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SECOND-ORDER SLICE-DERIVATIVE OF CONVEX FUNCTIONS IN NORMED SPACES

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Abstract: Epi-derivatives have many applications in variational analysis and optimization. In particular, second-order epi-derivatives play an important role in optimality conditions statements, and sensitivity results. Introduced by R.T. Rockafellar in finite dimensional spaces, these notions have been studied for convex functions acting on reflexive Banach spaces by Chi Ngoc Do. The purpose of this article is to extend these results to general normed spaces, replacing the Mosco convergence by a stronger notion which is called epigraphical slice convergence. New results are obtained for the second-order slice-derivative of convex functions and the proto-derivative of the subdifferential operators. We show that the conjugate of the second-order slice-derivative of a convex function is the second-order slice-derivative of its conjugate function. We establish that a function is twice slice-differentiable if and only if its subdifferential is proto-differentiable, as a set-valued mapping. We also give formulas for the second-order slice-derivative of a composite function of the form $f \circ A$ with f convex and A linear.

Key words: *epi-derivative, second-order slice-derivative, subdifferential, convex optimization, epigraphical slice convergence, proto-derivative*

Mathematics Subject Classification: *26B05, 49A52, 58C05, 58C20, 90C30*

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1 Introduction

In [15, 17] R.T. Rockafellar introduced second-order epi-derivatives to establish necessary and sufficient conditions for optimality. This notion is based on the epi-convergence of the second-order difference quotients, hence its name. It turns out that this a is useful concept for several classes of nonsmooth functions, including convex functions, convex-concave saddle functions, strongly subsmooth functions, or composite function of the form $f \circ A$, with f convex and A of class C^2. These concepts have been developed further in [3, 7, 8, 14, 17]. Second-order optimality conditions in nonlinear programming have been obtained in terms of epi-derivatives in [12, 15, 18].

Many authors have tried to define second-order derivatives in a different way. Most definitions have been limited to finite-valued functions (see for example [4, 17, 19]). In [10], the second-order epi-derivatives have been studied in the case of reflexive Banach spaces, where epiconvergence is replaced by the notion of Mosco-epi-convergence.

The purpose of this article is to extend the theory to general normed spaces, replacing Mosco-epi-convergence by a stronger convergence notion which is called epigraphical slice convergence. The slice-convergence is a concept introduced by Beer in [5, 6], which is equivalent to Mosco convergence if and only if the underlying space is reflexive. This notion has been developed by many authors (see [19–22]), and has proved effective in analyzing the stability of minimization problems in normed spaces.

The document is organized as follows. In Section 2, we set the notations and recall some classical definitions and results. In Section 3, we present the definition and the first properties of the second-order slice-derivative on a normed space X. We also give our first main result (Theorem 3.7) concerning the continuity of the Legendre-Fenchel transform, which allows us to establish the equivalence between the differentiability properties of f and f^*. In Section 4, we give our second important result (Theorem 4.3): we establish that a lower semicontinuous proper convex function f on X is twice slice-differentiable at x relatively to z if and only if its subdifferential mapping ∂f is proto-differentiable at x relatively to z. We also give a formula for the second-order slice-derivative of a composite function of the form $f \circ A$ with f convex and A linear.

2 Notation and Definitions

Recall some definitions and basic concepts in convex analysis and optimization. For more details, one can refer to [1,9,11]. Let $(X, \|.\|)$ be a normed linear space and $(X^*, \|.\|_*)$ its topological dual. The duality pairing between $y \in X^*$ and $x \in X$ is denoted by $\langle x, y\rangle$. Let $f : X \to \bar{\mathbb{R}}$ be an extended real-valued function acting on X. We set briefly $f \in \overline{\mathbb{R}}^X$. For such a function, the set

$$epif = \{(x, \alpha) \in X \times \mathbb{R} \mid f(x) \leq \alpha\}$$

is called the *epigraph* of f. The function f is called *convex* (resp. *lower semicontinuous*) if its epigraph is a convex (resp. closed) subset of $X \times \mathbb{R}$. Furthermore, f is called *proper* if its epigraph is nonempty.

As usual, $\Gamma(X)$ denotes the set of proper, lower semi continuous convex functions acting on X , and in a dual way, $\Gamma^*(X^*)$ denotes the set of proper, weak* lower semicontinuous convex functions acting on X^*.

For $f \in \Gamma(X)$, its conjugate $f^* \in \Gamma^*(X^*)$ is defined by the formula: for any $y \in X^*$

$$f^*(y) = \sup_{x \in X} \{\langle x, y\rangle - f(x)\} \tag{2.1}$$

For $h \in \Gamma^*(X^*)$, we adopt the usual convention that h^* be defined only on X rather than on all of X^{**}, so that $h^* \in \Gamma(X)$. with this in mind, the Fenchel transform $f \to f^*$ is a bijective involation between $\Gamma(X)$ and $\Gamma^*(X^*)$.

The subdifferential of $f \in \overline{R}^X$ at x_0, denoted by $\partial f(x_0)$, is defined by :

$$\begin{aligned}\partial f(x_0) &= \{y \in X^* \mid f(x) \geq f(x_0) + \langle\, x - x_0, y\rangle \;\forall x \in X\} \\ &= \{y \in X^* \mid f(x) + f^*(y) - \langle x, y\rangle = 0\}\,.\end{aligned} \tag{2.2}$$

This set is convex (closed) if f is convex (lower semicontinuous), and one has the following equivalent

$$y \in \partial f(x_0) \;\Leftrightarrow\; \exists\, \varepsilon > 0, \forall\, x \in B\,(x, \varepsilon)\,, f\,(x) \geq f(x_0) + \langle x - x_0, y\rangle. \tag{2.3}$$

We write $C(X)$ for the class of nonempty closed convex subsets of X, and $CB(X)$ for the nonempty closed and bounded convex subsets of X. For any A and C in $C(X)$ the gap $D(A,C)$ between the two sets is given by

$$D\left(A,C\right)=\inf\left\{\|a-c\|\,:\,a\in A\text{ and }c\in C\right\}. \tag{2.4}$$

The Wijsman topology on $C(X)$ is the topology generated by the family of functions $\{d\left(x,.\right):\,x\in X\}$, where $d\left(x,A\right)\;=\;\inf\left\{\|x-a\|\,:\,a\in A\right\}$. Note that such a distance functional is also a gap functional, namely, $d\left(x,.\right)\;=\;D\left(\{x\}\,,.\right)$. Having this in mind, the topology on $C(X)$ generated by

$$\{\,D\left(B,\ .\right)\,:\,B\,\in\,CB(X)\}$$

is the topology of interest for this paper, and is called the slice topology τ_s (see [2, 5, 6]).

The classical notion of convergence for sequences of closed sets in a topological space X is the convergence in the *Kuratowski-Painlevé sense.* Let us recall its definition: Given a sequence of nonempty closed subsets $\{A_n,A;\ n\in\mathbb{N}\}$ of X, we have $A=\lim_n A_n$ provided $A=\lim_n\sup_n A_n=\liminf_n A_n$, where the lower and upper limits of the sequence $\{A_n;\ n\in\mathbb{N}\}$ are defined by the formulas

$$\liminf_n A_n:=\{x\in X\mid\exists(x_n)_{n\in N};x_n\in A_n;x_n\to x\}$$

$$\limsup_n A_n:=\{x\in X\mid\exists(n_k)_{k\in N};\ \exists(x_k)_{k\in N};\ \forall k\in\mathbb{N}\ x_k\in A_{n_k};\ x_k\to x\}\,.$$

Graph convergence (see [1]) Let $\{f_n,f;n\in\mathbb{N}\}$ be a sequence of functions of $\Gamma(X)$. We say that the sequence (∂f_n) is graph-convergent to ∂f and we write $\partial f=G{-}\lim_n\partial f_n$, if the sequence $\{graph\,f_n;n\in\mathbb{N}\}$ converges in the Kuratowski-Painlevé sense to the set $gph\,\partial f$ in $(X\times R)$, i.e., $\partial f=\liminf_n\partial f_n=\limsup_n\partial f_n$, where the topological limits are taken with respect to the strong topology on $X\times X^*$.

Convergence in $C(X)$ (see [5])

- Let X be a normed space and let $\{A_n,A;\ n\in\mathbb{N}\}$ be subsets of $C(X)$. The sequence $\{A_n;n\in N\}$ is said to be *slice-convergent* to A, and we write $A=\tau_s-\lim_n A_n$, if for each closed and bounded convex subsets B of X, we have

$$D\left(B,A\right)=\lim_{n\to+\infty}D\left(B,A_n\right). \tag{2.5}$$

- When X is a reflexive Banach space and the sets belong to $C(X)$, the sequence $\{A_n;n\in\mathbb{N}\}$ is said to be *Mosco-convergent* to A, and we write $A\;=\;M-\lim_n A_n$, if for each weakly compact and convex subset K of X, we have

$$D\left(K,A\right)=\lim_{n\to+\infty}D\left(K,A_n\right) \tag{2.6}$$

Since the closed convex and bounded subsets of a reflexive Banach space are weakly compact, these two notions of convergence coincide in the reflexive setting.

Mosco-epigraphical convergence ([13]) Let X be a reflexive Banach space and let $\{f_n, f; n \in \mathbb{N}\}$ be a sequence of functions in $\Gamma(X)$. We say that the sequence $\{f_n; n \in \mathbb{N}\}$ is Mosco-epi-convergent to f and we write $f = M{-}epi{-}\lim_n f_n$, if the sequence $\{epi\, f_n;\, n \in \mathbb{N}\}$ Mosco-converges to $epi\, f$ in $(X \times \mathbb{R})$.

This is equivalent to say that, for any $x \in X$, the two following statements hold:

M1) for any sequence $(x_n)_{n\in\mathbb{N}} : x_n \xrightarrow[n]{w} x \;\Rightarrow\; f(x) \leq \liminf_n f_n(x_n)$;

M2) there exists a sequence $(\zeta_n)_{n\in\mathbb{N}}$ such that $\zeta_n \xrightarrow[n]{s} x$and

$f(x) \geq \limsup_n f_n(\zeta_n)$.

In the above statements, s (resp. w) is the strong (resp. weak) topology on X.

Slice-epigraphical convergence ([5, 6]) Let X be a normed space and $\{f_n, f\,;\, n \in \mathbb{N}\}$ be a sequence of proper lower semicontinuous convex functions on X. We say that the sequence $\{f_n\,;\, n \in \mathbb{N}\}$ is epigraphically slice convergent to f and we write $f = \tau_s - \lim_n f_n or\ f_n \xrightarrow{\tau_s} f$, if the sequence $\{epi\, f_n\,;\, n \in \mathbb{N}\}$ slice-converges to $epi\, f$ in $X \times \mathbb{R}$. This is equivalent to say that for any $x \in X$, the two following statements hold:

S1) for any $(y, \eta) \in epi\, f^* \ with\ \eta > f^*(y)$ and each bounded sequence (x_n), there exists $n_0 \in \mathbb{N}$ such that for each $n > n_0$, we have

$$f_n(x_n) > \langle x_n,\, y\rangle - \eta.$$

S2) for each $x \in X$, there exists $(x_n)_{n\in\mathbb{N}}$ that converges strongly to x and such that

$$f(x) = \lim_{n \to +\infty} f_n(x_n).$$

Similarly, for $\{h_n, h;\ n \in \mathbb{N}\}$ a sequence of proper weak* lower semicontinuous convex functions on X^*, we have $h = \tau_s^* - \lim_n h_n or\ h_n \xrightarrow{\tau_s^*} h$ if and only if :

$S^*1)$ for any $(x, \alpha) \in epi\, h^*\ with \alpha > h^*(x)$ and each bounded sequence (y_n), there exists $n_0 \in \mathbb{N}$ such that for each $n > n_0$, we have

$$h_n(y_n) > \langle x,\, y_n\rangle - \alpha.$$

$S^*2)$ for each $y \in X^*$, there exists $(y_n)_{n\in\mathbb{N}}$ that converges strongly to y and which satisfies

$$h(y) = \lim_{n \to +\infty} h_n(y_n).$$

We recall some fundamental results concerning the slice convergence.

Proposition 2.1 ([19]). *Let X be a normed space and $\{f_n, f;\ n \in \mathbb{N}\}$ be a sequence in $\Gamma(X)$ such that $f_n \xrightarrow{\tau_s} f$. Then for each $\xi_n \underset{n}{\to} \xi$, we have*

$$f\,(\xi) \leq \liminf_n f_n\,(\xi_n)$$

Theorem 2.2 ([6]). *Let $\{f_n\,,\, f\,;n \in \mathbb{N}\}$ be a sequence of functions in $\Gamma(X)$. Then the following two conditions are equivalent :*

(i) $f = \tau_s - \lim_n f_n$

(ii) $f^* = \tau_s^* - \lim_n f_n^*$

Theorem 2.3 ([2]). *Let X be a Banach space. Let $\{\, f_n\,,\, f\,;\, n \in \mathbb{N}\}$ be a sequence of closed proper convex functions. Then $f = \tau_s - \lim_n f_n$ if and only if the following two conditions are satisfied :*

(i) $\partial f = G - \lim_n \partial f_n$

(ii) *there exists $(u, z) \in \partial f$ and a sequence $(u_n, z_n) \in \partial f_n$, such that*

$$(u, f(u), z) = \lim_{n \to +\infty} (u_n, f_n(u_n), z_n)\,.$$

The previous notions have natural extensions in case of a family of functions $(\varphi_t)_{t>0}$ parametrized by $t > 0$. The slice convergence of φ_t to φ as $t \downarrow 0$ is defined by saying that $\varphi_{t_n} \xrightarrow{\tau_s} \varphi$ for every sequence $t_n \downarrow 0$. i.e $\varphi = \tau_s - \lim_{n\to+\infty} \varphi_{t_n}$. In view of S1) and S2), this is equivalent to say that:

St1) for any $(y, \eta) \in epi\, \varphi^*$ with $\eta > \varphi^*(y)$ and each bounded sequence (x_n), there exists $n_0 \in \mathbb{N}$ such that for each $n > n_0$, we have $\varphi_{t_n}(x_n) > \langle x_n, y\rangle - \eta$.

St2) for each $x \in X$, there exists $(x_n)_{n\in\mathbb{N}}$ that converges strongly to x for which

$$\varphi(x) = \lim_{n \to +\infty} \varphi_{t_n}(x_n).$$

3 Second-Order Epi-Derivatives

Throughout this section, X is a normed space and f denotes a closed proper convex function.

Definition 3.1. Let $f : X \to \overline{\mathbb{R}}$ be finite at $x \in X$. Let $z \in X^*$ and consider the second-order difference quotient functions $\varphi^f_{t,x,z} : X \to \overline{\mathbb{R}}$

$$\varphi^f_{t,x,z}(\xi) = \frac{1}{t^2}\,\{f(x + t\xi) - f(x) - t\langle \xi, z\rangle\}\,; \qquad (t > 0) \tag{3.1}$$

If the net of functions $(\varphi^f_{t,x,z})_t$ slice-converges as $t \downarrow 0$ to some function $\varphi \in \Gamma(X)$, then we say that f is twice slice- differentiable at x relatively to z, and φ is called the second-order slice-derivative of f at x relatively to z. We then write $f''_{x,\,z}$ instead of φ, i.e.,

$$f''_{x,\,z} = \tau_s - \lim_{t\downarrow 0} \varphi^f_{t,x,z}.$$

In terms of sequences ,

$$f''_{x,\,z} = \tau_s - \lim_{n\to+\infty} \varphi^f_{t_n,x,z}\,,\ \forall\ t_n \downarrow 0. \tag{3.2}$$

Some elementary properties entailed by these definitions are explored in the following propositions.

Proposition 3.2. *If $f \in \Gamma(X)$, then for each $t > 0$ the function $\varphi^f_{t,x,z} \in \Gamma(X)$.*

Proof. Since f is proper, lower semicontinuous and convex then, as a direct consequence of the formula (3.1), we have $\varphi^f_{t,x,z} \in \Gamma(X)$, which completes the proof. □

Proposition 3.3. *The second-order slice-derivative function $f''_{x,z}$ is positively homogeneous of degree 2 and $f''_{x,z}(0) = 0$.*

Proof. The positive homogeneity of degree 2 of $f''_{x,z}$, is immediate from the form of the functions $\varphi^f_{t,x,z}$ in (3.1). On the other hand, since $\varphi^f_{t,x,z}(0) = 0$ for every t, and by Proposition 2.1 , one has $f''_{x,z}(0) \leq \liminf_n \varphi^f_{t,x,z}(0) = 0$. Hence $f''_{x,z}(0) \leq 0$, so f is closed and proper. Also, by homogeneity, $f''_{x,z}(0) = \lambda f''_{x,z}(0)$, then the finiteness of $f''_{x,z}(0)$ gives $f''_{x,z}(0) = 0$, which complete the proof. □

Proposition 3.4. *If f is twice slice-differentiable at x relative to z, then $z \in \partial f(x)$ the subdifferential of f at x. Furthermore, one has $f''_{x,z} \geq 0$, and 0 is minimal point of $f''_{x,z}$, i.e., $0 \in \partial f''_{x,z}(0)$.*

Proof. We proceed by contradiction. Suppose that f is twice slice- differentiable at x relatively to z and that $z \notin \partial f(x)$. Since f is twice slice-differentiable at x relatively to z, then for all $t_n \downarrow 0$, $f''_{x,z} = \tau_s - \lim_{t_n \downarrow 0} \varphi^f_{t_n,x,z}$. Moreover, since $z \notin \partial f(x)$, then, by (2.3), we deduce that for all $\varepsilon > 0$ there exists $h \in B(x,\varepsilon)$ such that :

$$f(h) < f(x) + \langle h - x, z \rangle.$$

Let $h = \xi_n + x$ with $\varepsilon = \frac{1}{n}$ and $\xi_n \underset{n}{\longrightarrow} 0$, such that

$$f(x + \xi_n) - f(x) - \langle \xi_n, z \rangle = \alpha_n < 0.$$

Set

$$\varphi(\xi) := f(x + \xi) - f(x) - \langle \xi, z \rangle.$$

The function φ defined in this way is lower semicontinuous, convex and $\varphi(0) = 0$. As a consequence

$$\begin{aligned} 0 \leq\ & \liminf_{\xi_n \underset{n}{\longrightarrow} 0} \varphi(\xi_n) \\ =\ & \liminf_n \alpha_n \\ \leq\ & \limsup_n \alpha_n \\ =\ & 0, \end{aligned}$$

i.e., $\liminf_n \alpha_n = \limsup_n \alpha_n = 0$.

hence $\lim_n \alpha_n = 0$. We may assume that $|\alpha_n| < 1$ for all n. By convexity of φ, we have

$$\begin{aligned} \varphi(-\alpha_n \xi_n) &= \varphi(-\alpha_n \xi_n + (1 + \alpha_n)(0)) \\ &\leq -\alpha_n \varphi(\xi_n) + (1 + \alpha_n)\varphi(0) \\ &= -\alpha_n \varphi(\xi_n) \\ &= -\alpha_n^2. \end{aligned}$$

Set $t_n = -\alpha_n > 0$. For all $n \in \mathbb{N}$, the difference quotients for the function f satisfy

$$\begin{aligned} \varphi^f_{t_n,x\,,\,z}(\xi_n) &= \frac{f(x-\alpha_n\xi_n) - f(x) + \alpha_n\langle \xi_n, z\rangle}{(-\alpha_n)^2} \\ &= \frac{\varphi(-\alpha_n\xi_n)}{\alpha_n^2} \\ &\leq -1. \end{aligned} \tag{3.3}$$

By the slice-convergence of $\varphi^f_{t_n,x,z}$ to f''_{x,x^*}, combined with Proposition 2.1 and inequality (3.3), we obtain:

$$\begin{aligned} 0 &= f''_{x,z}(0) \\ &\leq \liminf_n \varphi^f_{t_n,x\,,\,z}(\xi_n) \\ &\leq -1 \end{aligned} \tag{3.4}$$

a clear contradiction. Hence $z \in \partial f(x)$.

On the other hand, the above proof also implies that $\varphi \geq 0$, and hence $f''_{x,z} \geq 0$; We conclude by Proposition 3.3 that 0 is a minimum point of $f''_{x,\,z}$, i.e., $0 \in \partial f''_{x,z}(0)$, which completes the proof. □

Proposition 3.5. *Let $f : X \to \overline{\mathbb{R}}$ be convex and C^2 (Fréchet) in a neighborhood of $x \in X$. Then the function*

$$\varphi(\xi) = \frac{1}{2}\langle \xi,\ D^2 f(x)\,\xi\rangle \tag{3.5}$$

is positive, convex, strongly continuous, and weakly lower semicontinuous.

Proof. Since f is convex and C^2, the bilinear form $D^2 f(x)$ is positive, hence $\varphi \geq 0$. The strong continuity of φ follows from the strong continuity of $D^2 f(x)$. It is easy to see that

$$\begin{aligned} \varphi(t\xi + (1-t)\eta) &= t\varphi(\xi) + (1-t)\varphi(\eta) - t(1-t)\varphi(\xi-\eta) \\ &\leq t\varphi(\xi) + (1-t)\,\varphi(\eta), \qquad \forall t \in [0,1]. \end{aligned}$$

This gives the convexity of φ, and therefore the lower semicontinuity of φ for the weak topology, which completes the proof. □

Proposition 3.6. *Let $f : X \to \overline{\mathbb{R}}$ be a C^2 convex function in a neighborhood of $x \in X$. Then f is twice slice-differentiable at x relatively to $Df(x)$, and the the second-order slice-derivative $f''_{x,Df(x)}$ is given by*

$$f''_{x,Df(x)}(\xi) = \frac{1}{2}\langle \xi,\ D^2 f(x)\,\xi\rangle. \tag{3.6}$$

Proof. By Taylor's formula, we have

$$f(x+t\xi) - f(x) - t\langle \xi, Df(x)\,\xi\rangle = \frac{t^2}{2}\langle \xi, D^2 f(x)\,\xi\rangle + 0\left(\|t\xi\|^2\right) \tag{3.7}$$

with $\lim_{t\downarrow 0} \frac{0\left(\|t\xi\|^2\right)}{t^2} = 0$. From (3.7) we obtain

$$\varphi^f_{t,x,Df(x)}(\xi) = \frac{1}{2}\langle \xi, D^2 f(x)\,\xi\rangle + \frac{0\left(\|t\xi\|^2\right)}{t^2}.$$

Let us show that $\varphi^f_{t,x,Df(x)}$ slice-converges (as $(t\downarrow 0)$ to $f''_{x,\,Df(x)}$.

We start by proving S1): for every $(t_n \downarrow 0)$ and for every $(y,\eta) \in epi\,(f''_{x,Df(x)})^*$ with $\eta > (f''_{x,Df(x)})^*(y)$, we have $f''_{x,\,Df(x)}(\xi) > \langle y,\xi\rangle - \eta,\ \forall \xi \in X$. As a consequence, for each bounded sequence (ξ_n) we have

$$\begin{aligned}\varphi^f_{t_n,x,Df(x)}(\xi_n) &= \frac{1}{2}\langle D^2 f(x)\,\xi_n,\ \xi_n\rangle + \frac{0\left(\|t_n\xi_n\|^2\right)}{t_n^2}\\ &> \langle y,\xi_n\rangle - \eta.\end{aligned}$$

Let us now examine S2). Let $x \in X$, and $(\xi_n)_{n\in N}$ that converge strongly to ξ. We have

$$\lim_n \varphi^f_{t_n,x,Df(x)}(\xi_n) = \lim_n \frac{1}{2}\langle D^2 f(x)\,\xi_n, \xi_n\rangle + \lim_n \frac{0\left(\|t_n\xi_n\|^2\right)}{t_n^2}$$

Hence, $\lim_n\ \varphi^f_{t_n,\,Df(x)}(\xi_n) = f''_{x,\,Df(x)}(\xi)$. i.e., f is twice slice-differentiable at x relatively to $Df(x)$, and the the second-order slice-derivative is given by (3.5), which completes the proof. □

Theorem 3.7 (Conjugacy)**.** *Let $f : X \to \overline{\mathbb{R}}$ be a closed proper convex function. Then one has*

a) $z \in \partial f(x)$ if and only if $x \in \partial f^*(z)$.

b) The second-order difference quotient functions of f and f^* are conjugates from each, namely

$$\left(\varphi^f_{t,x\,,\,z}\right)^*(\xi^*) = \left(\varphi^{f^*}_{t,z\,,\,x}\right) \tag{3.8}$$

c) f is twice slice-differentiable at x relatively to z if and only if f^* is twice slice-differentiable at z relatively to x. More precisely

$$\varphi^f_{t,x\,,\,z} \xrightarrow{\tau_s} f''_{x,z} \Leftrightarrow \varphi^{f^*}_{t,z\,,\,x} \xrightarrow{\tau_s^*} \left(f^*_{z,x}\right)''. \tag{3.9}$$

Moreover,

$$\left(f''_{x,z}\right)^* = \left(f^*_{z,x}\right)'' \tag{3.10}$$

Proof. part (a) : It is well-known in convex analysis. For closed proper convex function, one has the following equivalence :

$$z \in \partial f(x) \Leftrightarrow x \in \partial f^*(z) \Leftrightarrow f(x) + f^*(z) = \langle x,\ z\rangle.$$

part (b) : First we compute the conjugate function to $\varphi^f_{t,x\,,\,z}$ and we obtain the relation (3.8). Part (c) is a consequence of (b) and Theorem 2.2, which completes the proof. □

4 Proto-Derivatives

Throughout this section, X and Y are Banach spaces .

Definition 4.1. Let $\Gamma : X \rightrightarrows Y$ be any multifunction and let $x \in X$ with $\Gamma(x) \neq \phi$ and $y \in \Gamma(x)$. We consider the difference quotient multifunction :

$$\Delta^{\Gamma}_{t,x,y}(\xi) \;=\; \frac{1}{t}\,\{\Gamma(x+t\xi) - y\}\,;\; \xi \in X,\; (t>0) \tag{4.1}$$

If the graphs of $\Delta^{\Gamma}_{t,x,y}$ strongly-converge (as $t \downarrow 0$) to some multifunction $A : X \rightrightarrows Y$, then we shall say that Γ is proto- differentiable at x relatively to y, and A is called the proto-derivative of Γ at x relatively to y. We then write $\Gamma'_{x,y}$ instead of A, i.e.,

$$\Gamma'_{x,y} \;=\; G - \lim_{t\downarrow 0} \Delta^{\Gamma}_{t,x,y}$$

In terms of sequences ,

$$\Gamma'_{x,y} \;=\; G - \lim_{n\to+\infty} \Delta^{\Gamma}_{t_n,x,y}\,,\; \forall\, t_n \downarrow 0.$$

We easily prove that $0 \in \Gamma'_{x,y}(0)$ and $\Gamma'_{x,y}(\lambda\xi) = \lambda\Gamma'_{x,y}(\xi)$ for all $\xi \in X$ and $\lambda > 0$.

Theorem 4.2. *Let $f : X \to \overline{\mathbb{R}}$ be a closed proper convex function, $x \in X$ such that $f(x)$ is finite and $z \in X^*$. Then the following two conditions are equivalent*

(a) *f is twice slice-differentiable at x relatively to z.*

(b) *$z \in \partial f(x)$ and ∂f is proto-differentiable at x relatively to z, and the proto-derivative of ∂f at x relatively to z is the subdifferential of $f''_{x,z}$. More precisely,*

$$\partial\left(f''_{x,z}\right) \;=\; (\partial f)'_{x,z} \tag{4.2}$$

Proof. For any $t > 0$, let

$$\varphi^{f}_{t,x\,,\,z}(\xi) \;=\; \frac{1}{t^2}\,\{f(x+t\xi) - f(x) - \langle z, x\rangle\}\,;\; \xi \in X.$$

Then

$$\partial\varphi^{f}_{t,x\,,\,z}(\xi) = \frac{1}{t}\,\{f(x+t\xi) - z\} := \Delta^{\partial f}_{t,x,z}(\xi) \tag{4.3}$$

which is the difference quotient for the multifunction ∂f.

Now, suppose that f is twice slice-differentiable at x relatively to z. Then $z \in \partial f(x)$ by Proposition 3.4 and for every sequence $t_n \downarrow 0$, we have

$$f''_{x,z} = \tau_s - \lim_{n\to+\infty} \varphi^{f}_{t_n,x,z}$$

By (i) of Theorem 2.3, we have

$$\partial f''_{x,z} = G - \lim_{n\to+\infty} \partial\varphi^{f}_{t_n,x,z}$$

Hence, from (4.3) it follows that

$$\partial f''_{x,z} \ = \ G - \lim_{n\to+\infty} \Delta^{\partial f}_{t_n,x,z}$$

We conclude that the graph limit of $\Delta^{\partial f}_{t,x,z}$ exists and

$$\partial f^{''}_{x,z} \ = G \ - \lim_{t\downarrow 0} \Delta^{\partial f}_{t,x,z} \ = \ (\partial f)^{'}_{x,z}$$

This proves (a) and (4.2).

Conversely, suppose that (b) of the theorem holds, that is, $z \in \partial f(x)$ and there is a maximal monotone operator A such that

$$A \ = G - \lim_{n\to+\infty} \varphi^{\partial f}_{t_n,x,z} \ , \forall \ t_n \downarrow 0$$

Since the class of subdifferentials of closed proper convex functions is closed under Kuratowski-Painlevé convergence in the class of maximal monotone operators, one has $A \ = \ \partial\varphi$ for some closed proper convex function. Hence by (4.3) , we have

$$\partial\varphi \ = \ G - \lim_{n\to+\infty} \partial\varphi^{f}_{t_n,x,z} \tag{4.4}$$

In fact, since $z \in \partial f(x)$ then $0 \in \partial\varphi^{f}_{t_n,x,z}(0) = \Delta^{\partial f}_{t_n,x,z}(0)$ and by the convergence in (4.4) we obtain $0 \in \partial\varphi(0)$. Since $\varphi(0)$ is finite, we can assume that $\varphi(0) \ = 0$ and by definition (3.1), one has $\varphi^{f}_{t,x,z}(0) \ = 0$. Hence, we see that (ii) of theorem 2.3 holds for $\varphi^{f}_{t,x,z}$ and φ by taking $u_n = z_n = \ 0, \forall n \in \mathbb{N}$. To sum up, there exist $(0,0) \in \partial\varphi, (0,0) \in \partial\varphi^{f}_{t,x,z}$ such that $(0, \varphi(0), 0) = \ \lim_{n\to\infty}(u_n, \varphi^{f}_{t_n,x,z}(u_n), z_n)$.

Then by (4.4), we conclude from theorem 2.3 that, for any sequence $t_n \downarrow 0$

$$\varphi = \ \tau_s - \lim_{n\to+\infty} \varphi^{f}_{t_n,x,z} \ ; \ \text{i.e.}$$

$$\varphi = \ \tau_s - \lim_{t\downarrow 0} \varphi^{f}_{t,x,x^*}$$

with $\varphi(0) \ = \ 0$. This proves (b) , which completes the proof. □

Let us now consider the slice-diffferentiability properties of composite functions of the form $f \circ A$ with f convex and A linear in a normed space.

Proposition 4.3. *Let X, Y be two normed spaces and let $A : X \to Y$ be a linear operator isomorphism. Let $f \in \Gamma(x)$, and $x \in X$ such that $\partial f(A(x)) \neq 0$, and let $z \in \partial f(Ax)$.*

Then:

(a) $A^* z \in \partial (f \circ A)(x)$.

(b) *If f is twice slice-differentiable at Ax relatively to z, then $f \circ A$ is twice slice-differentiable at x relatively to $A^* z$. More precisely,*

$$(f \circ A)''_{x,A^* z}(\xi) = f''_{Ax,z}(A\xi) \, . \tag{4.5}$$

Proof. Part (a): For any $z \in \partial f(Ax)$, we have $f(\xi') > f(Ax) + \langle z, \xi' - Ax\rangle$.

Since A is an isomorphism, there exists $\xi \in X$ such that $A\xi = \xi'$ and

$$\begin{aligned}(f \circ A)(\xi) &> (f \circ A)(x) + \langle z, A\xi - Ax\rangle \\ &= (f \circ A)(x) + \langle z, A(\xi - x)\rangle \\ &= (f \circ A)(x) + \langle A^* z, \xi - x\rangle\end{aligned}$$

that is, $A^* z \in (f \circ A)(x)$.

Part (b): Let $\left(\varphi_t^{f\circ A}\right)_{x,A^*z}$, $\left(\varphi_t^{f}\right)_{Ax,z}$ be the difference quotients of $(f \circ A)$ (at x relatively to A^*z), f (at Ax relatively to z), respectively. Then for all $\xi \in X$

$$\begin{aligned}\left(\varphi_t^{f\circ A}\right)_{x,A^*z}(\xi) &= \frac{1}{t^2}\left\{(f \circ A)(x + t\xi) - (f \circ A)(x) - t\langle A^* z, \xi\rangle\right\} \\ &= \frac{1}{t^2}\left\{f(Ax + tA\xi) - f(Ax) - t\langle z, A\xi\rangle\right\} \\ &= \left(\varphi_t^{f}\right)_{Ax,z}(A\xi). \qquad (4.6)\end{aligned}$$

Now, we will examine the two conditions S1) and S2) for the slice epigraphical convergence:

S1) Suppose that f is twice slice-differentiable at Ax relatively to z. Let $(y, \eta) \in$ *epi* $\left(f''_{Ax,z} \circ A\right)^*$ with $\left(f''_{Ax,z} \circ A\right)^*(y) < \eta$. For each bounded sequence $(A\xi_n) = (\xi'_n)$, and for each $n \in \mathbb{N}$, we have

$$\begin{aligned}\langle y, \xi_n\rangle - \eta &< \left(f''_{Ax,z} \circ A\right)(\xi_n) \\ &= f''_{Ax,z}(\xi_n).\end{aligned}$$

For all sequence $t_n \downarrow 0$ and since f is twice slice-differentiable at Ax relatively to z, there exists $n_0 \in \mathbb{N}$ such that for each $n > n_0$, we have:

$$\left(\varphi_{t_n}^{f}\right)_{Ax,z}(A\xi_n) > \langle y, \xi_n\rangle - \eta \qquad (4.7)$$

Comparing (4.6) with (4.7), for all $t_n \downarrow 0$ and for each $n > n_0$, we obtain

$$\left(\varphi_{t_n}^{f\circ A}\right)_{x,A^*z}(\xi_n) > \langle y, \xi_n\rangle - \eta.$$

S2) Since f is twice slice-differentiable at Ax relatively to z, for each $\xi \in X$, there exists $(A\xi_n) = (\xi'_n)$ that converges strongly to $A\xi = \xi'$ for which

$$\begin{aligned}\left(f''_{Ax,z} \circ A\right)(\xi) &= \lim_n \left(\varphi_{t_n}^{f}\right)_{Ax,z}(A\xi_n) \\ &= \lim_n \left(\varphi_{t_n}^{f\circ A}\right)_{x,A^*z}(\xi_n).\end{aligned}$$

We conclude from S1) and S2) then $f \circ A$ is twice slice-differentiable at x relatively to A^*z, and its second-order slice derivative is given by (4.5), which completes the proof. □

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