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THE STRONG STABILITY OF A NONLINEAR TIME-DELAY SYSTEM IN BATCH PROCESS*

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Abstract: In this paper, a nonlinear time-delay system in batch culture of glycerol bioconversion to 1,3-propanediol induced by *Klebsiella pneumonia* and its some important properties are investigated. For this system, the corresponding linear variational system is then discussed. On this basis, we prove strong stability with respect to perturbance of initial condition for the nonlinear time-delay system.

Key words: *nonlinear time-delay system, linear variational system, fundamental matrix solution, strong stability, batch fermentation*

Mathematics Subject Classification: *49J15, 49M37, 65K10*

1 Introduction

1,3-Propanediol (1,3-PD) is an important raw material widely used in pharmaceutical, chemical, food and cosmetic industries [1]. Production methods for 1,3-PD can be divided into two categories: chemical synthesis and microbial fermentation. Compared with chemical synthesis, 1,3-PD microbial production is particularly attractive since the process does not generate toxic byproducts and uses renewable feedstock such as glycerol, a byproduct of biodiesel production [15].

Glycerol can be converted to 1,3-PD via one of three microbial fermentation modes: batch mode, continuous mode and fed-batch mode. In batch mode, the bacteria and substrate are added to the reactor at the beginning of the process, and nothing is added to and removed from the reactor during the culture process. Previous research indicates that glycerol fermentation in batch culture can obtain the highest production concentration and molar yield 1,3-PD to glycerol [4].

1,3-PD batch production is a complex bioprocess subject to multiple inhibitions of substrate and products, and containing time-delays [22,28]. Thus, precise mathematical models

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are required for this process. A kinetic model describing substrate consumption and production formation is proposed in [28]. This model is further modified to describe the excessive influence behavior in the process of glycerol conversion in [23]. An enzyme-catalytic kinetic model is proposed in [16]. Based on these mathematical model, parameter identification and optimal control problems are discussed in [2, 7, 18, 20, 21, 24, 25, 27]. Recently, stability of these nonlinear dynamical systems are widely discussed. Stability of an impulsive system is considered in [29]. ϕ_0-stability of an impulsive system is investigated in [30]. The strong stability of a nonlinear multistage system is discussed in [31]. However, time-delays are ignored in the above nonlinear dynamic systems. In fact, like most real systems, batch bioreactors are also influenced by time-delays. As a result, a nonlinear time-delay system is proposed to formulate the batch process in [11]. For this system, parameter identification and optimal control problems are discussed in [3, 8–10, 12–14, 17, 26]. However, stability analysis of the nonlinear time-delay system has not been reported in the literature.

In this paper, we consider the strong stability of nonlinear time-delay system arising in 1,3-PD batch fermentation. We first discuss the nonlinear time-delay system and its some important properties. Then, the corresponding linear variational system is presented. On this basis, the strong stability of the nonlinear time-delay system is proved.

The rest of this paper is organized as follows. Section 2 gives the nonlinear time-delay system. Section 3 provides the linear variational system. Strong stability of the nonlinear time-delay system is proved in Section 5. Finally, Section 6 provides the main conclusions.

2 Nonlinear Time-Delay System

- I_n denotes the set $\{1, 2, \cdots, n\}$.
- R denotes the set of real numbers.
- R_+ denotes the set of nonnegative real numbers.
- $x(t)$, $x_\tau(t) = x(t-\tau) \in R_+^5$ denote the state and delayed state vectors.
- $\tau > 0$ denotes a given state-delay.
- $x_0 \in R_+^5$ denotes the initial state trajectory vector.
- t_0 denotes the starting moment of the batch culture.
- t_f denotes the terminal moment of the batch culture.
- $D_0 := [t_0, t_f]$.
- $B^1([-\tau, t_f], R_+^5) = \{f : [-\tau, t_f] \to R_+^5 | f$ is bounded and continuously differ- ential.$\}$
- $\varphi \in B^1([-\tau, 0], R_+^5)$ denotes the history function.
- $C(D_0, R^5)$ denotes the set of continuous functions from D_0 to R^5.
- $C^1(D_0, R^5)$ denotes the set of continuously differentiable functions from D_0 to R^5.
- μ_m denotes the maximum specific growth rate.
- k_2 denotes the Monod saturation constant.
- m_2 denotes the maintenance term of substrate consumption under substrate-limited conditions.
- Y_2 denotes the maximum growth yield.

Table 1: The time-delay and kinetic parameters in system (2.1) [11].

τ	μ_m	k_2	m_2	Y_2
0.26	0.994	0.368	3.24	0.0085
m_3	Y_3	m_4	Y_4	m_5
3.679	76	0.491	35.54	7.309
Y_5	n_2	n_3	n_4	n_5
14.78	1	3	3	3

- m_i, $i = 3, 4, 5$, denote the maintenance terms of 1,3-PD, acetate and ethanol under substrate-limited conditions, respectively.
- Y_i, $i = 3, 4, 5$, denote the maximum yields of 1,3-PD, acetate and ethanol, respectively.

Based on the previous work [11], mass balance relationships for biomass, glycerol, 1,3-PD, acetate and ethanol in the batch process can be expressed as the following nonlinear time-delay system:

$$\begin{cases} \dot{x}(t) &= h(x(t), x_\tau(t)),\ t \in D_0, \\ x(0) &= x_0, \\ x(t) &= \varphi(t),\ t \in [-\tau, 0], \end{cases} \tag{2.1}$$

where

$$\begin{aligned} h(x(t), x_\tau(t)) &= [\mu x_1(t-\tau), q_2 x_1(t), q_3 x_1(t-\tau), q_4 x_1(t-\tau), q_5 x_1(t-\tau)] \\ &= [\mu x_{\tau 1}(t), q_2 x_1(t), q_3 x_{\tau 1}(t), q_4 x_{\tau 1}(t), q_5 x_{\tau 1}(t)]. \end{aligned} \tag{2.2}$$

In (2.2), μ is the specific growth rate of cells. q_2 is the consumption rate of substrate. $q_i, i = 3, 4, 5$ are the specific formation rates of 1,3-PD, acetate and ethanol, respectively. These quantities are defined by:

$$\mu := \mu_m \Big(\frac{x_2(t)}{x_2(t) + k_2}\Big) \prod_{i=2}^{5} \Big(1 - \frac{x_i(t)}{x_i^*}\Big)^{n_i}, \tag{2.3}$$

$$q_2 := m_2 + \mu / Y_2, \tag{2.4}$$

$$q_i := m_i + \mu Y_i,\ i = 3, 4, 5. \tag{2.5}$$

Under anaerobic conditions at 37°C and pH 7.0, the time-delay and kinetic parameters in system (2.1) are listed in Table 1.

Since each component of the state trajectory vector represents a certain substance concentration, the concentrations of biomass, glycerol and products should be restricted in a certain range according to the practical fermentation process. Thus, the admissible set of state trajectory vector is defined by

$$x(t), x_\tau(t) \in W_a = [x_*, x^*] = \prod_{i=1}^{5} [x_{*i}, x_i^*] \subset R_+^5, \tag{2.6}$$

where x_* and x^* are, respectively, the lower and upper bounded of the state trajectory vector, and defined by

$$x_* = [0.01, 150, 0, 0, 0], \quad x^* = [15, 2039, 939.5, 1026, 360.9].$$

According to the experiment process, we define the admissible set of initial state trajectory vectors as

$$W_0 = [0.01, 0.25] \times [150, 520] \times \{0\} \times \{0\} \times \{0\} \subset R^5_+. \tag{2.7}$$

Note that, in the sequel, the norm of vector $x \in R^n$ is $\|x\| := \sum_{i=1}^n |x_i|$; the norm of matrix $A = [a_{ij}]_{n\times n} \in R^{n\times n}$ is $\|A\| := \max_{1\le j\le n} \sum_{i=1}^n |a_{ij}|$; and the norm of vector function $x : D_0 \to R^n$ is $\|x(t)\| := \max_{t\in D_0} \sum_{i=1}^n |x_i(t)|$.

For system (2.1), we give the following important properties. The proofs of Properties 1-3 are similar to that given for Theorems 1 and 2 in [11].

Property 1. The function $h(x(t), x_\tau(t))$ defined in system (2.1) is Lipschitz continuous in W_a. Furthermore, it satisfies the linear growth condition, namely, there exists a constant $L > 0$, such that

$$\|h(x(t), x_\tau(t))\| \le L(\|x(t)\| + \|x_\tau(t)\| + 1),\ \forall x(t), x_\tau(t) \in W_a.$$

Property 2. For each $(x_0, \varphi) \in W_0 \times B^1([-\tau, 0], R^5_+)$, system (2.1) admits a unique solution in $D_0 \subset R_+$. Namely, $x(t) = x(t|t_0, x_0, \varphi)$ satisfies

$$x(t) = x_0 + \int_{t_0}^t h(x(s), x_\tau(s))ds,\ \forall t \in D_0, \tag{2.8}$$

and $x(t) = \varphi(t), \forall t \in [-\tau, 0]$.

The solution set S_0 of system (2.1) for an initial condition $(t_0, x_0, \varphi) \in D_0 \times W_0 \times B^1([-\tau, 0], R^5_+)$ is defined as

$$\begin{aligned} S_0 = \{x(t|t_0, x_0, \varphi) \in C(D_0, R^5) | x(t|t_0, x_0, \varphi) \text{ is the solution of system (2.1)} \\ \text{for } (t_0, x_0, \varphi) \in D_0 \times W_0 \times B^1([-\tau, 0], R^5_+)\}. \end{aligned} \tag{2.9}$$

Property 3. For each $(t_0, x_0, \varphi) \in D_0 \times W_0 \times B^1([-\tau, 0], R^5_+)$, the solution $x(t|t_0, x_0, \varphi)$ of system (2.1) is continuous in $(x_0, \varphi) \in W_0 \times B^1([-\tau, 0], R^5_+)$.

Let

$$S_{0a} = \{x(t|t_0, x_0, \phi) \in S_0 | x(t|t_0, x_0, \varphi) \in W_a\}. \tag{2.10}$$

Then, the sets S_0 and S_{0a} have the following property.

Property 4. Sets S_0 and S_{0a} defined in (2.9) and (2.10) are all compact in $C^1(D_0, R^5_+))$.

Proof. It follows from (2.7) that the set W_0 is nonempty and compact. By Property 3, the mapping $x_0 \in W_0 \mapsto x(t|t_0, x_0, \varphi) \in S_0$ is continuous. Thus, S_0 is a nonempty subset in $C^1(D_0, R^5_+))$. Let $\{x^k(t|t_0, x_0^k, \varphi)\}_{k=1}^\infty$ be any sequence of S_{0a}. Since $S_{0a} \subseteq S_0$, $\{x^k(t|t_0, x_0^k, \varphi)\}_{k=1}^\infty$ is the sequence of compact set S_0. Thus, there exists a convergent subsequence, denoted by $\{x^{k_j}(t|t_0, x_0^{k_j}, \varphi)\}_{k_j=1}^\infty$, satisfying $x^{k_j}(t|t_0, x_0^{k_j}, \varphi) \to \bar{x}(t|t_0, \bar{x}_0, \varphi), x_0^{k_j} \to \bar{x}_0$, as $k_j \to \infty$. In view of $x^{k_j}(t|t_0, x_0^{k_j}, \varphi) \in S_{0a}$, we obtain

$$\begin{cases} \dot{x}^{k_j}(t|t_0, x_0^{k_j}, \varphi) = h(x^{k_j}(t), x_\tau^{k_j}(t)),\ t \in D_0, \\ x^{k_j}(t|t_0, x_0^{k_j}, \varphi) = \varphi(t),\ t \in [-\tau, 0]. \end{cases}$$

Since $x^{k_j}(t|t_0, x_0^{k_j}, \varphi) \in W_a$, $x_{*i} \le x^{k_j}(t|t_0, x_0^{k_j}, \varphi) \le x_i^*, i \in I_5$.

By Properties 1 and 2, $x^{k_j}(t|t_0, x_0^{k_j}, \varphi)$ is continuously differential in t, t_0 and $x_0^{k_j}$. It follows that $\dot{\bar{x}}(t|t_0, \bar{x}_0, \varphi) = h(\bar{x}(t|t_0, \bar{x}_0, \varphi), \bar{x}_\tau(t|t_0, \bar{x}_0, \varphi))$ as $k_j \to \infty$. Furthermore, we obtain that $\bar{x}(t|t_0, \bar{x}_0, \varphi) \in S_{0a}$ according to the definition of S_{0a}. This means that $\{x^k(t|t_0, x_0^k, \varphi)\}_{k=1}^\infty$ is convergent in S_{0a} and its limitation satisfies $\bar{x}(t|t_0, \bar{x}_0, \varphi) \in S_{0a}$. Thus, the set S_{0a} is compact in $C^1(D_0 \times W_0 \times B^1([-\tau, 0], R^5_+))$. □

3 Linear Variational System

In this section, we will construct the corresponding linear variational system of system (2.1) since the partial derivation of function $h(x(t), x_\tau(t))$ is continuous in $x(t)$ and $x_\tau(t)$.

Let $x(t) = x(t|t_0, x_0, \varphi) \in S_{0a}$ be the solution of system (2.1) with the initial condition $(t_0, x_0, \varphi) \in D_0 \times W_0 \times B^1([-\tau, 0], R^5)$. Furthermore, we consider another solution of system (2.1):

$$z(t) + x(t|t_0, x_0, \varphi),\ t \in [-\tau, t_f], \tag{3.1}$$

with the initial condition $(t_0, x_0, \varphi) \in D_0 \times W_0 \times B^1([-\tau, 0], R^5_+)$. It satisfies that

$$\begin{cases} \dot{z}(t) + \dot{x}(t|t_0, x_0, \varphi) = h(z(t) + x(t|t_0, x_0, \varphi), z_\tau(t) + x_\tau(t|t_0, x_0, \varphi)),\ t \in D_0, \\ z(t_0) + x(t_0|t_0, x_0, \varphi) = x_0, \\ z(t) + x(t|t_0, x_0, \varphi) = \varphi(t),\ t \in [-\tau, 0]. \end{cases} \tag{3.2}$$

In (3.2), the differentiation with respect to t is

$$\begin{aligned} \dot{z}(t) + \frac{dx(t|t_0, x_0, \varphi)}{dt} &= \dot{z}(t) + h(x(t|t_0, x_0, \varphi), x_\tau(t|t_0, x_0, \varphi)) \\ &= h(z(t) + x(t|t_0, x_0, \varphi), z_\tau(t) + x_\tau(t|t_0, x_0, \varphi)), t \in D_0, \end{aligned}$$

and we have

$$\begin{aligned} \dot{z}(t) &= h(z(t) + x(t|t_0, x_0, \varphi), z_\tau(t) + x_\tau(t|t_0, x_0, \varphi)) - h(x(t|t_0, x_0, \varphi), x_\tau(t|t_0, x_0, \varphi)) \\ &= h(z(t) + x(t), z_\tau(t) + x_\tau(t)) - h(x(t), x_\tau(t) + z_\tau(t)) + h(x(t), z_\tau(t) + x_\tau(t)) \\ &\quad - h(x(t), x_\tau(t)) \\ &= \frac{\partial h(x(t), x_\tau(t) + z_\tau(t))}{\partial x(t)} z(t) + \frac{\partial h(x(t), x_\tau(t))}{\partial x_\tau(t)} z_\tau(t) + o(\|z(t)\| + \|z_\tau(t)\|). \end{aligned}$$

The above equation becomes

$$\dot{z}(t) = \frac{\partial h(x(t), x_\tau(t))}{\partial x(t)} z(t) + \frac{\partial h(x(t), x_\tau(t))}{\partial x_\tau(t)} z_\tau(t),\ t \in D_0, \tag{3.3}$$

when $\|z(t)\|, \|z_\tau(t)\|$ are sufficiently small and close to zero. System (3.3) is called the linear variational system corresponding to the solution of the system (2.1).

Let $m > 0$ be an integer and satisfy $m\tau \le t_f < (m+1)\tau$. The interval of $D_0 \subset R_+$ can be divided into $m+1$ subinterval, that is, $[0, \tau), [\tau, 2\tau), \ldots, [(m-1)\tau, m\tau), [m\tau, t_f]$. In view of system (2.1), we consider three cases of system (3.3) on the subintervals of $[(j-1)\tau, j\tau) \subset D_0, j \in I_m$ and $[m\tau, t_f]$ as follows:

Case 1. When $j = 1$, i.e. $t \in [0, \tau)$, system (3.3) is

$$\dot{z}(t) = \frac{\partial h(x(t), x_\tau(t))}{\partial x(t)} z(t) + \frac{\partial h(x(t), x_\tau(t))}{\partial x_\tau(t)} z_\tau(t), t \in [0, \tau). \tag{3.4}$$

The function $z_\tau(t) = z(t - \tau) = \varphi(t)$ is given on $t \in [t_0, \tau)$, so the second part of (3.4) is known. Thus, system (3.4) is a non-homogeneous linear system on $z(t)$, and the corresponding homogeneous linear system is

$$\dot{z}(t) = \frac{\partial h(x(t), x_\tau(t))}{\partial x(t)} z(t),\ t \in [0, \tau). \tag{3.5}$$

By Theorem 3.3 in [5], the matrix $\frac{\partial x(t|t_0,x_0,\varphi)}{\partial x_0} \in R^{5\times 5}$ is the fundamental matrix solution of system (3.5) with initial state $\frac{\partial x(t|t_0,x_0,\varphi)}{\partial x_0} = I$, where $I \in R^{5\times 5}$ is an unit matrix.

By Theorem 1.1 in [5], the fundamental matrix solution of system (3.4) on $[0,\tau)$ is

$$\begin{aligned}&\Phi_{1,i}(t|t_0,x_0,\varphi)\\&= \frac{\partial x(0|t_0,x_0,\varphi)}{\partial x_0}e_i + \int_0^t \frac{\partial x(s|t_0,x_0,\varphi)}{\partial x_0} \cdot \frac{\partial h(x(s|t_0,x_0,\varphi),x_\tau(s|t_0,x_0,\varphi))}{\partial x_\tau(t)} z_\tau(s)ds,\\&\qquad t\in[0,\tau),\ t_0=0,\ i\in I_5,\end{aligned} \tag{3.6}$$

where $e_i \in R^5$ is the ith column of unit matrix, and the terminal state of system (3.4) on $[0,\tau)$ is $\Phi_{1,i}(\tau|t_0,x_0,\varphi) \in R^{5\times 5}, i \in I_5$. Thus, $\Phi_1(t|t_0,x_0,\varphi) = [\Phi_{1,1}(t|t_0,x_0,\varphi),\dots,\Phi_{1,5}(t|t_0,x_0,\varphi)] \in R^{5\times 5}$ is the fundamental matrix solution of system (3.4).

Case 2. When $j = 2$, i.e. $t \in [\tau,2\tau)$, $z_\tau(t)$ of system (3.3) on $[\tau,2\tau) \subset D_0$ has been decided by $[t_0,\tau)$. Thus, $z_\tau(t)$ is given. As a result, system (3.3) is non-homogeneous linear system with respect to $z(t)$ on $[\tau,2\tau)$. Using the similar method as for Case 1, the fundamental matrix solution of system (3.3) on $[\tau,2\tau) \subset D_0$ is

$$\begin{aligned}\Phi_{2,i}(t|\tau,x_0,\varphi) &= \frac{\partial x(\tau|t_0,x_0,\varphi)}{\partial x_0}\Phi_{1,i}(\tau|t_0,x_0,\varphi)\\&\quad+\int_\tau^t \frac{\partial x(s|t_0,x_0,\varphi)}{\partial x_0} \cdot \frac{\partial h(x(s|t_0,x_0,\varphi),x_\tau(s|t_0,x_0,\varphi))}{\partial x_\tau(t)} z_\tau(s)ds\\&\qquad t\in[\tau,2\tau), t_0=\tau, i\in I_5.\end{aligned} \tag{3.7}$$

Thus, $\Phi_2(t|\tau,x_0,\varphi) = [\Phi_{2,1}(t|\tau,x_0,\varphi),\dots,\Phi_{2,5}(t|\tau,x_0,\varphi)] \in R^{5\times 5}$ is the fundamental matrix solution of non-homogeneous linear system (3.3) for $t \in [\tau,2\tau)$. At the same time, the fundamental matrix solution at the terminal time 2τ on $[\tau,2\tau)$ is $\Phi_{2,i}(2\tau|\tau,x_0,\varphi), i \in I_5$.

Case 3. When $j \geq 3$ and $j \leq m+1$, i.e. $t \in [(j-1)\tau, j\tau) \subset D_0$, due to the time-delay, the system (3.3) is still a non-homogeneous linear system of $z(t)$ on $[(j-1)\tau, j\tau) \subset D_0$.

Similar to Cases 1 and 2, the fundamental matrix solution of non-homogeneous linear system (3.3) on $[(j-1)\tau, j\tau) \subset D_0$ is

$$\begin{aligned}&\Phi_{j,i}(t|(j-1)\tau,x_0,\varphi)\\&= \frac{\partial x((j-1)\tau|t_0,x_0,\varphi)}{\partial x_0}\Phi_{j-1,i}((j-1)\tau|t_0,x_0,\varphi)\\&+\int_{(j-1)\tau}^t \frac{\partial x(s|t_0,x_0,\varphi)}{\partial x_0} \cdot \frac{\partial h(x(s|t_0,x_0,\varphi),x_\tau(s|t_0,x_0,\varphi))}{\partial x_\tau(t)} z_\tau(s)ds,\\&\quad t\in[(j-1)\tau,j\tau)\subset D_0,\ t_0=(j-1)\tau,\ i\in I_5,\ j\in\{3,4,\dots,m+1\},\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&\Phi_j(t|(j-1)\tau,x_0,\varphi)\\&= [\Phi_{j1}(t|(j-1)\tau,x_0,\varphi),\dots,\Phi_{j5}(t|(j-1)\tau,x_0,\varphi)] \in R^{5\times 5},\ t\in[(j-1)\tau,j\tau).\end{aligned}$$

According to (3.6), (3.7) and (3.8), the fundamental matrix solution of system (3.3) on D_0 is

$$\Phi_i(t|t_0,x_0,\varphi) = \sum_{j=1}^{m+1} \chi_{[(j-1)\tau,j\tau)}(t)\cdot\Phi_{j,i}(t|(j-1)\tau,x_0,\varphi) \in R^{5\times 5}, i\in I_5,\ t\in D_0,$$

where

$$\chi_{[(j-1)\tau, j\tau)}(t) = \begin{cases} 1, & t \in [(j-1)\tau, j\tau), \\ 0, & \text{otherwise}. \end{cases}$$

Let

$$\Phi(t|t_0, x_0, \varphi) = [\Phi_1(t|t_0, x_0, \varphi), \ldots, \Phi_5(t|t_0, x_0, \varphi)] \in R^{5\times 5}, \ t \in D_0. \tag{3.9}$$

Then, $\Phi(t|t_0, x_0, \varphi)$ is the fundamental matrix solution of system (3.3) for $x(t|t_0, x_0, \varphi)$ of system (2.1).

By Theorem 2.6.4 in [6], we give the following lemma.

Lemma 3.1. *Let $x(t|t_0, x_0, \varphi)$ and $x(t|t_0, y_0, \psi)$ be the solutions of system (2.1) with the given initial conditions of (t_0, x_0, φ) and $(t_0, y_0, \psi) \in D_0 \times W_0 \times B^1([-\tau, 0], R^5_+)$. Then,*

$$\begin{aligned} & x(t|t_0, y_0, \psi) - x(t|t_0, x_0, \varphi) \\ & = \int_0^1 \Phi(t|t_0, x_0 + s(y_0 - x_0), \varphi + s(\psi - \varphi))ds \cdot (y_0 - x_0), \ t \in D_0, \end{aligned} \tag{3.10}$$

where $\Phi(t|t_0, x_0 + s(y_0 - x_0), \varphi + s(\psi - \varphi)) \in R^{5\times 5}$ is the fundamental matrix solutions of system (3.3) corresponding to solution $x(t|t_0, x_0 + s(y_0 - x_0), \varphi + s(\psi - \varphi))$ for system (2.1) with the initial condition of $(t_0, x_0 + s(y_0 - x_0), \varphi + s(\psi - \varphi)) \in D_0 \times W_0 \times B^1([-\tau, 0], R^5_+)$.

4 Strong Stability of Nonlinear Time-Delay System

In this section, we will discuss the strong stability of nonlinear time-delay system in batch fermentation of glycerol to 1,3-PD. First, we introduce the definition of the strong stability for a dynamic system.

Definition 4.1. Let $x(t|t_0, x_0, \varphi)$ be the solution of system (2.1) with initial condition $(t_0, x_0, \varphi) \in D_0 \times W_0 \times B^1([-\tau, 0], R^5_+)$. For any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for any $(y_0, \psi) \in W_0 \times B^1([-\tau, 0], R^5_+)$, the following inequality holds:

$$\|x(t|t_0, x_0, \varphi) - x(t|t_0, y_0, \psi)\| < \varepsilon, \quad \forall \ \|\varphi - \psi\| < \delta(\varepsilon), \ \|x_0 - y_0\| < \delta(\varepsilon),$$

where $x(t|t_0, y_0, \psi)$ denotes the solution of system (2.1) with initial condition $(t_0, y_0, \varphi) \in D_0 \times W_0 \times B^1([-\tau, 0], R^5_+)$. Then, the solution $x(t|t_0, x_0, \varphi)$ of system (2.1) is said to be strongly stable.

Recall that the function $h(x(t), x_\tau(t))$ in system (2.1) is continuously differentiable in $x(t), x_\tau(t) \in W_a$. According to the comparison principle in [6], we obtain the following theorem.

Theorem 4.2. *Let $x(t|t_0, x_0, \varphi)$ be the solution of system (2.1) with the initial condition $(t_0, x_0, \varphi) \in D_0 \times W_0 \times B^1([-\tau, 0], R^5_+)$ and $\Phi(t|t_0, x_0, \varphi) \in R^{5\times 5}$ be the fundamental matrix solutions of system (3.3). Then, $\Phi(t|t_0, x_0, \varphi)$ is bounded in D_0.*

Proof. Let $\Phi(t|t_0, x_0, \varphi) = [\Phi_1(t|t_0, x_0, \varphi), \ldots, \Phi_5(t|t_0, x_0, \varphi)] \in R^{5\times 5}$ be the fundamental matrix solution of system (3.3). Thus, for each $t \in D_0, (t_0 = 0)$

$$\begin{aligned} & \dot{\Phi}_i(t|t_0, x_0, \varphi) = \frac{\partial h(x(t), x_\tau(t))}{\partial x(t)} \Phi_i(t|t_0, x_0, \varphi) + \frac{\partial h(x(t), x_\tau(t))}{\partial x_\tau(t)} \Phi_{\tau i}(t|t_0, x_0, \varphi), \\ & \Phi_i(t_0|t_0, x_0, \varphi) = e_i, \end{aligned} \tag{4.1}$$

$$\Phi_i(t|t_0,x_0,\varphi) = \phi(t),\ t\in[-\tau,t_0],\ \phi(0)=e_i,\ i\in I_5,$$

where $e_i \in R^5$ is the ith column of the identity matrix $I\in R^{5\times 5}$; $\phi(t)$ is a given function. Since $h(x(t),x_\tau(t))$ is continuously differentiable in $x(t),x_\tau(t)\in W_a$ and $W_a\subset R^5$ is a compact set, $\frac{\partial h(x(t),x_\tau(t))}{\partial x(t)}$ and $\frac{\partial h(x(t),x_\tau(t))}{\partial x_\tau(t)}$ are bounded in W_a. Namely, there exists a constant $M>0$ such that for all $t\in D_0$,

$$\left\|\frac{\partial h(x(t),x_\tau(t))}{\partial x(t)})\right\| \le M \text{ and } \left\|\frac{\partial h(x(t),x_\tau(t))}{\partial x_\tau(t)}\right\| \le M.$$

Let

$$\begin{aligned} u_i(t) = \arg\min\big\{&\|v_i(t)\|\ |v_i(t)\in C^1(D_0,R_+),\\ &\max_{1\le j\le 5}\|\Phi_{ij}(t|t_0,x_0,\varphi)\|\le v_i(t),\forall t\in D_0,\\ &\max_{1\le j\le 5}\|\Phi_{ij}(t-\tau|t_0,x_0,\varphi)\|\le v_{i\tau}(t):=v_i(t-\tau),\forall t\in D_0\big\},\\ W_i(t,u_i(t)) = 5M&\cdot u_i(t)+5M\cdot u_{i\tau}(t),\ i\in I_5. \end{aligned}$$

Obviously, $W_i(t,u_i(t))$ is continuous in $D_0\times R_+$. Thus, there exists a unique solution $u_i(t)\ge 1, t\in D_0$, to the following system

$$\begin{cases} \dot{u}_i(t) = W_i(t,u_i(t)), & t\in D_0,\\ u_i(0)=1. \end{cases} \tag{4.2}$$

For the right-hand side of system (4.1), we obtain that due to $\|\Phi_i(t|t_0,x_0,\varphi)\| = \|e_i\| = 1$ and $\|u_i(0)\|=1$,

$$\begin{aligned} &\left\|\frac{\partial h(x(t),x_\tau(t))}{\partial x(t)}\Phi_i(t|t_0,x_0,\varphi)+\frac{\partial h(x(t),x_\tau(t))}{\partial x_\tau(t)}\Phi_{\tau i}(t|t_0,x_0,\varphi)\right\|\\ &\le \left\|\frac{\partial h(x(t),x_\tau(t))}{\partial x(t)}\right\|\cdot\|\Phi_i(t|t_0,x_0,\varphi)\| + \left\|\frac{\partial h(x(t),x_\tau(t))}{\partial x_\tau(t)}\right\|\cdot\|\Phi_{\tau i}(t|t_0,x_0,\varphi)\|\\ &\le 5M\cdot u_i(t)+5M\cdot u_{i\tau}(t) = W_i(t,u_i(t)),\ t\in D_0,\ i\in I_5. \end{aligned}$$

Comparing system (4.1) and (4.2) and by Theorem 6.1 and Corollary 6.3 in [6], we have

$$\|\Phi_i(t|t_0,x_0,\varphi)\| \le u_i(t) \le \max_{t\in D_0} u_i(t),\ \forall t\in D_0. \tag{4.3}$$

Since $u_i(t)\in C^1(D_0,R_+)$, $u_i(t)$ is bounded in D_0. Name, there exists a $m_i>0$, such that $\|u_i(t)\|\le m_i<\infty, i\in I_5$. Furthermore, by (4.3), we have

$$\begin{aligned} \|\Phi_i(t|t_0,x_0,\varphi)\| &\le u_i(t)\le m_i, i\in I_5,\\ \|\Phi(t|t_0,x_0,\varphi)\| = \max_{s\in D_0}\|\Phi(s|t_0,x_0,\varphi)\| &= \max_{s\in D_0}\sum_{i=1}^{5}|\Phi_i(s|t_0,x_0,\varphi)|\\ &\le \sum_{i=1}^{5} m_i = M_0, \forall t\in D_0, \end{aligned}$$

which completes the proof. □

Based on Theorem 4.2, we can prove the following theorem.

Theorem 4.3. *Let $x(t|t_0, x_0, \varphi)$ be the solution of system (2.1) with the initial condition $(t_0, x_0, \varphi) \in D_0 \times W_0 \times B^1([-\tau, 0], R_+^5)$. Then, the solution $x(t|t_0, x_0, \varphi)$ is strongly stable.*

Proof. Let $x(t|t_0, x_0, \varphi)$ and $x(t|t_0, y_0, \psi)$ be two solutions of system (2.1) with initial conditions $(t_0, x_0, \varphi), (t_0, y_0, \psi) \in D_0 \times W_0 \times B^1([-\tau, 0], R_+^5)$, respectively. For any $\varepsilon > 0$, let $\delta(\varepsilon) = \frac{\varepsilon}{M_0}$. Then, suppose that $(y_0, \psi) \in W_0 \times B^1([-\tau, 0], R_+^5)$ satisfies $\|x_0 - y_0\| < \delta(\varepsilon)$ and $\|\varphi - \psi\| < \delta(\varepsilon)$. By Lemma 1, we have

$$x(t|t_0, y_0, \psi) - x(t|t_0, x_0, \varphi) = \int_0^1 \Phi(t|t_0, x_0 + s(y_0 - x_0), \varphi + s(\psi - \varphi)) \cdot (y_0 - x_0) ds.$$

Hence,

$$\begin{aligned}
&\|x(t|t_0, y_0, \psi) - x(t|t_0, x_0, \varphi)\| \\
&\leq \|(y_0 - x_0)\| \int_0^1 \|\Phi(t|t_0, x_0 + s(y_0 - x_0), \varphi + s(\psi - \varphi))\| ds \\
&\leq M_0 \cdot \varepsilon / M_0 = \varepsilon.
\end{aligned}$$

By Definition 4.1, we obtain that $x(t|t_0, x_0, \varphi)$ is strongly stable. □

5 Conclusions

This paper has studied the strong stability of the solution for a nonlinear time-delay system arising in 1,3-PD batch fermentation. We first discuss the nonlinear time-delay system and its some properties. Then, the fundamental matrix solution corresponding linear variational system of the nonlinear time-delay system is discussed. Finally, we prove the strong stability of the nonlinear time-delay system. In the future, our effort will focus on robust optimal control of nonlinear time-delay systems.

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