0,\xi,f^*(\xi)) + \sum_{t\in T(\lambda)} \lambda_t(v_t,0,f_t^*(v_t)) + (u,-\xi,(\xi h)^*(u))\\ &+ (x^* - u - \sum_{t\in T(\lambda)} \lambda_t v_t, 0, r - f^*(\xi) - (\xi h)^*(u) - \sum_{t\in T(\lambda)} \lambda_t f_t^*(v_t))\\ \in&\{0\}\times \mathrm{epi}(f)^* + \{(p,0,s) : (p,s) \in \mathrm{cone}\left\{\bigcup_{t\in T} \mathrm{epi} f_t^*\right\}\\ &+ \bigcup_{\lambda\in K^\oplus} \{(p,-\lambda,s) : (p,s)\in \mathrm{epi}((\lambda h)^*)\} + \{(p,0,s) : (p,s) \in \mathrm{epi}\delta_C^*\}\\ =&M,\end{aligned}$$

which implies that $\mathrm{epi}(F+G+H)^* \cap (X^*\times\{0\}\times\mathbb{R}) \subseteq M \cap (X^*\times\{0\}\times\mathbb{R})$. The proof is complete. □

Next, we obtain a non-asymptotic stable Farkas lemma of the problem (P). Non-asymptotic stable Farkas lemmas for convex and/or lower semicontinuous functions have been investigated in [11, 12, 16, 22–25].

Theorem 3.13. *If (DQC) holds, then for every $x^* \in X^*$ and every $\alpha \in \mathbb{R}$ the following statements are equivalent:*

(i) $x \in C,\ f_t(x) \le 0,\ t \in T \Rightarrow f(h(x)) \ge \langle x^*, x\rangle + \alpha.$

(ii) $(x^*, 0, -\alpha) \in M$.

(iii) *There exists $\lambda \in \mathbb{R}_+^T$ such that*

$$f(h(x)) + \sum_{t\in T} \lambda_t f_t(x) \ge \langle x^*, x\rangle + \alpha,\ \forall\, x \in C.$$

Proof. (i) ⇒ (ii): It is a straightforward consequence of Proposition 3.1(ii) and (DQC).

(ii) ⇒ (iii): Let $(x^*, 0, -\alpha) \in M$. By Proposition 3.1(ii) and (DQC), one has $(x^*, -\alpha) \in \mathrm{epi}(f\circ h + \delta_A)^*$. It follows that $(f\circ h+\delta_A)^*(x^*) \le -\alpha$. By Theorem 3.10(ii), there exist $\xi \in K^\oplus$, $\lambda \in \mathbb{R}_+^T$ and $u, v_t \in X^*$ with $t \in T(\lambda)$ such that

$$f^*(\xi) + (\xi h)^*(u) + \sum_{t\in T(\lambda)} \lambda_t f_t^*(v_t) + \delta_C^*\left(x^* - u - \sum_{t\in T(\lambda)} \lambda_t v_t\right) \le -\alpha. \tag{3.5}$$

By the definition of conjugate functions, for any $x \in X$, we have

$$\begin{aligned}f^*(\xi) + (\xi h)^*(u) + \sum_{t\in T(\lambda)} \lambda_t f_t^*(v_t) + \delta_C^*\left(x^* - u - \sum_{t\in T(\lambda)} \lambda_t v_t\right) &\ge \langle x^*, x\rangle - f(h(x))\\ &- \sum_{t\in T(\lambda)} \lambda_t f_t(x) - \delta_C(x).\end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we get

$$f(h(x)) + \sum_{t\in T(\lambda)} \lambda_t f_t(x) + \delta_C(x) \ge \langle x^*, x\rangle + \alpha.$$

This implies that (iii) holds.

Since the implication (iii) $\Rightarrow$ (i) is obvious, the proof is complete. $\square$

Example 3.14. Let X, Y, K, T, C, f, h and f_t be as in Example 3.11. Then (DQC) holds. For every $x^* \in \mathbb{R}$ and every $\alpha \in \mathbb{R}$,

$$\begin{aligned}
&[x \in C,\ f_t(x) \le 0,\ t \in T \Rightarrow f(h(x)) \ge \langle x^*, x\rangle + \alpha] \\
&\Rightarrow x^* \le 0 \text{ and } \alpha \le 0 \\
&\Rightarrow (x^*, 0, -\alpha) \in M \\
&\Rightarrow \text{there exists } \lambda \in \mathbb{R}^T_+ \text{ such that } f(h(x)) + \sum_{t\in T} \lambda_t f_t(x) \ge \langle x^*, x\rangle + \alpha,\ \forall\, x \in C.
\end{aligned}$$

Therefore, Theorem 3.13 holds.

$\boxed{3.3}$ A Special Case

In this subsection, we consider a particular case of the composite semi-infinite programming problem (P) with $Y = X$, $h(x) = x$ for any $x \in X$ and $K = \{0\}$ when (P) reduces to the following semi-infinite programming problem:

$$\begin{aligned}
(\mathrm{P}_0)\quad &\text{Minimize} \quad f(x), \\
&\text{subject to} \quad f_t(x) \le 0,\ t \in T, \\
&\qquad\qquad\quad\ x \in C.
\end{aligned}$$

Semi-infinite programming problem (P_0) has been considered recently in several papers with various requirements of f, f_t, $t \in T$, and spaces due to its extensive applications in many fields such as reverse Chebyshev approximation, robust optimization, minimax problems, design centering and disjunctive programming; see, e.g., [20, 37, 38]. A large number of results have appeared in the literature; see, e.g., [11, 12, 16, 18, 29, 30, 32] and the references therein.

Since $Y = X$, $h(x) = x$ for any $x \in X$ and $K = \{0\}$, the set M becomes

$$M' := \{0\} \times \mathrm{epi} f^* + \{(p, 0, r) : (p, r) \in K_1\} + \bigcup_{\lambda\in X^*} \{(\lambda, -\lambda, r) : r \ge 0\}.$$

Definition 3.15. We say that the problem (P_0) satisfies the data qualification condition $(\mathrm{DQC})_0$ if

$$\mathrm{epi}(F + G + H)^* \cap (X^* \times \{0\} \times \mathbb{R}) = M' \cap (X^* \times \{0\} \times \mathbb{R}),$$

where the functions F, G, H are defined in Corollary 3.2.

Corollary 3.16. *For the problem (P_0), $(DQC)_0$ holds if and only if the following statements are equivalent:*

(i) *For every $x^* \in X^*$ and every $\alpha \in \mathbb{R}$,*

$$x \in C,\ f_t(x) \le 0,\ t \in T \Rightarrow f(x) \ge \langle x^*, x\rangle + \alpha.$$

(ii) *For every $x^* \in X^*$ and every $\alpha \in \mathbb{R}$, there exist $\lambda \in \mathbb{R}^T_+$ and $u, v_t \in X^*$ with $t \in T(\lambda)$ such that*

$$f^*(u) + \sum_{t\in T(\lambda)} \lambda_t f_t^*(v_t) + \delta_C^*\Big(x^* - u - \sum_{t\in T(\lambda)} \lambda_t v_t\Big) \le -\alpha.$$

Remark 3.17. In [39, Theorem 4.6], Sun obtained a stable Farkas lemma in dual form for the problem (P_0). He assumes that f, f_t are convex and C is a convex set in [39]. However, Corollary 3.16 does not require these assumptions. The following example shows that there are situations in which Corollary 3.16 can be applied while Theorem 4.6 of [39] does not apply.

Example 3.18. Let $X = \mathbb{R}$, $T = \{0, 1, 2, 3, \cdots\}$, $K = \{0\}$ and $C = \mathbb{Q}$. Define $f_t, f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ respectively by $f_t(x) = \delta_{\mathbb{Q}}(tx)$, $t \in T$ and

$$f(x) = \begin{cases} 0, & \text{if } x > 0; \\ 1, & \text{if } x \le 0. \end{cases}$$

By a simple computation, $X^* = \mathbb{R}$, $K^{\oplus} = \mathbb{R}$, $A = \mathbb{Q}$, $f^* = \delta_{\{0\}}$, $f_t^* = \delta_{\{0\}}$ and $\delta_C^* = \delta_{\{0\}}$. It follows that

$$\begin{aligned} M' &= \{0\} \times \mathrm{epi} f^* + \{(p, 0, r) : (p, r) \in K_1\} + \bigcup_{\lambda \in X^*} \{(\lambda, -\lambda, r) : r \ge 0\} \\ &= \{0\} \times \{0\} \times [0, +\infty) + \{0\} \times \{0\} \times [0, +\infty) + \bigcup_{\lambda \in X^*} \{(\lambda, -\lambda, r) : r \ge 0\} \\ &= \bigcup_{\lambda \in X^*} \{(\lambda, -\lambda, r) : r \ge 0\}. \end{aligned}$$

Note that

$$(F + G + H)^*(x^*, y^*) = \begin{cases} 0, & \text{if } x^* + y^* = 0; \\ +\infty, & \text{otherwise.} \end{cases}$$

Then

$$\mathrm{epi}(F + G + H)^* = \bigcup_{\lambda \in X^*} \{(\lambda, -\lambda, r) : r \ge 0\}.$$

This implies that $(\mathrm{DQC})_0$ holds. It is easy to see that Corollary 3.16 holds. However, Theorem 4.6 in [39] is not applicable since C is not a convex set and f_t is not convex for any $t \in T$.

Corollary 3.19. *For the problem (P_0), if (DQC)$_0$ holds, then for every $x^* \in X^*$ and $\alpha \in \mathbb{R}$ the following statements are equivalent:*

(i) $x \in C,\ f_t(x) \le 0,\ t \in T \Rightarrow f(x) \ge \langle x^*, x\rangle + \alpha.$

(ii) $(x^*, 0, -\alpha) \in M'$.

(iii) *There exists $\lambda \in \mathbb{R}_+^T$ such that*

$$f(x) + \sum_{t \in T(\lambda)} \lambda_t f_t(x) \ge \langle x^*, x\rangle + \alpha,\ \forall\, x \in C.$$

Remark 3.20. In [16, Corollary 4.6], Fang et al. obtained a non-asymptotic stable Farkas lemma for the problem (P_0) under the assumptions that f, f_t are convex functions and C is a convex set. However, Corollary 3.19 removes these assumptions. Example 3.4 shows that Corollary 3.19 can be applied in situations where Corollary 4.6 of [16] does not apply. We also point out that Corollary 3.19 improves Theorem 2 of Dinh et al. [12].

$\boxed{4}$ Stable Duality

In this section, using results obtained in Section 3, we derive some stable duality results for composite semi-infinite programming problems, in which the functions involved need not be convex and lower semicontinuous.

First, we obtain a stable Fenchel-Lagrange duality for the problem (P).

Theorem 4.1 (Stable Fenchel-Lagrange Duality)**.** *The following statements are equivalent:*

(i) *(DQC) holds.*

(ii) *For any $x^* \in X^*$, we have*

$$\begin{aligned}&\inf_{x\in A}\{f(h(x)) - \langle x^*, x\rangle\}\\ &= \max_{\substack{\lambda\in\mathbb{R}_+^{(T)},\xi\in K^{\oplus}\\ u,v_t\in X^*,t\in T(\lambda)}} \left[-f^*(\xi) - (\xi h)^*(u) - \sum_{t\in T(\lambda)} \lambda_t f_t^*(v_t) - \delta_C^*\left(x^* - u - \sum_{t\in T(\lambda)} \lambda_t v_t\right)\right].\end{aligned}$$

Proof. From Theorem 3.10(ii), we only need to show that

$$\inf_{x\in A}\{f(h(x)) - \langle x^*, x\rangle\} = -(f\circ h + \delta_A)^*(x^*).$$

In fact, by the definition of conjugate function,

$$\begin{aligned}\inf_{x\in A}\{f(h(x)) - \langle x^*, x\rangle\} &= \inf_{x\in X}\{f(h(x)) + \delta_A(x) - \langle x^*, x\rangle\}\\ &= -\sup_{x\in X}\{\langle x^*, x\rangle - (f(h(x)) + \delta_A(x))\}\\ &= -(f\circ h + \delta_A)^*(x^*).\end{aligned}$$

The proof is complete. □

If $Y = X$, $h(x) = x$ for any $x \in X$ and $K = \{0\}$, then we have the following corollary.

Corollary 4.2. *For the problem (P_0), (DQC)$_0$ holds if and only if for any $x^* \in X^*$,*

$$\inf_{x\in A}\{f(x) - \langle x^*, x\rangle\} = \max_{\substack{\lambda\in\mathbb{R}_+^{(T)},u\in X^*\\ v_t\in X^*,t\in T(\lambda)}} \left[-f^*(u) - \sum_{t\in T(\lambda)} \lambda_t f_t^*(v_t) - \delta_C^*\left(x^* - u - \sum_{t\in T(\lambda)} \lambda_t v_t\right)\right].$$

Remark 4.3. Corollary 4.2 extends Theorem 4.5 of Sun [39] to a non-convex case.

Next, we derive a stable Lagrange duality for the problem (P).

Theorem 4.4 (Stable Lagrange Duality)**.** *If (DQC) holds, then for any $x^* \in X^*$, we have*

$$\inf_{x\in A}\{f(h(x)) - \langle x^*, x\rangle\} = \max_{\lambda\in\mathbb{R}_+^{(T)}} \inf_{x\in C}\left\{f(h(x)) + \sum_{t\in T}\lambda_t f_t(x) - \langle x^*, x\rangle\right\}.$$

Proof. Assume that (DQC) holds. Let $\alpha := \inf_{x\in A}\{f(h(x)) - \langle x^*, x\rangle\}$. Then $f(h(x)) - \langle x^*, x\rangle \geq \alpha$, for all $x \in A$. By Theorem 3.13, there exists $\lambda \in \mathbb{R}_+^T$ such that

$$(\forall\, x \in C) \quad f(h(x)) + \sum_{t\in T} \lambda_t f_t(x) - \langle x^*, x\rangle \geq \alpha,$$

which implies that

$$\inf_{x\in A}\{f(h(x)) - \langle x^*, x\rangle\} = \alpha \leq \max_{\lambda\in\mathbb{R}_+^{(T)}} \inf_{x\in C}\{f(h(x)) + \sum_{t\in T} \lambda_t f_t(x) - \langle x^*, x\rangle\}. \tag{4.1}$$

On the other hand, for every $x \in A$ and $\lambda \in \mathbb{R}_+^T$, we have

$$f(h(x)) - \langle x^*, x\rangle \geq f(h(x)) + \sum_{t\in T} \lambda_t f_t(x) - \langle x^*, x\rangle.$$

It follows that

$$\begin{aligned}\inf_{x\in A}\{f(h(x)) - \langle x^*, x\rangle\} &\geq \inf_{x\in A}\{f(h(x)) + \sum_{t\in T} \lambda_t f_t(x) - \langle x^*, x\rangle\} \\ &\geq \inf_{x\in C}\{f(h(x)) + \sum_{t\in T} \lambda_t f_t(x) - \langle x^*, x\rangle\},\end{aligned}$$

from which

$$\inf_{x\in A}\{f(h(x)) - \langle x^*, x\rangle\} \geq \max_{\lambda\in\mathbb{R}_+^{(T)}} \inf_{x\in C}\{f(h(x)) + \sum_{t\in T} \lambda_t f_t(x) - \langle x^*, x\rangle\}. \tag{4.2}$$

Combining (4.1) and (4.2) we get the conclusion. □

Corollary 4.5. *For the problem (P_0), if ($DQC)_0$ holds, then for every $x^* \in X^*$ we have*

$$\inf_{x\in A}\{f(x) - \langle x^*, x\rangle\} = \max_{\lambda\in\mathbb{R}_+^{(T)}} \inf_{x\in C} \left\{ f(x) + \sum_{t\in T} \lambda_t f_t(x) - \langle x^*, x\rangle \right\}.$$

Remark 4.6. Corollary 4.5 extends Theorem 5.2 of Fang et al. [16] to a non-convex case. Example 3.18 shows that Corollary 4.5 may apply in situations where Theorem 5.2 of Fang et al. [16] cannot be applied.

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