Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 15, No. 1, pp. 145 – 155, 2019

ON THE CONVEXITY AND EXISTENCE OF SOLUTIONS TO QUADRATIC PROGRAMMING PROBLEMS WITH CONVEX CONSTRAINT*

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Abstract: Sequential quadratic programming (SQP) algorithm is a very effective algorithm for solving constrained nonlinear programming. The search direction at each iteration of an SQP algorithm is usually generated by solving one or more quadratic programming (QP) problems with closed convex constraint. Therefore, an significant problem is discussing the convexity and the existence of solutions to QP. In this paper, we present several conditions for the (strict) convexity and existence of solutions to a class of QP problems.

Key words: *convex constraint, quadratic programming, convexity, existence of solutions*

Mathematics Subject Classification: *90C20, 90C25*

1 Introduction

Since the effective numerical performance, the research on the SQP method is very active. The important achievements on SQP methods are abundant, see Refs. [3, 5, 6, 8–15], but far more than these. As we all know, at the each iteration of an SQP method, the core technique and computation are to solve one or more appropriate QP subproblems with a closed convex constraint.

Therefore, for SQP algorithms, the existence of solutions to QP is a basic and critical problem. Gould [7] discussed and established the existence and uniqueness of the solutions to QP with *equality constraints*. Cambini and Sodini [4] discussed a kind of particular QP with a *polyhedron set*, where the Hessian matrix of the objective function has no more than one nonpositive eigenvalue. Based on the Hessian matrix and gradient of objective function as well as the recession cone of feasible set, Cambini and Sodini [4, Theorem 2.2] established a sufficient and necessary condition for the existence of the optimal solution to the discussed QP above. The key to the proof of [4, Theorem 2.2] is that the constraint set is a polyhedron. [1, Theorem 3.4.1] established a sufficient and necessary condition for the existence of the optimal solutions to the general optimization problems based on the recession function in R^n. Obviously, this result can be applied to QP, but the condition is

*This work was supported by the Natural Science Foundation of China (Nos.11771383,11601095), and the Natural Science Foundation of Guangxi Province (Nos. 2016GXNSFBA380185, 2016GXNSFDA380019 and 2014GXNSFFA118001) as well as Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing scientific research project (No.2016CSOBDP0204).
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not easy to be verified. This paper aims to provide several sufficient conditions which can be checked easily for the existence of solutions to QP with a closed convex constraint.

For these reasons, the aim of this work is to discuss the convexity and existence of solutions to QP with a closed convex constraint. In Section 2, we present several sufficient conditions which ensure the (strict) convexity of a QP. In Section 3, we present one necessary condition and three sufficient conditions for the existence of solutions to QP with a closed convex constraint.

2 The Convexity of QP Problems

In this work, we discuss the convexity and solution existence of the following QP problem:

$$\mathrm{QP}(H,S) \qquad \min\ q(x) := c^{\mathrm{T}}x + \tfrac{1}{2}x^{\mathrm{T}}Hx, \quad \text{s.t. } x \in S,$$

where $c \in R^n$, H is an n-order real symmetric matrix, and S is a nonempty convex set in R^n.

Firstly, to obtain suitable conditions for the convexity of the objective function $q(x)$ on the convex feasible set S, referring to the analysis of [2, Section 3.2], we have the following theorem. Since the set $\mathcal{F}$ (see Theorem 2.1) is not necessarily open, the following theorem is different from the textbook material, where requires $\mathcal{F}$ to be an open set.

Theorem 2.1. *Let $\mathcal{F}^+$ be a nonempty open set in R^n, and function $f : \mathcal{F}^+ \to \mathcal{R}$ be differentiable on $\mathcal{F}^+$. Assume that $\mathcal{F}$ is a nonempty convex subset (not necessarily open) of $\mathcal{F}^+$. Then the following two conclusions hold true.*

(i) *The function $f(x)$ is convex on the convex set $\mathcal{F}$ if and only if*

$$f(y) \geq f(x) + \nabla f(x)^T(y-x), \forall\ x, y \in \mathcal{F}. \tag{2.1}$$

(ii) *The function $f(x)$ is strictly convex on the convex set $\mathcal{F}$ if and only if*

$$f(y) > f(x) + \nabla f(x)^T(y-x), \forall\ x, y \in \mathcal{F},\ x \neq y. \tag{2.2}$$

Proof (ia) [Necessity of claim (i)] Let $f(x)$ be convex on $\mathcal{F}$. Then, for two given points $x,\ y \in \mathcal{F}$ and any $\lambda \in (0,1)$, according to the convexity and the first-order Taylor expansion of f at x, we obtain

$$\begin{aligned} \lambda f(y) + (1-\lambda)f(x) &\geq f(\lambda y + (1-\lambda)x) = f(x + \lambda(y-x)) \\ &= f(x) + \lambda\nabla f(x)^T(y-x) + \lambda\|y-x\|\alpha(\lambda), \end{aligned}$$

where $\alpha : R \to R$ satisfies $\lim\limits_{\lambda \to 0^+} \alpha(\lambda) = 0$. By rearranging the terms and eliminating the factor λ, the above relation further shows that

$$f(y) - f(x) \geq \nabla f(x)^T(y-x) + \|y-x\|\alpha(\lambda).$$

Letting $\lambda \to 0^+$ in above inequality, it follows that $f(y) \geq f(x) + \nabla f(x)^T(y-x)$, which shows that the formula (2.1) is true.

(ib) [Sufficiency of claim (i)] Suppose that relation (2.1) is satisfied. Let $x_\lambda := \lambda y + (1-\lambda)x \in \mathcal{F}$ for any two points $x,\ y \in \mathcal{F}$ and any $\lambda \in (0,1)$, then

$$f(y) \quad \geq \quad f(x_\lambda) + \nabla f(x_\lambda)^T(y - x_\lambda) = f(x_\lambda) + (1-\lambda)\nabla f(x_\lambda)^T(y-x), \tag{2.3}$$

$$f(x) \geq f(x_\lambda) + \nabla f(x_\lambda)^T (x - x_\lambda) = f(x_\lambda) + \lambda \nabla f(x_\lambda)^T (x - y). \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$f(x_\lambda) = f(\lambda y + (1-\lambda)x) \leq \lambda f(y) + (1-\lambda) f(x).$$

This shows that $f(x)$ is convex on $\mathcal{F}$.

(iia) [Necessity of claim (ii)] Let $f(x)$ be strictly convex on $\mathcal{F}$. Then $f(x)$ is convex and relation (2.1) holds true. We suppose, by contradiction, that there exist two dissimilar points $\bar{x}, \bar{y} \in \mathcal{F}$ such that (2.2) does not hold. Then $f(\bar{y}) = f(\bar{x}) + \nabla f(\bar{x})^T (\bar{y} - \bar{x})$ by (2.1), i.e., $\nabla f(\bar{x})^T (\bar{y} - \bar{x}) = f(\bar{y}) - f(\bar{x})$. Therefore, for each $\lambda \in (0,1)$, using relation (2.1) at points $\lambda \bar{y} + (1-\lambda)\bar{x} \in \mathcal{F}$ and $\bar{x} \in \mathcal{F}$, one has

$$\begin{aligned} f(\lambda \bar{y} + (1-\lambda)\bar{x}) &\geq f(\bar{x}) + \nabla f(\bar{x})^T (\lambda \bar{y} + (1-\lambda)\bar{x} - \bar{x}) \\ &= f(\bar{x}) + \lambda \nabla f(\bar{x})^T (\bar{y} - \bar{x}). \end{aligned}$$

This, together with $\nabla f(\bar{x})^T (\bar{y} - \bar{x}) = f(\bar{y}) - f(\bar{x})$, further gives

$$f(\lambda \bar{y} + (1-\lambda)\bar{x}) \geq f(\bar{x}) + \lambda (f(\bar{y}) - f(\bar{x})) = \lambda f(\bar{y}) + (1-\lambda) f(\bar{x}).$$

This contradicts the strict convexity of f on $\mathcal{F}$.

(iib) [Sufficiency of claim (ii)] The proof is similar to the one of claim (i). □

For the objective function $q(y)$ of QP(H,S), one always has $q(y) = q(x) + \nabla q(x)^T (y - x) + \frac{1}{2}(y-x)^T H (y-x)$ holding true for any $x, y \in R^n$. Now, applying Theorem 2.1 to QP(H,S), we have the following result immediately.

Corollary 2.2. (i) *The objective function $q(x)$ of QP(H,S) is convex on the convex feasible set S if and only if*

$$d^T H d \geq 0, \ \forall \ d \in (S - S),$$

i.e., the matrix H is positive semidefinite on the convex set $(S-S) := \{d = x - y : \ x, \ y \in S\}$.

(ii) *The objective function $q(x)$ of QP(H,S) is strictly convex on the convex feasible set S if and only if*

$$d^T H d > 0, \ \forall \ d \in (S - S), \ d \neq 0,$$

i.e., the matrix H is positive definite on $(S - S)$.

Remark 2.3. If the interior of S is nonempty, i.e., int $S \neq \emptyset$, then $0 \in$ int $(S - S)$. Thus, H is positive definite (positive semidefinite) on $(S - S)$ if and only if it is positive definite (positive semidefinite) on R^n when int $S \neq \emptyset$. Therefore, the objective function $q(x)$ of QP(H,S) is convex (strictly convex) on its feasible set S if and only if H is positive semidefinite (positive definite) on the whole space R^n when int $S \neq \emptyset$.

The following example shows that, in case int $S = \emptyset$, the positive definiteness (positive semidefiniteness) of matrix H on the convex set $(S - S)$ is weaker than on the whole space R^n.

Example 2.4. Let

$$H_1 = \begin{pmatrix} 4 & -1 \\ -1 & -2 \end{pmatrix}, \quad S_1 = \{x \in R^2 : \ x_1 - x_2 = 0, x_1 \geq 0, x_2 \geq 0\}.$$

It is easy to show that $(S_1 - S_1) = \{x = (\lambda, \lambda), \ \lambda \in R\}$. Further, $x^T H_1 x = 0$ holds for any $x \in (S_1 - S_1)$, i.e., H_1 is positive semidefinite on $(S_1 - S_1)$. Obviously, H_1 is not positive semidefinite on R^2.

For the application of Corollary 2.2, we analyze the structure of the set $(S - S)$. Thus, we review the recession cone (see [16]) of the set S as follows

$$0^+S = \{d \in R^n : \ x + \lambda d \in S, \ \forall \ x \in S, \ \forall \ \lambda \geq 0\}.$$

Specially, let $0^+S = \emptyset$ in case $S = \emptyset$. The following result can be shown easily.

Proposition 2.5. *For any closed convex set $S \subseteq R^n$, the recession cone 0^+S is a closed convex cone. Furthermore, if $\sum_{i=1}^{\infty} \lambda_i d_i$ ($\lambda_i \geq 0$, $d_i \in 0^+S$) is convergent, then $\sum_{i=1}^{\infty} \lambda_i d_i \in 0^+S$. In particular, the nonnegative combination of any finite terms in 0^+S belongs to 0^+S.*

As special cases of S, we consider three kinds of polyhedral sets S as follows

$$S_{\rm e} = \{x \in R^n : \ Ax = a\}, \ S_{\rm i} = \{x \in R^n : \ Bx \leq b\},$$

$$S_{\rm ei} = \{x \in R^n : \ Ax = a, \ Bx \leq b\},$$

where matrices $A \in R^{m \times n}$, $B \in R^{q \times n}$ and vectors $a \in R^m$, $b \in R^q$. It is easy to get that $0^+S_{\rm e} = \{d \in R^n : \ Ad = 0\}$, $0^+S_{\rm i} = \{d \in R^n : \ Bd \leq 0\}$ and $0^+S_{\rm ei} = \{d \in R^n : \ Ad = 0, \ Bd \leq 0\}$.

Now, we use the recession cone 0^+S to describe the structure of $(S - S)$.

Proposition 2.6. (i) *Relation $(S - S) \supseteq 0^+S$ holds true for any set S.*

(ii) $(S_e - S_e) = 0^+S_e$.

Proof (i) If $0^+S = \emptyset$, then the conclusion holds true. Without loss of generality, we suppose that $0^+S \neq \emptyset$. Let $d \in 0^+S$. Choose a point $x^0 \in S \neq \emptyset$, then, by the definition of 0^+S, both $x^0 + d$ and x^0 belong to S. Thus $d = (x^0 + d) - x^0 \in (S - S)$. This shows that $(S - S) \supset 0^+S$.

(ii) From claim (i), it is sufficient to show $(S_{\rm e} - S_{\rm e}) \subseteq 0^+S_{\rm e}$. For any point $d = (x - y) \in (S_{\rm e} - S_{\rm e})$, it follows that $Ad = Ax - Ay = a - a = 0$, i.e., $d \in 0^+S_{\rm e}$. Thus, $(S_{\rm e} - S_{\rm e}) \subseteq 0^+S_{\rm e}$. □

Taking into account of Proposition 2.6 and Corollary 2.2, one has the following result.

Corollary 2.7. (i) *If the objective function $q(x)$ of $\mathrm{QP}(H, S)$ is convex (strictly convex) on S, then the matrix H is positive semidefinite (positive definite) on the recession cone 0^+S.*

(ii) *The objective function $q(x)$ of $\mathrm{QP}(H, S_e)$ is convex (strictly convex) on its feasible set S_e if and only if matrix H is positive semidefinite (positive definite) on the recession cone 0^+S_e.*

The following example shows that, even if $S = S_{\rm i}$, relation $(S - S) = 0^+S$ is not necessary true, and the positive definiteness of H on $0^+S_{\rm i}$ is not a sufficient condition for the convexity of $q(x)$ on the feasible set S.

Example 2.8. Let

$$H_2 = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}, \ S_2 = \{(x_1, x_2) : \ x_2 - x_1 \leq b_1, \ -x_2 \leq b_2\}.$$

Then

$$0^+S_2 = \{(d_1, d_2) : \ d_2 - d_1 \leq 0, \ -d_2 \leq 0\} = \{(d_1, d_2) : \ d_1 \geq d_2 \geq 0\}.$$

For $\hat{x} = (12 - b_1 - b_2, 3 - b_2) \in S_2$ and $\hat{y} = (11 - b_1 - b_2, 10 - b_2) \in S_2$, one has

$$\hat{d} := \hat{x} - \hat{y} = (1, -7) \in (S_2 - S_2) \setminus 0^+S_2.$$

This shows that $(S_2 - S_2) \neq 0^+S_2$.

Furthermore, for any $d \in 0^+S_2 \setminus \{0\}$, we have $d_1 > 0$, and

$$d^T H_2 d = 4d_1^1 - d_2^2 \geq 4d_1^2 - d_1^2 = 3d_1^2 > 0.$$

Thus the matrix H_2 is positive definite on 0^+S_2. However, H_2 is not positive definite on $(S_2 - S_2)$ since $\hat{d}^T H_2 \hat{d} = -45 < 0$. Therefore, by Corollary 2.2, one can conclude that the objective function $q(x)$ of $\text{QP}(H_2, S_2)$ is nonconvex on its feasible set S_2.

3 The Existence of Solutions to QP Problems

In this section, we focus our attention on discussing the necessary and sufficient conditions for the existence of solutions to the QP problem $\text{QP}(H, S)$. It is easy to see that, if S is a nonempty bounded closed set then $\text{QP}(H, S)$ has at least one optimal solution. Throughout this section, we always assume that S is a nonempty unbounded closed convex set in R^n. Let E_S denote the extreme point set of S, i.e., $E_S = \{p : \ p \text{ is an extreme point of } S\}$. Let $\text{conv}(E_S)$ denote the convex hull of the set E_S and $\Omega_H^+(0^+S) := \{d \in 0^+S : \ d^T Hd = 0\}$, $\Omega_H(0^+S) := \{d \in 0^+S : \ Hd = 0\}$.

If $S = R^n$ (imply $0^+S = R^n$) and H is positive semidefinite on S, then $d^T Hd = 0 \Leftrightarrow Hd = 0$ (see [2, Corollary 1 in Section 11.2]), Thus $\Omega_H^+(0^+S) = \Omega_H(0^+S)$. However, in case $S \neq R^n$, $\Omega_H^+(0^+S)$ and $\Omega_H(0^+S)$ might be not the same. This can be seen by Example 2.4, where $0^+S = \{(\lambda, \lambda) : \ \lambda \geq 0\}$, $\Omega_H^+(0^+S) = 0^+S$ and $\Omega_H(0^+S) = \{0\}$.

First, we have the following necessary condition.

Theorem 3.1. *If the problem QP(H, S) has an optimal solution, i.e., the solution set of QP(H, S) is nonempty, then*

(i) *the matrix H is positive semidefinite on the recession cone 0^+S;*

(ii) *$(c + Hx)^T d \geq 0$ for all $x \in S$ and all $d \in \Omega_H^+(0^+S)$, which further implies that $c^T d \geq 0$ for all $d \in \Omega_H(0^+S)$.*

Proof (i) Suppose by contraction that the stated conclusion is not true. Then there exists a $\bar{d} \in 0^+S$ such that $\bar{d}^T H\bar{d} < 0$. Let x^* be an optimal solution to $\text{QP}(H, S)$ and $\bar{x}$ be any given point in S. Then $\bar{x} + \lambda\bar{d} \in S$ for all $\lambda \geq 0$ and

$$\begin{aligned} q(x^*) \leq q(\bar{x} + \lambda\bar{d}) &= q(\bar{x}) + \lambda\nabla q(\bar{x})^T\bar{d} + \frac{1}{2}\lambda^2\bar{d}^T H\bar{d} \\ &= q(\bar{x}) + \lambda(c + H\bar{x})^T\bar{d} + \frac{1}{2}\lambda^2\bar{d}^T H\bar{d} \qquad (3.1) \\ &\to -\infty, \text{ as } \lambda \to +\infty. \end{aligned}$$

This contradicts that x^* is an optimal solution to $\text{QP}(H, S)$. Thus H is positive semidefinite on the recession cone 0^+S.

(ii) Suppose by contraction that there exist $\bar{d} \in \Omega_H^+(0^+S) \subset 0^+S$ and $\bar{x} \in S$ such that $(c + H\bar{x})^T\bar{d} < 0$. Then, in view of $\bar{d}^T H\bar{d} = 0$, in a same fashion to part (i), from (3.1), one can bring a contraction. □

Remark 3.2. The necessary conditions (i)-(ii) in Theorem 3.1 are not sufficient conditions for the existence of solution to $\text{QP}(H, S)$, even if the inequality in condition (ii) is strict for $d \in \Omega_H^+(0^+S) \setminus \{0\}$. See the following example.

Example 3.3. In R^2, consider

$$c = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ H_3 = \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix},\ \text{i.e., } q(x) = c^T x + \tfrac{1}{2} x^T H_3 x = x_1 + x_2 - 2x_1^2,$$
$$S_3 = \{x \in R^2 :\ x_2 \geq x_1^2,\ x_1 \geq 1\} \text{ (a nonempty closed convex).}$$

It is easy to see that $0^+S_3 = \{d :\ d = \lambda(0,1)^T,\ \lambda \geq 0\}$. Thus, $H_3 d = 0$ and $d^T H_3 d = 0$ hold for all $d \in 0^+S_3$. Thus H_3 is positive semidefinite on 0^+S_3, and $\Omega^+_{H_3}(0^+S_3) = \Omega_{H_3}(0^+S_3) = 0^+S_3$. Furthermore, it follows that $(c + H_3 x)^T d = c^T d = \lambda > 0$ for all $d = \lambda(0,1)^T \in \Omega^+_{H_3}(0^+S_3)\backslash\{0\}$ and all $x \in S_3$. However, for the sequence $\{x_k = (k, k^2)^T\} \subset S_3$, $q(x_k) = k - k^2 \to -\infty$ as $k \to +\infty$. Therefore, the associated QP(H_3, S_3) has no optimal solution.

In the rest of this section, if a sequence $\{x_k\} \subset S$ such that $\|x_k\| \to \infty$, we assume that $x_k \neq 0$ for all $k \in \{1, 2, \cdots\}$. Next, we present some sufficient conditions for the existence of solution to QP(H, S). For this purpose, we first have the following lemma.

Lemma 3.4. *If a sequence $\{x_k\} \subset S$ such that $\|x_k\| \to \infty$, then each accumulation $\bar{d}$ of $\{x_k/\|x_k\|\}$ belongs to 0^+S and $\bar{d} \neq 0$.*

Proof Let $\bar{d}$ be any given accumulation of $\{x_k/\|x_k\|\}$. Then there exists an infinite index set K such that $x_k/\|x_k\| \xrightarrow{K} \bar{d}$. Obviously, $\|\bar{d}\| = 1$ and $\bar{d} \neq 0$. For any given point $\bar{x} \in S$, we need to show that $\bar{x} + \lambda\bar{d} \in S$ for any $\lambda \geq 0$. In view of $\bar{x},\ x_k \in S$ and $\|x_k\| \to \infty$ as well as the convexity of S, we know $(1 - \frac{\lambda}{\|x_k\|})\bar{x} + \frac{\lambda}{\|x_k\|} x_k \in S$ when k is large enough. Further,

$$\begin{aligned}
\bar{x} + \lambda\bar{d} &= \lim_{k\to+\infty, k\in K} \left(\bar{x} + \lambda\frac{x_k}{\|x_k\|}\right) = \lim_{k\to+\infty, k\in K} \left(\bar{x} + \lambda\frac{(x_k - \bar{x})}{\|x_k\|}\right) \\
&= \lim_{k\to+\infty, k\in K} \left((1 - \frac{\lambda}{\|x_k\|})\bar{x} + \frac{\lambda}{\|x_k\|} x_k\right).
\end{aligned}$$

This, along with the closedness of S, shows that $\bar{x} + \lambda\bar{d} \in S$ for any $\lambda \geq 0$. The proof is completed. □

Theorem 3.5. *For the QP problem QP(H, S), if the symmetric matrix H is positive definite on the recession cone 0^+S, then*

(i) *for any sequence $\{x_k\} \subseteq S$ with $\|x_k\| \to \infty$, one has $\lim_{k\to\infty} q(x_k) = +\infty$;*

(ii) *the QP problem QP(H, S) has at least one optimal solution.*

Proof (i) Suppose by contraction that the stated conclusion is not true. Then there exists an infinite sequence $\{x_k\}$ such that

$$\{x_k\} \subset S,\quad \lim_{k\to+\infty} \|x_k\| = \infty,\ \lim_{k\to+\infty} q(x_k) \neq +\infty.$$

Thus, there exists an infinite index set K such that $\{q(x_k) :\ k \in K\}$ has an upper bound. Therefore

$$a := \sup\{q(x_k) :\ k \in K\} < +\infty.$$

That is

$$a \geq q(x_k) = c^T x_k + \frac{1}{2} x_k^T H x_k,\ k \in K.$$

Dividing both sides of the above inequality by $\|x_k\|^2$, one has

$$\frac{a}{\|x_k\|^2} \geq \frac{c^T x_k}{\|x_k\|^2} + \frac{1}{2}\frac{x_k^T}{\|x_k\|} H \frac{x_k}{\|x_k\|},\ k \in K. \tag{3.2}$$

In view of the boundedness of the sequence $\{x_k/\|x_k\| :\ k \in K\}$, one can suppose, without loss of generality, that it converges to $\bar{d}$, i.e., $x_k/\|x_k\| \overset{K}{\to} \bar{d}$. Again, by Lemma 3.4, we know that $\bar{d} \in 0^+S$ and $\bar{d} \neq 0$. Thus, letting $k \to \infty$ and $k \in K$ in (3.2) and taking into account $\|x_k\| \to \infty$, we have $\bar{d}^T H \bar{d} \leq 0$. This, together with $\bar{d} \in 0^+S$ and $\bar{d} \neq 0$, contradicts the positive definiteness assumption about H on 0^+S.

(ii) Let $\{x_k\} \subset S$ be such that $q(x_k) \to q^* := \inf\{q(x) :\ x \in S\} < +\infty$. Then, from result (i), we know that $\{x_k\}$ must possess a bounded infinite subsequence, namely, $\{x_k :\ k \in K'\}$. Without loss of generality, suppose that $x_k \overset{K'}{\to} x^* \in S$ (due to the closedness of S). Thus, $q^* = q(x^*) > -\infty$ and x^* is an optimal solution of QP(H, S). □

It is known that, in discussing and solving the QP problem QP(H, S), both the convexity of QP(H, S) and the existence of solutions to QP(H, S) are very important. Now, summarizing Corollaries 2.2, 2.7 and Theorem 3.5, we have three corollaries as follows.

Corollary 3.6. *The problem QP(H, S) is not only strictly convex but also has a unique optimal solution if and only if the matrix H is positive definite on $(S - S)$. Particularly, if* int $S \neq \emptyset$, *then this condition is equivalent to that H is positive definite on R^n.*

Proof (i) From Corollary 2.2, one has the equivalence between the strict convexity of the problem QP(H, S) and the positive definiteness of matrix H on $(S - S)$. Next, we prove the following result:

If the matrix H is positive definite on $(S - S)$, then the problem QP(H, S) has a unique optimal solution.

Since $0^+S \subset (S - S)$ and H is positive definite on $(S - S)$, the matrix H is positive definite on 0^+S. From Theorem 3.5, one has QP(H, S) has at least one optimal solution. Furthermore, take into account the strict convexity of the problem QP(H, S), QP(H, S) has a unique optimal solution. □

Corollary 3.7. *Suppose that the equality constrained set S_e is nonempty. Then, the associated problem QP(H, S_e) is not only strictly convex but also has a unique optimal solution if and only if the matrix H is positive definite on the recession cone $0^+S_e = \{d \in R^n :\ Ad = 0\}$. Furthermore, in this case, its optimal solution and Karush-Kuhan-Tucker (KKT) point are the same.*

Corollary 3.8. *Let the equality and inequality constrained set S_{ei} be nonempty. Assume that the matrix H is positive definite on the recession cone $0^+S_{ei} = \{d \in R^n :\ Ad = 0,\ Bd \leq 0\}$. Then the associated problem QP(H, S_{ei}) has at least one optimal solution. Further, its optimal solution is necessary a KKT point. However, due to $q(x)$ is not necessary convex on S_{ei}, the converse is not necessarily true, i.e., a KKT point is not necessary an optimal solution.*

To illustrate Corollary 3.8, we consider the following example, which is a special case of Example 2.8.

Example 3.9. (Illustrate Corollary 3.8) In R^2, let

$$H_4 = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix},\ c = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$S_4 = \{(x_1, x_2) \in R^2 : \ x_2 - x_1 - 1 \le 0, \ -x_2 - 2 \le 0\}.$$

From Example 2.8, one knows that the matrix H_4 is positive definite on $0^+S_4 = \{(d_1, d_2) : d_1 \ge d_2 \ge 0\}$. Therefore, from Corollary 3.8, we know that the associated problem $\mathrm{QP}(H_4, S_4)$ has at least an optimal solution, which is necessary a KKT point of $\mathrm{QP}(H_4, S_4)$. Again, by solving the KKT system of $\mathrm{QP}(H_4, S_4)$, all the KKT points are as follows:

$$(x^*; \lambda^*) = (0, -2; 0, 2)^T, \ (\hat{x}; \hat{\lambda}) = (0, 0; 0, 0)^T, \ (\bar{x}; \bar{\lambda}) = \left(\frac{1}{3}, \frac{4}{3}; \frac{4}{3}, 0\right)^T.$$

It is not difficult to check that the KKT point $x^* = (0, -2)^T$ is an optimal solution to $\mathrm{QP}(H_4, S_4)$, and the other two KKT points $\hat{x}$ and $\bar{x}$ are not optimal.

Next, motivated by Theorem 3.1 and based on the positive semidefiniteness of H on 0^+S plus other suitable conditions, we further discuss sufficient conditions for the existence of solution to $\mathrm{QP}(H, S)$.

Lemma 3.10. *Assume that the QP problem $QP(H, S)$ has no optimal solution. If $\{x_k\} \subset S$ such that $q(x_k) \to q^* := \inf\{q(x) : \ x \in S\}$, then $\|x_k\| \to \infty$.*

Proof If $\|x_k\| \not\to \infty$, then it possesses a bounded subsequence, say, $\{x_k : \ k \in K\}$. Without loss of generality, we suppose that $x_k \xrightarrow{K} x^* \in S$ (due to the closedness of S). Thus $q^* = q(x^*)$ and x^* is an optimal solution of $\mathrm{QP}(H, S)$, which conflicts with the fact that $\mathrm{QP}(H, S)$ has no optimal solution. □

Theorem 3.11. *For the QP problem $QP(H, S)$, suppose that the extreme point set E_S is nonempty and bounded. Assume that the matrix H is positive semidefinite on the recession cone 0^+S, and that one of the two following conditions holds:*

(C1) $(c + Hx)^T d > 0$ *holds for all* $x \in cl(conv(E_S))$ *and all* $d \in \Omega_H^+(0^+S) \backslash \{0\}$*;*

(C2) $(c + Hx)^T d \ge 0$ *holds for all* $x \in E_S$ *and all* $d \in 0^+S$*.*

Then,

(i) *the QP problem $QP(H, S)$ has at least one optimal solution;*

(ii) *there exists one optimal solution in the convex hull $conv(E_S)$, if condition* (C2) *is satisfied.*

Proof First of all, it is easy to know that the condition (C2) above has an equivalence as follows:

$$\text{Condition (C2)} \ \Leftrightarrow \ (c + Hx)^T d \ge 0, \ \forall \ x \in \mathrm{conv}(E_S), \forall \ d \in 0^+S. \tag{3.3}$$

(i) Suppose by contraction that $\mathrm{QP}(H, S)$ has no optimal solution.

First, there exists an infinite sequence $\{x_k\} \subset S$ such that $q(x_k) \to q^* := \inf\{q(x) : \ x \in S\} < +\infty$. According to Lemma 3.10, it follows that $\|x_k\| \to \infty$. Again, in view of the boundedness of the sequence $\{x_k/\|x_k\|\}$, there exists a convergent subsequence $\{x_k/\|x_k\| : k \in K\}$. Assume that $x_k/\|x_k\| \xrightarrow{K} \bar{d}$. Therefore, by Lemma 3.4, we know that $\bar{d} \in 0^+S$ and $\bar{d} \ne 0$.

Second, taking into account $q(x_k) \to q^* < +\infty$, there exists a positive constant $M > 0$ such that

$$M \ge q(x_k) = c^T x^k + \frac{1}{2} x_k^T H x_k.$$

This further implies that

$$\frac{M}{\|x_k\|^2} \ge \frac{c^T x_k}{\|x_k\|^2} + \frac{1}{2} \frac{x_k^T}{\|x_k\|} H \frac{x_k}{\|x_k\|}.$$

Passing to the limit in the above inequality for $k \in K$, it follows that $\bar{d}^T H \bar{d} \le 0$. This, together with the positive semidefiniteness of H on $0^+ S$ and $\bar{d} \in 0^+ S$, gives that $\bar{d}^T H \bar{d} = 0$ and $\bar{d} \in \Omega_H^+(0^+S) \backslash \{0\}$.

Third, from the fact that E_S is nonempty, it is not difficult to show that S contains no line. Otherwise, suppose that there exists a line $\{\hat{x} + \lambda \hat{d} : \ \lambda \in R\} \subset S$, where $\hat{d} \neq 0$. Therefore, both $\hat{d}$ and $-\hat{d}$ belong to 0^+S (see [16, Theorem 8.3]). Thus, for any point $x \in S$, it follows that both $x + \hat{d}$ and $x - \hat{d}$ belong to S, and $x = \frac{1}{2}(x + \hat{d}) + \frac{1}{2}(x - \hat{d})$. This shows that E_S is empty. Therefore, by [16, Theorem 18.5] and Proposition 2.5, we know that the set S can be expressed as the sum of the convex hull of E_S and the recession cone 0^+S, namely,

$$S = \operatorname{conv}(E_S) \oplus 0^+ S := \{x + d : \ x \in \operatorname{conv}(E_S), \ d \in 0^+ S\}. \tag{3.4}$$

Fourth, by (3.4), there exist $y_k \in \operatorname{conv}(E_S)$ and $\hat{d}_k \in 0^+S$ such that

$$x_k = y_k + \hat{d}_k, \ k \in K.$$

Since $\|x_k\| \to \infty$ and $\{y_k\}$ is bounded (since E_S is bounded), without loss of generality, one could assume that $\|\hat{d}_k\| \neq 0$ for all $k \in K$. And

$$x_k = y_k + \lambda_k d_k, \quad \text{where } \lambda_k = \|\hat{d}_k\| \text{ and } d_k = \hat{d}_k / \|\hat{d}_k\| \in 0^+S, \ k \in K.$$

Furthermore, in view of $x_k/\|x_k\| \to \bar{d}$ and $\|d_k\| = 1$, it follows that

$$\lambda_k = \|x_k - y_k\| \to +\infty, \ d_k = \frac{x_k - y_k}{\lambda_k} = \frac{x_k - y_k}{\|x_k - y_k\|} \to \bar{d}, \ k \in K.$$

Again, in view of the boundedness of $\{y_k : k \in K\}$, one can suppose that, without loss of generality, it converges to $y_* \in \operatorname{cl}(\operatorname{conv}(E_S)) \subset S$. Furthermore, since $\mathrm{QP}(H, S)$ has no optimal solution, one has $q^* < q(y^*)$. Thus

$$\begin{aligned} 0 \quad &> \quad q^* - q(y^*) = \lim_{k \in K} (q(x_k) - q(y_k)) \\ &= \quad \lim_{k \in K} (q(y_k + \lambda_k d_k) - q(y_k)) = \lim_{k \in K} \left(\lambda_k \left(c + H y_k\right)^T d_k + \frac{\lambda_k^2}{2} d_k^T H d_k \right). \end{aligned}$$

This, together with $d_k^T H d_k \ge 0$ (since $d_k \in 0^+S$), implies that

$$(c + H y_k)^T d_k < -\frac{\lambda_k}{2} d_k^T H d_k \le 0, \ \text{when } k \in K \text{ large enough}. \tag{3.5}$$

This contradicts the relationship (3.3) (i.e., condition (C2)). Furthermore, letting $k \to \infty$ and $k \in K$ in (3.5), one has

$$(c + H y_*)^T \bar{d} \le 0, \ y_* \in \operatorname{cl}(\operatorname{conv}(E_S)), \ \bar{d} \in \Omega_H^+(0^+S) \backslash \{0\}.$$

This also contradicts the condition (C1). Therefore, result (i) holds true.

(ii) Let x_* be an optimal solution to $\mathrm{QP}(H, S)$. Then, by (3.4), there exist $y_* \in \operatorname{conv}(E_S)$ and $d_* \in 0^+S$ such that $x_* = y_* + d_*$. Furthermore, by (3.3) and the positive semidefiniteness of H on 0^+S, it follows that

$$(c + H y_*)^T d_* \ge 0, \ d_*^T H d_* \ge 0.$$

Therefore, we have

$$q(x_*) = q(y_* + d_*) = q(y_*) + (c + H y_*)^T d_* + \frac{1}{2} d_*^T H d_* \ge q(y_*).$$

This shows that the element y_* of $\operatorname{conv}(E_S)$ is also an optimal solution to $\mathrm{QP}(H, S)$. $\square$

4 Conclusions

In this paper, we present several conditions for the (strict) convexity and existence of solution to a class of quadratic programming problems $QP(H, S)$, whose constraint is a closed convex set. $QP(H, S)$ is convex (strictly convex) if and only if the Hessian matrix of the objective function is positive semidefinite (positive definite) on the difference of feasible set. Based on the Hessian matrix of objective function and the recession cone of feasible set, we present one necessary condition (Theorem 3.1) and three sufficient conditions (see Theorems 3.5 and 3.11) for the existence of solution to $QP(H, S)$.

Acknowledgments

The authors would like to thank the anonymous referees and Prof. Chunming Tang for their constructive comments and valuable suggestions which greatly improved the paper.

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Manuscript received 17 June 2017
revised 18 January 2018
accepted for publication 10 February 2018

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