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STABILITY FOR PARAMETRIC EXTENDED TRUST REGION SUBPROBLEMS*

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Abstract: Our purpose in this paper is to investigate stability for parametric extended trust region subproblem (ETRS), in which the trust region intersects an ellipsoid solid with many linear inequality constraints. We establish conditions for the continuity of the solution map, the stationary solution map and the optimal value function to parametric ETRS. The special structure of ETRS allows us to obtain deeper and sharper results on stability of this problem.

Key words: *upper semicontinuity, lower semicontinuity, stationary solution, extended trust region subproblem, stability*

Mathematics Subject Classification: *90C20, 90C30, 90C31*

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1 Introduction

Consider the following *extended trust region subproblem (ETRS)*

$$\begin{array}{ll} \min\ f(x;Q,c) := \frac{1}{2}x^TQx + c^Tx \\ \text{s.t. } x \in \mathbb{R}^n : x^TDx \le \alpha,\ Ax + b \le 0, \end{array} \qquad (ET(\omega))$$

where $Q, D \in \mathbb{R}^{n\times n}$ are symmetric, D is positive definite, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $\alpha > 0$ and $\omega = (Q, c, D, \alpha, A, b)$.

Model problems of this type are widespread in real-world applications such as: nonlinear programming problems with linear inequality constraints, nonlinear optimization problems with discrete variables [5, 24] and robust optimization problems under matrix norm or polyhedral uncertainty, optimal control and system theory (see [14, 29]).

Without the first constraint, ETRS reduces the *linearly constrained quadratic programming (LCQP) problem*. A survey on stability for parametric LCQP was investigated by Lee et al. [18].

The special case of ETRS, where D is the unit matrix and $m = 0$, is the well-known *trust region subproblem (TRS)*, which plays an important role in trust region methods for nonlinear optimization (see [7]). The stability for parametric TRSs has been concerned by many authors. Lee et al. [19] obtained necessary and sufficient conditions for the upper/lower semicontinuity of the stationary solution map and the global solution map, explicit formulas

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for computing the directional derivative and the Fréchet derivative of the optimal value function. The locally Lipschitz-like property of the stationary solution map was characterized in [20, 25].

ETRS is a generalization of TRS and LCQP. It is a common agreement that linear and pure quadratic forms are relatively easy but their combination is not. Recently, some topics related to ETRS have been investigated:

(i) Beck and Eldar [2], Jeyakumar and Li [14] showed the necessary and sufficient optimality conditions for the global optimality of ETRS;

(ii) Some methods to find the global solution for the general problem of ETRS have been proposed (see [6, 29, 30]);

(iii) ETRS with a linear constraint ($m = 1$) has been mentioned in the literature (see [6, 22, 29]).

Since D is positive definite, the feasible set of $ET(\omega)$ is convex and compact. This guarantees that the problem $ET(\omega)$ always has a solution. Therefore, for qualitative properties of ETRSs, one often concerns optimal conditions, sensitivity analysis and stability.

Stability for parametric ETRSs plays an important role because they can be used for analyzing algorithms for solving this problem. Several stability results in nonlinear optimization have obtained under the assumption MFCQ, for example, Gauvin and Dubeau [11] gave the sufficient condition for the continuity of the value function; Dontchev and Rockafellar [8] characterized the local upper-Lipschitz property of the stationary solution map at a point in its graph; many aspects of stability and sensitivity analysis for nonlinear program can be found in [3, 16, 17, 26]. Note that, MFCQ, Robinson condition, and Zowe-Kurcyusz condition are equivalent (see, [27, 34]). One of them may be viewed as an equivalent statement of metric regularity (see [3]) for a suitable multi-valued mapping. Various stability properties have investigated under the weaker assumptions like metric subregularity and calmness (see [9, 12]). Since ETRSs form a subclass of nonlinear optimization problems, many interesting results on stability for nonlinear optimization and parametric quadratic programming problem (see [15, 23, 32]) can be applied to parametric ETRSs. As far as we know, up to now, there have been a few researches which approach directly the continuity for parametric ETRS.

In this paper, we use the special structure of ETRS (the objective function is quadratic and the trust region intersects an ellipsoid solid with many linear inequality constraints) to investigate in detail the stability of this problem. We obtain deeper and sharper results on the continuity of the global solution map, the local solution map, the stationary solution map and the optimal value function to parametric ETRSs with several illustrating examples. Our results develop and complement the published ones in [18, 19]. The approach adopted herein is quite different from that used in [20, 22, 25].

The rest of the paper is organized as follows. Section 2 gives some preliminaries. In Section 3, we characterize the continuity of the global solution and local solution maps. The necessary and/or sufficient conditions for lower and/or upper semicontinuity of the optimal value function are established in Section 4. Finally, in Section 5, we present necessary and/or sufficient conditions for lower and/or upper semicontinuities of the stationary solution map.

2 Preliminaries

The following notation will be adopted. Let $\mathbb{R}^n$ be n-dimensional Euclidean space equipped with the standard scalar product and the Euclidean norm, $\mathbb{R}_+$ be the set of negative real numbers, $\mathbb{R}_S^{n\times n}$ be the space of real symmetric $(n \times n)$-matrices equipped with the matrix norm induced by the vector norm in $\mathbb{R}^n$ and $\mathbb{R}_{S+}^{n\times n}$ be the set of positive definite real symmetric $(n\times n)$–matrices. Let

$$\Omega := \mathbb{R}_S^{n\times n} \times \mathbb{R}^n \times \mathbb{R}_{S+}^{n\times n} \times (\mathbb{R}_+ \setminus \{0\}) \times \mathbb{R}^{m\times n} \times \mathbb{R}^m \subset \mathbb{R}^s,$$

with $s = 2n^2 + mn + m + n + 1$ and $\omega = (Q, c, D, \alpha, A, b) \in \Omega$.

We denote by $\mathcal{F}(D, \alpha, A, b)$ the solution set of the following system

$$g_0(x; D, \alpha) := x^T Dx - \alpha \le 0,\ g_i(x; A_i, b_i) := A_i x + b_i \le 0,\ i = 1, \dots, m. \tag{2.1}$$

The *set of global solutions*, the *set of local solutions* and the *set of isolated local solutions* of $(ET(\omega))$ will be denoted by $G(\omega)$, $L(\omega)$ and $IL(\omega)$, respectively.

The function

$$\varphi : \Omega \longrightarrow \mathbb{R} \cup \{\pm\infty\}$$

defined by

$$\varphi(\omega) = \begin{cases} \inf\{f(x; Q, c) : x \in \mathcal{F}(D, \alpha, A, b)\} & \text{if } \mathcal{F}(D, \alpha, A, b) \neq \emptyset; \\ +\infty & \text{if } \mathcal{F}(D, \alpha, A, b) = \emptyset, \end{cases}$$

is called *the optimal value function* of $(ET(\omega))$.

We say that $(ET(\omega))$ satisfies *Slater's condition* if there exists $x^0 \in \mathbb{R}^n$ such that $g_0(x^0; D, \alpha) < 0$ and $g_i(x^0; A_i, b_i) < 0$ for every $i = 1, \dots, m$.

For some $\bar{x} \in \mathcal{F}(D, \alpha, A, b)$, let us define

$$I(\bar{x}; D, \alpha, A, b) := \{i \in \{0, 1, \dots, m\} : g_i(\bar{x}; D, \alpha, A, b) = 0\},$$

the *active constraint index* set.

The Mangasarian-Fromovitz Constraint Qualification (MFCQ) holds at $\bar{x} \in \mathcal{F}(D, \alpha, A, b)$ if there exists $v^0 \in \mathbb{R}^n$ such that

$$\nabla g_i(\bar{x}; D, \alpha, A, b)^T v^0 < 0 \ \ \forall i \in I(\bar{x}; D, \alpha, A, b).$$

Since $g_0(.; D, \alpha)$ and $g_i(.; A_i, b_i),\ i = 1, \dots, m$ are convex, $(ET(\omega))$ satisfies Slater's condition if and only if it satisfies MFCQ (see [31]).

Recall [17] that x is a *stationary solution* of $(ET(\omega))$ if there exists an $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}^m$ (called a Lagrange multiplier corresponding to x) satisfying the following *Karush-Kuhn-Tucker (KKT) condition*:

$$(Q + \lambda D)x + A^T \mu + c = 0, \tag{2.2}$$

$$\lambda \ge 0,\ \mu \ge 0,\ x^T Dx - \alpha \le 0,\ Ax + b \le 0, \tag{2.3}$$

$$\lambda(x^T Dx - \alpha) = 0,\ \mu_i(A_i x + b_i) = 0,\ i = 1, \dots, m. \tag{2.4}$$

The triplet (x, λ, μ) satisfying (2.2)–(2.4) is called a *KKT pair* of $(ET(\omega))$. Note that, for each stationary solution x, MFCQ is equivalent to the set of all Lagrange multipliers corresponding to x is bounded (see [10]).

The stationary solution set of $(ET(\omega))$ is denoted by $S(\omega)$. It is well-known that (see

[10]), under Slater's condition,

$$\emptyset \neq G(\omega) \subset L(\omega) \subset S(\omega) \subset \mathcal{F}(D, \alpha, A, b).$$

We next recall the notion of upper semicontinuity and lower semicontinuity of multifunctions.

A multifunction $F : X \subset \mathbb{R}^s \rightrightarrows \mathbb{R}^n$ is said to be *upper semicontinuous* at $\bar{p} \in \mathbb{R}^s$ if for each open set V containing $F(\bar{p})$ there exists $\delta > 0$ such that $F(p) \subset V$ for every $p \in \mathbb{R}^s$ satisfying $\|p - \bar{p}\| < \delta$.

A multifunction $F : X \subset \mathbb{R}^s \rightrightarrows \mathbb{R}^n$ is said to be *lower semicontinuous* at $\bar{p} \in \mathbb{R}^n$ if $F(\bar{p}) \neq \emptyset$ and, for each open set V satisfying $F(\bar{p}) \cap V \neq \emptyset$, there exists $\delta > 0$ such that $F(p) \cap V \neq \emptyset$ for every $p \in \mathbb{R}^s$ satisfying $\|p - \bar{p}\| < \delta$.

The notion of upper semicontinuity stated herein is quite standard (see, for instance, [33, p. 451]). It is very different from the concept of stability considered by Gowda and Pang [13, Def. 1]. The notion of lower semicontinuity stated herein agrees with that considered in [33, p. 451], but differs slightly from the one given in [1, p. 39].

Note that, Rockafellar and Wets [28] mentioned the relations between upper/lower semicontinuity and *outer/inner semicontinuity*. According to [28], the lower semicontinuity agrees with the inner semicontinuity, but the upper semicontinuity differs from the outer semicontinuity. For the multifunctions F into compact spaces, the upper semicontinuity is equivalent to the outer semicontinuity. If the multifunction F is not locally bounded, the upper semicontinuity and the outer semicontinuity are not equivalent. The outer and inner semicontinuities of the feasible multifunction $\mathcal{F}$ of $(ET(\omega))$ are implied from [28, Example 5.10]. In addition, lower semicontinuity of $\mathcal{F}$ is also showed in [21].

Since D is positive definite, the feasible set $\mathcal{F}(D, \alpha, A, b)$ of $ET(\omega)$ is convex and compact. This guarantees that the problem $ET(\omega)$ always has a global solution whenever $\mathcal{F}(D, \alpha, A, b)$ is nonempty. Therefore, for qualitative properties of ETRSs, one often concerns optimal conditions, sensitivity analysis and stability.

In what follows we fix $\bar{\omega} := (\bar{Q}, \bar{c}, \bar{D}, \bar{\alpha}, \bar{A}, \bar{b}) \in \Omega$.

3 Continuity of Global and Local Solution Maps

In this section, we characterize continuity of global and local solution maps problems parametric $(ET(\omega))$ under total perturbations. The continuity of global solution map G is presented in the following theorem.

Theorem 3.1. *The following assertions hold:*

(i) *G is upper semicontinuous at $\bar{\omega}$ if $(ET(\bar{\omega}))$ satisfies Slater's condition;*

(ii) *G is lower semicontinuous at $\bar{\omega}$ if and only if $(ET(\bar{\omega}))$ satisfies Slater's condition and $G(\bar{\omega})$ is a singleton;*

(iii) *G is continuous at $\bar{\omega}$ if and only if $(ET(\bar{\omega}))$ satisfies Slater's condition and $G(\bar{\omega})$ is a singleton.*

Proof. (i) Suppose that G is not upper semicontinuous at $\bar{\omega}$, that is, there exist an open set V containing $\mathrm{G}(\bar{\omega})$, a sequence $\{\omega^k\}$ converging to $\bar{\omega}$ and a sequence $\{x^k\}$ satisfying $x^k \in \mathrm{G}(\omega^k) \setminus V$ for every $k \in \mathbb{N}$. We claim that $\{x^k\}$ is bounded. Indeed, if $\{x^k\}$ is unbounded then we may assume that $x^k \neq 0$ for all k and $\|x^k\| \to \infty$ as $k \to \infty$. Then, $\{x^k/\|x^k\|\}$

has a convergent subsequence. Without loss of generality, we may assume that $\{x^k/\|x^k\|\}$ itself converges to some $\bar{v}$ with $\bar{v} \neq 0$. Since D^k is positive definite, $0 \leq (x^k)^T D^k x^k \leq \alpha^k$. Dividing this inequality by $\|x^k\|^2$ and letting $k \to \infty$, we obtain $\bar{v}^T \bar{D} \bar{v} = 0$. Hence, $\bar{v} = 0$ since $\bar{D}$ is positive definite. This contradicts the fact that $\bar{v} \neq 0$. Thus, $\{x^k\}$ is bounded. Without loss of generality, we may assume that $x^k \to \bar{x}$ for some $\bar{x} \in \mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b})$.

Since $(ET(\bar{\omega}))$ satisfies Slater's condition, $\mathcal{F}$ is lower semicontinuous at $\bar{\omega}$ (see [21, Lemma 4]). Hence, for each $z \in \mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b})$, there exists z^k converging to z such that $z^k \in \mathcal{F}(D^k, \alpha^k, A^k, b^k)$ for all $k \in \mathbb{N}$. From $x^k \in \mathrm{G}(\omega^k)$ it follows $f(x^k; Q^k, c^k) \leq f(z^k; Q^k, c^k)$. Letting $k \to \infty$ yields $f(\bar{x}; \bar{Q}, \bar{c}) \leq f(z; \bar{Q}, \bar{c})$. Hence, $\bar{x} \in \mathrm{G}(\bar{\omega}) \subset V$. This contradicts the fact that V is open and $x^k \notin V$ for all k.

(ii) *Necessity:* Suppose that G is lower semicontinuous at $\bar{\omega}$ but the number of elements of $G(\bar{\omega})$ is greater than 1, that is, there exist $\bar{x}, \bar{y} \in G(\bar{\omega})$ such that $\bar{x} \neq \bar{y}$. We choose $\tilde{c} \in \mathbb{R}^n$ such that $\|\tilde{c}\| = 1$ and

$$\tilde{c}^T \bar{x} > \tilde{c}^T \bar{y}. \tag{3.1}$$

Then, there exists an open set V containing $\bar{x}$ such that

$$\tilde{c}^T x > \tilde{c}^T \bar{y} \ \ \forall x \in V. \tag{3.2}$$

For given $\gamma > 0$, let $t \in (0, \gamma)$ and $\omega^t = (\bar{Q}, c^t, \bar{D}, \bar{\alpha}, \bar{A}, \bar{b})$ with $c^t = \bar{c} + t\tilde{c}$. Then, $\|\omega^t - \bar{\omega}\| = \|c^t - \bar{c}\| = t < \gamma$. We show $G(\omega^t) \cap V = \emptyset$. Indeed, for any $z \in \mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b}) \cap V$, from (3.2) it follows that

$$\frac{1}{2} z^T \bar{Q} z + (c^t)^T z > \frac{1}{2} \bar{x}^T \bar{Q} \bar{x} + \bar{c}^T \bar{x} + t \tilde{c}^T \bar{y} = \frac{1}{2} \bar{y}^T \bar{Q} \bar{y} + (c^t)^T \bar{y}.$$

This leads to $z \notin G(\omega^t)$. Hence, for the chosen neighborhood V of $\bar{x} \in G(\bar{\omega})$ and for every $\gamma > 0$, there exists $c^t \in \mathbb{R}^n$ satisfying $\|c^t - \bar{c}\| < \gamma$ and $G(\omega^t) \cap V = \emptyset$. This contradicts the assumption that G is lower semicontinuous at $\bar{\omega}$. Thus, $G(\bar{\omega})$ is a singleton.

Now, suppose that $(ET(\bar{\omega}))$ does not satisfy Slater's condition. Then, there exists a sequence $\{\omega^k\} \subset \Omega$ converging to $\bar{\omega}$ such that $\mathcal{F}(D^k, \alpha^k, A^k, b^k) = \emptyset$ for every k (see [21, Lemma 5]). This follows $G(\omega^k) = \emptyset$ for every k, contrary to the assumption that G is lower semicontinuous at $\bar{\omega}$.

Sufficiency: Suppose that $(ET(\bar{\omega}))$ satisfies Slater's condition and $G(\bar{\omega})$ is a singleton. Let U be an open set containing the unique solution $\bar{x} \in \mathrm{G}(\bar{\omega})$. By [21, Lemma 4], there exists $\varepsilon > 0$ such that $\mathcal{F}(\tilde{D}, \tilde{\alpha}, \tilde{A}, \tilde{b}) \neq \emptyset$ for every $\tilde{\omega}$ satisfying $\|\tilde{\omega} - \bar{\omega}\| < \varepsilon$. Hence, for such a ε, $\mathrm{G}(\tilde{\omega}) \neq \emptyset$. From (i) it follows that G is upper semicontinuous at $\bar{\omega}$, that is, $\mathrm{G}(\tilde{\omega}) \subset U$ for every $\tilde{\omega}$ satisfying $\|\tilde{\omega} - \bar{\omega}\| < \varepsilon$ for $\varepsilon > 0$ small enough. This follows that G is lower semicontinuous at $\bar{\omega}$.

(iii) This assertion follows from (i) and (ii).

The proof is complete. □

The following example is an illustration for Theorem 3.1.

Example 3.2. Consider $(ET(\bar{\omega}))$ with $n = 2, m = 2$,

$$\bar{Q} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\bar{D} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{\alpha} = 1, \quad \bar{A}_1^T = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \bar{b}_1 = 0, \quad \bar{A}_2^T = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \bar{b}_2 = -1.$$

Rewriting the above problem as follows

$$\min \left\{ f(x; \bar{Q}, \bar{c}) = -\frac{1}{2}x_1^2 + x_2 : x \in \mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b}) \right\},$$

where

$$\mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b}) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 1; -x_1 + x_2 \le 0; x_1 + x_2 - 1 \le 0\}.$$

Clearly, $(ET(\bar{\omega}))$ satisfies Slater's condition. It is easy to check that $G(\bar{\omega}) = \{(0, -1)\}$. Hence, by Theorem 3.1, the global solution map G is continuous at $\bar{\omega}$.

In the next result, we propose a necessary and sufficient condition for the lower semicontinuity of the local solution map L.

Theorem 3.3. *The local solution map L is lower semicontinuous at $\bar{\omega}$ if and only if $(ET(\bar{\omega}))$ satisfies Slater's condition and the set of local solutions coincides with the set of isolated local solutions, i.e., $L(\bar{\omega}) = IL(\bar{\omega})$.*

Proof. Necessity: Suppose that L is lower semicontinuous at $\bar{\omega}$ but $(ET(\bar{\omega}))$ does not satisfy Slater's condition. According to [21, Lemma 5], there exists a sequence $\omega^k \in \Omega$ converging to $\bar{\omega}$ such that $\mathcal{F}(D^k, \alpha^k, A^k, b^k) = \emptyset$ for every k. This implies $L(\omega^k) = \emptyset$ for every k, contrary to the assumption that L is lower semicontinuous at $\bar{\omega}$.

Suppose that $L(\bar{\omega}) \neq IL(\bar{\omega})$. Since $IL(\bar{\omega}) \subset L(\bar{\omega})$, there exists $\bar{x} \in L(\bar{\omega}) \setminus IL(\bar{\omega})$. Then, there are a open set U containing $\bar{x}$ such that $f(z; \bar{Q}, \bar{c}) \ge f(\bar{x}; \bar{Q}, \bar{c})$ for every $z \in \mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b}) \cap U$ and a local solution $\bar{y} \in L(\bar{\omega}) \cap U$ such that $\bar{x} \neq \bar{y}$. We can choose $\tilde{c} \in \mathbb{R}^n$ such that $\tilde{c}^T\bar{x} > \tilde{c}^T\bar{y}$ and $\|\tilde{c}\| = 1$. There exists a neighborhood V of $\bar{x}$ such that

$$\tilde{c}^T x > \tilde{c}^T \bar{y} \ \ \forall x \in V. \tag{3.3}$$

Let $W := U \cap V$. For given $\delta > 0$, let $t \in (0, \delta)$ and $\omega^t = (\bar{Q}, c^t, \bar{D}, \bar{\alpha}, \bar{A}, \bar{b})$ with $c^t = \bar{c} + t\tilde{c}$. Then, $\|\omega^t - \bar{\omega}\| = \|c^t - \bar{c}\| = t < \delta$. For every $z \in \mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b}) \cap W$, from (3.3), we obtain that

$$\begin{aligned} \tfrac{1}{2}z^T\bar{Q}z + (c^t)^T z &= \tfrac{1}{2}z^T\bar{Q}z + \bar{c}^T z + t\tilde{c}^T z \ge \tfrac{1}{2}\bar{x}^T\bar{Q}\bar{x} + \bar{c}^T\bar{x} + t\tilde{c}^T z \\ &> \tfrac{1}{2}\bar{x}^T\bar{Q}\bar{x} + \bar{c}^T\bar{x} + t\tilde{c}^T\bar{y} = \tfrac{1}{2}\bar{y}^T\bar{Q}\bar{y} + (c^t)^T\bar{y}. \end{aligned}$$

Hence, $z \notin L(\omega^t)$. This leads to $L(\omega^t) \cap W = \emptyset$. Hence, for the chosen neighborhood W of $\bar{x} \in L(\bar{\omega})$ and for every $\delta > 0$, there exists $c^t \in \mathbb{R}^n$ satisfying $\|c^t - \bar{c}\| < \delta$ such that $L(\omega^t) \cap W = \emptyset$. This contradicts the assumption that L is lower semicontinuous at $\bar{\omega}$. Thus, $L(\bar{\omega}) = IL(\bar{\omega})$.

Sufficiency: Suppose that $(ET(\bar{\omega}))$ satisfies Slater's condition and $L(\bar{\omega}) = IL(\bar{\omega})$. Take any open set $V \subset \mathbb{R}^n$ such that $L(\bar{\omega}) \cap V \neq \emptyset$. Fix $\bar{x} \in V \cap L(\bar{\omega})$. Since $L(\bar{\omega}) = IL(\bar{\omega})$, $\bar{x} \in V \cap IL(\bar{\omega})$. Hence there exists a open ball $B(\bar{x}, \epsilon) \subset V$ such that $f(x; \bar{Q}, \bar{c}) > f(\bar{x}; \bar{Q}, \bar{c})$ for every $x \in (\mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b}) \cap B(\bar{x}, \epsilon)) \setminus \{\bar{x}\}$. It follows that $\bar{x}$ is the unique global solution of the following problem

$$\min\{f(x;\bar{Q},\bar{c}) : x \in \mathcal{F}(\bar{D},\bar{\alpha},\bar{A},\bar{b}) \cap \bar{B}(\bar{x},\epsilon/2)\}, \tag{P}$$

where $\bar{B}(\bar{x},\epsilon/2)$ is the closure of $B(\bar{x},\epsilon/2)$.

Next, we show that the global solution map G_P of the problem (P) is upper semicontinuous at $\bar{\omega}$. Indeed, suppose that G_P is not upper semicontinuous at $\bar{\omega}$, that is, there exist an open set W containing $G_P(\bar{\omega})$, a sequence $\{\omega^k\}$ converging to $\bar{\omega}$ and a sequence $\{y^k\}$ satisfying $y^k \in G_P(\omega^k) \setminus W \quad$ for every $\; k \in \mathbb{N}$. Since $\{y^k\}$ is bounded, without loss of generality, we may assume that $y^k \to \bar{y}$ for some $\bar{y} \in \mathbb{R}^n$. From $y^k \in G_P(\omega^k)$ it follows that $y^k \in \mathcal{F}(D^k,\alpha^k,A^k,b^k) \cap \bar{B}(\bar{x},\epsilon/2)$. Letting $k \to \infty$, we have $\bar{y} \in \mathcal{F}(\bar{D},\bar{\alpha},\bar{A},\bar{b}) \cap \bar{B}(\bar{x},\epsilon/2)$. Since $(ET(\bar{\omega}))$ satisfies Slater's condition, $\mathcal{F}$ is lower semicontinuous at $\bar{\omega}$ (see [21, Lemma 4]). Hence, for $\bar{x} \in \mathcal{F}(\bar{D},\bar{\alpha},\bar{A},\bar{b}) \cap B(\bar{x},\epsilon/2)$, there exists x^k converging to $\bar{x}$ such that $x^k \in \mathcal{F}(D^k,\alpha^k,A^k,b^k) \cap B(\bar{x},\epsilon/2)$ for all $k \in \mathbb{N}$. From $y^k \in \mathrm{G}_P(\omega^k)$ it follows $f(y^k;Q^k,c^k) \le f(x^k;Q^k,c^k)$. Letting $k \to \infty$ yields $f(\bar{y};\bar{Q},\bar{c}) \le f(\bar{x};\bar{Q},\bar{c})$. Since $\bar{x}$ is the unique global solution of (P), we have $\bar{y} = \bar{x}$. Hence, $\bar{y} \in G_P(\bar{\omega}) \subset W$. This contradicts the fact that W is open and $y^k \notin W$ for all k. Therefore G_P is upper semicontinuous at $\bar{\omega}$.

Take any open set U such that $G_P(\bar{\omega}) \cap U \neq \emptyset$. By the lower semicontinuity of $\mathcal{F}$ and by $\mathcal{F}(\bar{\omega}) \cap B(\bar{x},\epsilon/2) \neq \emptyset$, there exists $\delta_1 > 0$ such that $\mathcal{F}(\tilde{D},\tilde{\alpha},\tilde{A},\tilde{b}) \cap B(\bar{x},\epsilon/2) \neq \emptyset$ for every $\tilde{\omega}$ satisfying $\|\tilde{\omega} - \bar{\omega}\| < \delta_1$. Hence, for such δ_1, we have $G_P(\tilde{\omega}) \neq \emptyset$. Since G_P is upper semicontinuous at $\bar{\omega}$, $G_P(\tilde{\omega}) \subset U$ for every $\tilde{\omega}$ satisfying $\|\tilde{\omega} - \bar{\omega}\| < \delta_1$ for $\delta_1 > 0$ small enough. This follows that G_P is lower semicontinuous at $\bar{\omega}$.

Since $G_P(\bar{\omega}) \cap B(\bar{x},\epsilon/2) \neq \emptyset$, there exists $\delta_2 > 0$ such that $G_P(\tilde{\omega}) \cap B(\bar{x},\epsilon/2) \neq \emptyset$ for every $\tilde{\omega} \in \Omega$ satisfying $\|\tilde{\omega} - \bar{\omega}\| < \delta_2$. This leads to $L(\tilde{\omega}) \cap V \neq \emptyset$ for every $\tilde{\omega} \in \Omega$ satisfying $\|\tilde{\omega} - \bar{\omega}\| < \delta_2$. Therefore L is lower semicontinuous at $\bar{\omega}$. The proof is complete. □

Next example tells us that the local solution map L is not lower semicontinuous if there exists a local solution which is not an isolated local solution.

Example 3.4. Consider $(ET(\bar{\omega}))$ with $n = 2, m = 0$,

$$\bar{Q} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad \bar{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \bar{D} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \bar{\alpha} = 2.$$

This problem can be rewritten as follows

$$\min\left\{ f(x;\bar{Q},\bar{c}) = \frac{1}{2}(-x_1^2 - 2x_2^2) : x \in \mathcal{F}(\bar{D},\bar{\alpha},\bar{A},\bar{b}) \right\},$$

where

$$\mathcal{F}(\bar{D},\bar{\alpha},\bar{A},\bar{b}) = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + 2x_2^2 \le 2\}.$$

Let $T := \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + 2x_2^2 = 2\}$. We have

$$f(x;\bar{Q},\bar{c}) = \frac{1}{2}(-x_1^2 - 2x_2^2) \ge \frac{1}{2}(-2) = -1,$$

and

$$f(x;\bar{Q},\bar{c}) = -1 \quad \forall x \in T.$$

Hence, $\emptyset \neq T \subseteq G(\bar{\omega}) \subseteq L(\bar{\omega})$ and $T \nsubseteq IL(\bar{\omega})$. This implies $L(\bar{\omega}) \neq IL(\bar{\omega})$. From Theorem 3.3 it follows that the local solution map L is not lower semicontinuous at $\bar{\omega}$.

4 Continuity of the Optimal value Function

Properties of the optimal value function play an important role in parametric nonlinear programming problems. In this section, we use the special structure of ETRS to obtain some results for the continuity of the optimal value function φ. The main result is presented in the following theorem.

Theorem 4.1. *The following assertions hold:*

(i) *φ is lower semicontinuous at $\bar{\omega}$;*

(ii) *φ is upper semicontinuous at $\bar{\omega}$ if $(ET(\bar{\omega}))$ satisfies Slater's condition;*

(iii) *φ is continuous at $\bar{\omega}$ if $(ET(\bar{\omega}))$ satisfies Slater's condition;*

(iv) *If $\mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b})$ is nonempty and φ is continuous at $\bar{\omega}$, then $(ET(\bar{\omega}))$ satisfies Slater's condition;*

(v) *If $\mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b})$ is empty, then φ is continuous at $\bar{\omega}$.*

Proof. (i) Let any sequence $\{\omega^k\} \subset \Omega$ such that $\omega^k \to \bar{\omega}$. We have to show that $\liminf_{k\to\infty} \varphi(\omega^k) \geq \varphi(\bar{\omega})$.

Suppose, on the contrary, that

$$\liminf_{k\to\infty} \varphi(\omega^k) < \varphi(\bar{\omega}).$$

Without loss of generality, we may assume that

$$\liminf_{k\to\infty} \varphi(\omega^k) = \lim_{k\to\infty} \varphi(\omega^k).$$

Then, there exist a real number $\delta > 0$ and an index k_0 such that

$$\varphi(\omega^k) \leq \delta < \varphi(\bar{\omega})$$

for every $k \geq k_0$. Since $\varphi(\omega^k) < +\infty$, we obtain that $\mathcal{F}(\omega^k) \neq \emptyset$ and $G(\omega^k) \neq \emptyset$. Hence, there exists a bounded sequence $\{x^k\}$ such that $x^k \in \mathcal{F}(\omega^k)$. We may assume that this sequence itself converges to a vector $\hat{x} \in \mathcal{F}(\bar{\omega})$. Since $x^k \in \mathcal{F}(\omega^k)$, we have

$$\varphi(\omega^k) = f(x^k; Q^k, c^k) \leq \delta.$$

Letting $k \to \infty$, we get $f(\hat{x}; \bar{Q}, \bar{c}) \leq \delta$. Hence,

$$\delta < \varphi(\bar{\omega}) \leq f(\hat{x}; \bar{Q}, \bar{c}) \leq \delta.$$

This is impossible. Therefore, $\liminf_{k\to\infty} \varphi(\omega^k) \geq \varphi(\bar{\omega})$.

(ii) Suppose that $(ET(\bar{\omega}))$ satisfies Slater's condition. Let any sequence $\{\omega^k\} \subset \Omega$ such that $\omega^k \to \bar{\omega}$. Since $(ET(\bar{\omega}))$ satisfies Slater's condition, we get $\mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b}) \neq \emptyset$ and $G(\bar{\omega}) \neq \emptyset$. Hence, there exists $\bar{x} \in G(\bar{\omega})$ such that $\varphi(\bar{\omega}) = f(\bar{x}; \bar{Q}, \bar{c})$.

According to [21, Lemma 4], the set-valued map $(D, \alpha, A, b) \mapsto \mathcal{F}(D, \alpha, A, b)$ is lower semicontinuous at $\bar{\omega}$. Thus, there exists $x^k \in \mathcal{F}(D^k, \alpha^k, A^k, b^k)$ such that $x^k \to \bar{x}$. We have

$$\limsup_{k\to\infty} \varphi(\omega^k) \leq \limsup_{k\to\infty} f(x^k; Q^k, c^k) = f(\bar{x}; \bar{Q}, \bar{c}) = \varphi(\bar{\omega}).$$

Therefore, φ is continuous at $\bar{\omega}$.

(iii) The desired conclusion follows from (i) and (ii).

(iv) Suppose that $\mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b})$ is nonempty, φ is continuous at $\bar{\omega}$ and $(ET(\bar{\omega}))$ does not satisfy Slater's condition. By [21, Lemma 5], there exists $\omega^k \to \bar{\omega}$ such that, for each k, $\mathcal{F}(D^k, \alpha^k, A^k, b^k) = \emptyset$. Then, $\varphi(\omega^k) = +\infty$. Since φ is continuous at $\bar{\omega}$, we obtain $\varphi(\bar{\omega}) = +\infty$, that is $\mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b}) = \emptyset$. This contradicts the assumption that $\mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b}) \neq \emptyset$. Therefore, $(ET(\bar{\omega}))$ satisfies Slater's condition.

(v) Suppose that the set $\mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b})$ is empty. We shall show that, for every sequence $\{\omega^k = (Q^k, c^k, D^k, \alpha^k, A^k, b^k)\} \subset \Omega$ converging to $\bar{\omega}$,

$$\liminf_{k\to\infty} \varphi(\omega^k) = +\infty.$$

Suppose that $\liminf_{k\to\infty} \varphi(\omega^k) < +\infty$. Then, there exist a number $\beta > 0$ and a subsequence $\{\omega^s\}$ of $\{\omega^k\}$ such that

$$\varphi(\omega^s) \leq \beta \ \ \forall s \in \mathbb{N}.$$

This leads to $G(\omega^s) \neq \emptyset$. Hence, for each s, there exits $x^s \in G(\omega^s)$, that is,

$$\varphi(\omega^s) \leq \beta;\ (x^s)^T D^s x^s \leq \alpha^s;\ A^s x^s + b^s \leq 0. \tag{4.1}$$

Since $\{x^s\}$ is bounded, without loss of generality, we may assume that $\{x^s\}$ converges to $\hat{x} \in \mathbb{R}^n$. Passing the second and the third inequalities in (4.1) to limits as $s \to +\infty$, we obtain that

$$\hat{x}^T D\hat{x} \leq \bar{\alpha};\ \bar{A}\hat{x} + \bar{b} \leq 0.$$

This follows $\mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b}) \neq \emptyset$, contrary to the assumption $\mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b}) = \emptyset$.

The proof is complete. □

Remark 4.2. By Theorem 4.1, under the assumption $\mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b}) \neq \emptyset$, we obtain the continuity of the optimal value function φ by checking whether $(ET(\bar{\omega}))$ satisfies Slater's condition. The advantage of proposed conditions is that they are verifiable.

Remark 4.3. For *unconstrained optimization* problems, under the assumption that the objective function $f(x; Q, q)$ is level-bounded in x locally uniformly in (Q, q), the optimal value function is lower semicontinuous (see [28, Theorem 1.17]).

5 Continuity of the Stationary Solution Map

In this section, we characterize the upper and lower semicontinuities of the stationary solution map of parametric ERTSs. The following theorem gives a sufficient condition for the upper semicontinuity of S.

Theorem 5.1. *S is upper semicontinuous at $\bar{\omega}$ if $(ET(\bar{\omega}))$ satisfies Slater's condition.*

Proof. Suppose, contrary to our claim, that S is not upper semicontinuous at $\bar{\omega}$, that is, there exist an open set U containing $S(\bar{\omega})$, a sequence $\{\omega^k = (Q^k, c^k, D^k, \alpha^k, A^k, b^k)\} \subset \Omega$ converging to $\bar{\omega}$ and a sequence $\{x^k\}$ with $x^k \in S(\omega^k) \setminus U$. Since $x^k \in S(\omega^k)$, there exists $(\lambda^k, \mu^k) \in \mathbb{R} \times \mathbb{R}^m$ satisfying:

$$(Q^k + \lambda^k D^k)x^k + (A^k)^T \mu^k + c^k = 0, \tag{5.1}$$

$$\lambda^k \geq 0,\ \mu^k \geq 0,\ (x^k)^T D^k x^k - \alpha^k \leq 0,\ A^k x^k + b^k \leq 0, \tag{5.2}$$

$$\lambda^k\big((x^k)^T D^k x^k - \alpha^k\big) = 0,\ \mu_i^k(A_i^k x^k + b_i^k) = 0,\ i = 1, \ldots, m. \tag{5.3}$$

By the assumption that $(ET(\bar{\omega}))$ satisfies Slater's condition, we conclude that $(ET(\omega^k))$ also satisfies Slater's condition for every k large enough. We claim that the sequence $\{\lambda^k, \mu^k\}$ of KKT multipliers bounded. Indeed, suppose that $\{\lambda^k, \mu^k\}$ is unbounded. Then we may assume that $(\lambda^k, \mu^k) \neq 0$ for every k and $\|(\lambda^k, \mu^k)\| \to \infty$ as $k \to \infty$. Hence $\{(\lambda^k, \mu^k)/\|(\lambda^k, \mu^k)\|\}$ has a convergent subsequence. Without loss of generality, we may assume that $\{(\lambda^k, \mu^k)/\|(\lambda^k, \mu^k)\|\}$ itself converges to some $(\bar{\lambda}, \bar{\mu})$ with $(\bar{\lambda}, \bar{\mu}) \neq 0$. By repeating the argument as in the proof of Theorem 3.1, we obtain that $\{x^k\}$ is bounded. Hence, there exists a sequence $\{k_j\} \subset \{k\}$ such that $x^{k_j} \to \bar{x}$ for some $\bar{x} \in \mathbb{R}^n$. Without loss of generality, we may assume that $\{k_j\} \equiv \{k\}$. Dividing both sides of (5.1)–(5.3) by $\|(\lambda^k, \mu^k)\|$ and letting $k \to \infty$ yields

$$\bar{\lambda}\bar{D}\bar{x} + \bar{\mu}^T \bar{A} = 0,$$

$$\bar{\lambda} \geq 0,\ \bar{\mu} \geq 0,\ \bar{x}^T \bar{D}\bar{x} - \bar{\alpha} \leq 0,\ \bar{A}\bar{x} + \bar{b} \leq 0,$$

$$\bar{\lambda}\big(\bar{x}^T \bar{D}\bar{x} - \bar{\alpha}\big) = 0,\ \bar{\mu}_i(\bar{A}_i\bar{x} + \bar{b}_i) = 0.$$

These give

$$\sum_{i \in I(\bar{x}; \bar{D}, \bar{\alpha}, \bar{A}, \bar{b})} \bar{\mu}_i \nabla g_i(\bar{x}; \bar{D}, \bar{\alpha}, \bar{A}, \bar{b}) = 0 \tag{5.4}$$

with $\bar{\mu}_0 = \bar{\lambda}$. On other hand, since $(ET(\bar{\omega}))$ satisfies Slater's condition, there exists $v^0 \in \mathbb{R}^n$ such that

$$\nabla g_i(\bar{x}; D, \alpha, A, b)^T v^0 < 0 \ \ \forall i \in I(\bar{x}; D, \alpha, A, b). \tag{5.5}$$

Combining (5.4) and (5.5) gives $\bar{\mu}_i = 0$ for every $i \in \{0, 1, 2, \ldots, m\}$, contrary to the fact that $(\bar{\lambda}, \bar{\mu}) \neq 0$. Thus $\{\lambda^k, \mu^k\}$ of KKT multipliers bounded. We may assume, without loss of generality, that (λ^k, μ^k) converges to $(\bar{\lambda}, \bar{\mu})$, for some $(\bar{\lambda}, \bar{\mu}) \in \mathbb{R} \times \mathbb{R}^m$.

Since $\{x^k\}$ is bounded, there exists a sequence $\{k_j\} \subset \{k\}$ such that $x^{k_j} \to \bar{x}$ for some $\bar{x} \in \mathbb{R}^n$. Passing (5.1)–(5.3) to the limits as $j \to \infty$, we obtain that

$$(\bar{Q} + \bar{\lambda}\bar{D})\bar{x} + \bar{\mu}^T \bar{A} + \bar{q} = 0,$$

$$\bar{\lambda} \geq 0,\ \bar{\mu} \geq 0,\ \bar{x}^T \bar{D}\bar{x} - \bar{\alpha} \leq 0,\ \bar{A}\bar{x} + \bar{b} \leq 0,$$
$$\bar{\lambda}\big(\bar{x}^T \bar{D}\bar{x} - \bar{\alpha}\big) = 0,\ \bar{\mu}_i(\bar{A}_i\bar{x} + \bar{b}_i) = 0.$$

From these it follows that $\bar{x} \in S(\bar{\omega}) \subset U$, contrary to the fact that $x^{k_j} \notin U$ and U is open. Therefore, S is upper semicontinuous at $\bar{\omega}$. The proof is complete. □

In general, Slater's condition is not necessary for the upper semicontinuity of the stationary solution map, which is illustrated in the following example.

Example 5.2. Consider $(ET(\bar{\omega}))$ with $n = 1, m = 1, \bar{Q} = 1, \bar{c} = 0, \bar{D} = 1, \bar{\alpha} = 1, \bar{A} = -1, \bar{b} = 1$. $(ET(\bar{\omega}))$ does not satisfy Slater's condition since $\mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b}) = \{1\}$. Let $\omega^t = (\bar{Q}, \bar{c}, \bar{D}, \bar{\alpha}, \bar{A}, \bar{b} + t)$ for $t \in \mathbb{R}$. We obtain that

$$S(\omega^t) = \begin{cases} \{1 + t\} & \text{if } t \leq 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

For each open set V containing $S(\bar{\omega}) = \{1\}$, there exists $\epsilon > 0$ such that $S(\omega^t) \subset V$ for every t satisfying $|t| < \epsilon$. Hence, $S(\bar{Q}, \bar{c}, \bar{D}, \bar{\alpha}, \bar{A}, .)$ is upper semicontinuous at $\bar{b}$.

Next, we investigate the lower semicontinuity of S. The following lemma is useful for proving the main results.

Lemma 5.3. *If $(ET(\omega))$ does not satisfy Slater's condition then there exists $b^k \to b$ such that, for each k, the set $\mathcal{F}(D, \alpha, A, b^k)$ is empty.*

Proof. Let $b^k \downarrow b$. Fix any $x \in \mathbb{R}^n$. Since $(ET(\bar{\omega}))$ does not satisfy Slater's condition, we obtain either $g_0(x; D, \alpha) = 0$ or $g_i(x; A_i, b_i) = 0$ for some $i \in \{1, \ldots, m\}$.

If there exists $i \in \{1, \ldots, m\}$ such that $g_i(x; A_i, b_i) = 0$, then $A_i x + b_i^k > A_i x + b_i = 0$. Hence, $x \notin \mathcal{F}(\alpha, A, b^k)$.

If $g_0(x; D, \alpha) = 0$ and $g_i(x; A_i, b_i) < 0$ for every $i \in \{1, \ldots, m\}$, then there exists a sequence $\{x^s\}$ such that $x^s \to x$ and $g_0(x^s; \alpha) < 0$. Thus, for s large enough, $Ax^s + b < 0$. This implies that $(ET(\omega))$ satisfies Slater's condition, contrary to the assumption.

The proof is complete. □

The necessary condition for the lower semicontinuity of $S(\bar{Q}, ., \bar{D}, \bar{\alpha}, \bar{A}, .)$ is characterized in the following theorem.

Theorem 5.4. *If $\bar{A}$ has full rank and $S(\bar{Q}, ., \bar{D}, \bar{\alpha}, \bar{A}, .)$ is lower semicontinuous at $(\bar{c}, \bar{b})$, then $(ET(\bar{\omega}))$ satisfies Slater's condition and $S(\bar{\omega})$ is a nonempty set which contains at most 2^m points.*

Proof. We first show that $(ET(\bar{\omega}))$ satisfies Slater's condition. Indeed, if $(ET(\bar{\omega}))$ does not satisfy Slater's condition, there exists $b^k \to \bar{b}$ such that, for each k, $\mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, b^k)$ is empty by Lemma 5.3. Then $S(\bar{Q}, \bar{c}, \bar{\alpha}, \bar{A}, b^k) = \emptyset$ for all $k \in \mathbb{N}$ and $S(\bar{Q}, ., \bar{D}, \bar{\alpha}, \bar{A}, .)$ cannot be lower semicontinuous at $(\bar{c}, \bar{b})$. This contradicts the assumption. Thus $(ET(\bar{\omega}))$ satisfies Slater's condition.

For each $\emptyset \neq S \subset \{1, \ldots, m\}$ and for each $t \in \mathbb{R}$, set

$$\mathcal{A}_S := \begin{pmatrix} \bar{Q} & \bar{A}_S \\ \\ \bar{A}_S^T & 0 \end{pmatrix} \text{ and } \mathcal{A}_{0S}(t) := \begin{pmatrix} \bar{Q} + t\bar{D} & \bar{A}_S \\ \\ \bar{A}_S^T & 0 \end{pmatrix},$$

where $\bar{A}_S = (\bar{A}_{ij})_{i \in S,\, j=1,\ldots,m}$. If $S = \emptyset$ then we let $\mathcal{A}_S = \bar{Q}$ and $\mathcal{A}_{0S}(t) = \bar{Q} + t\bar{D}$.

Since $\bar{D}$ is positive definite, there exists an orthogonal matrix C such that $C^{-1}\bar{D}C = I$, where I denotes the $n \times n$ unit matrix. We denote the set of eigenvalues of $C^{-1}\bar{Q}C$ by T.

For each $S \subset \{1, \ldots, m\}$, let:

$$\mathcal{P}_S := \Big\{ (u, v) \in \mathbb{R}^n \times \mathbb{R}^m : \begin{pmatrix} u \\ v_S \end{pmatrix} = \mathcal{A}_S \begin{pmatrix} x \\ \mu_S \end{pmatrix} \text{ for some } (x, \mu) \in \mathbb{R}^n \times \mathbb{R}^m \Big\};$$

$$\mathcal{P}_{0S}(t) := \Big\{ (u, v) \in \mathbb{R}^n \times \mathbb{R}^m : \begin{pmatrix} u \\ v_S \end{pmatrix} = \mathcal{A}_{0S}(t) \begin{pmatrix} x \\ \mu_S \end{pmatrix} \text{ for some } (x, \mu) \in \mathbb{R}^n \times \mathbb{R}^m \Big\};$$

$$\mathcal{P} := \cup \big\{ \mathcal{P}_S : S \subset \{1, \ldots, m\}, det\mathcal{A}_S = 0 \big\} \cup \big\{ \mathcal{P}_{0S}(t) : S \subset \{1, \ldots, m\}, t \in T \big\}.$$

For each $S \subset \{1, \ldots, m\}$, if $det\mathcal{A}_S = 0$, then $\mathcal{P}_S$ is a proper linear subspace of $\mathbb{R}^n \times \mathbb{R}^m$. If $t \in T$ then $det\mathcal{A}_S(t) = 0$; hence, $\mathcal{P}_{0S}(t)$ is also a proper linear subspace of $\mathbb{R}^n \times \mathbb{R}^m$. According to Baire's lemma (see [4, p.15]), there exists a sequence $\{(c^k, b^k)\} \subset \mathbb{R}^n \times \mathbb{R}^m$ converging to $(\bar{c}, \bar{b})$ such that $(-c^k, -b^k) \notin \mathcal{P}$ for all k.

From the assumption that $S(\bar{Q}, ., \bar{D}, \bar{\alpha}, \bar{A}, .)$ is lower semicontinuous at $(\bar{c}, \bar{b})$, $S(\bar{Q}, \bar{c}, \bar{D}, \bar{\alpha}, \bar{A}, \bar{b})$ is nonempty. Fix any $\bar{x} \in S(\bar{Q}, \bar{c}, \bar{D}, \bar{\alpha}, \bar{A}, \bar{b})$. Then, there exists a sequence $\{x^k\} \subset \mathbb{R}^n$ converging to $\bar{x}$ such that $x^k \in S(\bar{Q}, c^k, \bar{D}, \bar{\alpha}, \bar{A}, b^k)$ for all k. For each k, there exists $(\lambda^k, \mu^k) \in \mathbb{R} \times \mathbb{R}^m$ such that:

$$(\bar{Q} + \lambda^k \bar{D})x^k + (\mu^k)^T \bar{A} + c^k = 0, \tag{5.6}$$

$$\lambda^k \geq 0,\ \mu^k \geq 0,\ (x^k)^T \bar{D} x^k - \bar{\alpha} \leq 0,\ \bar{A}x^k + b^k \leq 0, \tag{5.7}$$

$$\lambda^k((x^k)^T \bar{D} x^k - \bar{\alpha}) = 0,\ \mu_i^k(\bar{A}_i x^k + b_i^k) = 0,\ \ i = 1, \ldots, m. \tag{5.8}$$

Let $S_k = \{i \in \{1, \ldots, m\} : \mu_i^k > 0\}$. Then, there exists a set $J \subset \{1, \ldots, m\}$ such that $S_k = J$ for infinitely many k. Without loss of generality we may assume that $S_k = J$ for every k. Hence, (5.6)–(5.8) reduces to

$$(\bar{Q} + \lambda^k \bar{D})x^k + \bar{A}_J^T \mu_J^k + c^k = 0, \tag{5.9}$$

$$\bar{A}_J x^k + b_J^k = 0. \tag{5.10}$$

This can be rewritten as follows

$$\begin{pmatrix} -c^k \\ -b_J^k \end{pmatrix} = \mathcal{A}_{0J}(\lambda^k) \begin{pmatrix} x^k \\ \mu_J^k \end{pmatrix}. \tag{5.11}$$

Consider the following three cases:

Case 1: $\lambda^k = 0$ *for infinitely many* k. There is no loss of generality in assuming $\lambda^k = 0$ for every k. From (5.11) it follows

$$\begin{pmatrix} -c^k \\ -b_J^k \end{pmatrix} = \mathcal{A}_J \begin{pmatrix} x^k \\ \mu_J^k \end{pmatrix}. \tag{5.12}$$

This gives $(-c^k, -b^k) \in \mathcal{P}_J$. If $\det \mathcal{A}_J = 0$ then $(-c^k, -b^k) \in \mathcal{P}$, contrary to the fact that $(-c^k, -b^k) \notin \mathcal{P}$. Hence, $\det \mathcal{A}_J \neq 0$ and

$$\begin{pmatrix} x^k \\ \mu_J^k \end{pmatrix} = (\mathcal{A}_J)^{-1} \begin{pmatrix} -c^k \\ -b_J^k \end{pmatrix}$$

follows from (5.12). It implies that μ_J^k converges to some $\bar{\mu}_J \in \mathbb{R}$. Hence,

$$\begin{pmatrix} \bar{x} \\ \bar{\mu}_J \end{pmatrix} = (\mathcal{A}_J)^{-1} \begin{pmatrix} -\bar{c} \\ -\bar{b}_J \end{pmatrix}.$$

Therefore, $\bar{x}$ is defined uniquely by J.

Case 2: $\lambda^k \in T$ *for infinitely many* k. Since T is finite, $\lambda^k = \bar{\lambda}$ for infinitely many k, for some $\bar{\lambda} \in T$. There is no loss of generality in assuming that $\lambda^k = \bar{\lambda}$ for every k. Then, (5.11) leads to

$$\begin{pmatrix} -c^k \\ -b^k \end{pmatrix} = \mathcal{A}_{0J}(\bar{\lambda}) \begin{pmatrix} x^k \\ \mu^k \end{pmatrix}.$$

Combining this with $\bar{\lambda} \in T$ gives $(-c^k, -b^k) \in \mathcal{P}$, contrary to the fact that $(-c^k, -b^k) \notin \mathcal{P}$. Thus this case does not occur.

Case 3: $\lambda^k \notin T \cup \{0\}$ *for infinitely many* k. There is no loss of generality in assuming that $\lambda^k \notin T \cup \{0\}$ for every k. Since λ^k is not an eigenvalue of $C^{-1}\bar{Q}C$, we obtain that

$$\det(\bar{Q} + \lambda^k \bar{D}) = \det(C^{-1}\bar{Q}C + \lambda^k I) \neq 0$$

and

$$\det \mathcal{A}_{0J}(\lambda^k) = \det(\bar{Q} + \lambda^k \bar{D}) \det(-\bar{A}_J^T(\bar{Q} + \lambda^k \bar{D})^{-1} \bar{A}_J).$$

By the assumption that $\bar{A}$ has full rank, so is $\bar{A}_J$. Then,

$$\operatorname{rank}(\bar{A}_J^T(\bar{Q} + \lambda^k \bar{D})^{-1} \bar{A}_J) = |J|,$$

that is, $\det(\bar{A}_J^T(\bar{Q} + \lambda^k \bar{D})^{-1} \bar{A}_J) \neq 0$. This leads to $\det \mathcal{A}_{0J}(\lambda^k) \neq 0$ for every k. From (5.11) we get

$$\begin{pmatrix} x^k \\ \mu_J^k \end{pmatrix} = (\mathcal{A}_{0J}(\lambda^k))^{-1} \begin{pmatrix} -c^k \\ -b_J^k \end{pmatrix}. \tag{5.13}$$

From the assumption that $(ET(\bar{\omega}))$ satisfies Slater's condition, $(ET(\omega^k))$ also satisfies Slater's condition for every k large enough. According to [10], for each k large enough, $\{(\lambda^k, \mu^k)\}$ is bounded. Hence, $\{(\lambda^k, \mu^k_J)\}$ is bounded. Without loss of generality, one may assume that $(\lambda^k, \mu^k_J) \to (\hat{\lambda}, \hat{\mu}_J)$ for some $(\hat{\lambda}, \hat{\mu}_J) \in \mathbb{R} \times \mathbb{R}^{|J|}$. Then, the sequence on the right hand side of (5.13) is convergent. Passing both sides of the equality (5.13) to the limits as $k \to \infty$, we deduce that $\bar{x}$ is defined uniquely by J.

By the above cases, we conclude that $\bar{x}$ is defined uniquely by J, for some $J \subset \{1, \dots, m\}$. Therefore, the number of elements of $S(\bar{\omega})$ can not be greater than 2^m. The proof is complete. □

Theorem 5.4 leads to the following corollary.

Corollary 5.5. *If $\bar{A}$ has full rank and S is lower semicontinuous at $\bar{\omega}$, then $(ET(\bar{\omega}))$ satisfies Slater's condition and $S(\bar{\omega})$ is a nonempty set which contains at most 2^m points.*

Denote

$$\partial\mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b}) := \big\{x \in \mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b}) : (x^T\bar{D}x - \bar{\alpha})\prod_{i=1}^{m}(\bar{A}_i x - \bar{b}_i) = 0\big\}.$$

The following theorem shows some sufficient conditions for the lower semicontinuity of S.

Theorem 5.6. *If at least one of the following conditions is satisfied:*

(i) *$\bar{Q} + \lambda\bar{D}$ is positive definite for every KKT pair (x, λ, μ) and $(ET(\bar{\omega}))$ satisfies Slater's condition;*

(ii) *$S(\bar{\omega})$ is a singleton and $(ET(\bar{\omega}))$ satisfies Slater's condition;*

(iii) *$S(\bar{\omega})$ is a singleton and φ is continuous at $\bar{\omega}$;*

(iv) *G is lower semicontinuous at $\bar{\omega}$;*

(v) *$S(\bar{\omega})$ is finite and $S(\bar{\omega}) \cap \partial\mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b}) = \emptyset$;*

(vi) *$S(\bar{\omega}) \neq \emptyset$, $\bar{Q}$ is nonsingular, and $S(\bar{\omega}) \cap \partial\mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b}) = \emptyset$,*

then S is lower semicontinuous at $\bar{\omega}$.

Proof. In oder to prove that S is lower semicontinuous at $\bar{\omega}$, we have to show that for any $z \in S(\bar{\omega})$ and for any open neighborhood U_z of z, there exists $\delta > 0$ such that

$$S(\tilde{\omega}) \cap U_z \neq \emptyset \tag{5.14}$$

for every $\tilde{\omega} \in \Omega$ satisfying $\|\tilde{\omega} - \bar{\omega}\| < \delta$.

We now fix any $x \in S(\bar{\omega})$ with the corresponding Lagrange multiplier (λ, μ). Let U_x be an open neighborhood of x.

Firstly, suppose that (i) holds. Let

$$L(y; \bar{\omega}, \lambda, \mu) := f(y; \bar{Q}, \bar{c}) + \lambda g_0(y; \bar{D}, \bar{\alpha}) + \sum_{i=1}^{m}\mu_i g_i(y; \bar{A}_i, \bar{b}_i).$$

From system (2.2)–(2.4) it follows that $\nabla L_y(x, \bar{\omega}, \lambda, \mu) = 0$. For every $\tilde{x} \in \mathcal{F}(\bar{D}, \bar{\alpha}, \bar{A}, \bar{b})$ and $\tilde{x} \neq x$, we have

$$\begin{aligned} f(\tilde{x}; \bar{Q}, \bar{c}) - f(x; \bar{Q}, \bar{c}) &\geq L(\tilde{x}, \bar{\omega}, \lambda, \mu) - L(x, \bar{\omega}, \lambda, \mu) \\ &= \frac{1}{2}(\tilde{x} - x)^T(\bar{Q} + \lambda\bar{D})(\tilde{x} - x) + \nabla L_y(x, \bar{\omega}, \lambda, \mu)^T(\tilde{x} - x) \\ &> 0 \end{aligned}$$

by the assumption that $\bar{Q} + \lambda\bar{D}$ is positive definite. Hence, x is the unique solution of $(ET(\omega))$ with $\omega = \bar{\omega}$.

According to Theorem 3.1, G is upper semicontinuous at $\bar{\omega}$. Hence, there exists $\epsilon^3 > 0$ such that $G(\tilde{\omega}) \cap U_x \neq \emptyset$ for every $\tilde{\omega} \in \Omega$ satisfying $\|\tilde{\omega} - \bar{\omega}\| < \epsilon^3$. The latter leads to (5.14).

We now suppose that (ii) holds, i. e., $(ET(\bar{\omega}))$ satisfies Slater's condition and $S(\bar{\omega}) = \{x\}$. According to Lemma 4 in [21], there exists $\delta > 0$ such that $\mathcal{F}(\tilde{D}, \tilde{\alpha}, \tilde{A}, \tilde{b}) \neq \emptyset$ for every $(\tilde{D}, \tilde{\alpha}, \tilde{A}, \tilde{b})$ satisfying $\|(\tilde{D}, \tilde{\alpha}, \tilde{A}, \tilde{b}) - (\bar{D}, \bar{\alpha}, \bar{A}, \bar{b})\| < \delta$. Since $\mathcal{F}(\tilde{D}, \tilde{\alpha}, \tilde{A}, \tilde{b})$ is nonempty and compact, $S(\tilde{\omega}) \neq \emptyset$ for every $\tilde{\omega}$ satisfying $\|\tilde{\omega} - \bar{\omega}\| < \delta$. By Theorem 5.1, S is upper semicontinuous at $\bar{\omega}$. Hence, $S(\tilde{\omega}) \subset U_x$ for every $\tilde{\omega}$ satisfying $\|\tilde{\omega} - \bar{\omega}\| < \delta$. It follows that $S(\tilde{\omega}) \cap U_x \neq \emptyset$ for every $\tilde{\omega}$ satisfying $\|\tilde{\omega} - \bar{\omega}\| < \delta$. Thus, S is lower semicontinuous at $\bar{\omega}$.

By the (ii) and Theorem 4.1, we obtain (iii).

The assertion (iv) follows from (ii) and Theorem 3.1.

We next consider the case where (v) holds, i. e., $x^T\bar{D}x < \bar{\alpha}$, $\bar{A}x + \bar{b} < 0$ and $S(\bar{\omega})$ is finite. It follows that $(\lambda, \mu) = (0, 0) \in \mathbb{R} \times \mathbb{R}^m$ and x is a solution of the following linear system

$$\bar{Q}y = -\bar{c}. \tag{5.15}$$

Since $x^T\bar{D}x < \bar{\alpha}$ and $\bar{A}x + \bar{b} < 0$, there exist $\epsilon^1 > 0$ and an open neighborhood $V_x \subset U_x$ such that $V_x \subset \mathcal{F}(\tilde{D}, \tilde{\alpha}, \tilde{A}, \tilde{b})$ for every $(\tilde{D}, \tilde{\alpha}, \tilde{A}, \tilde{b})$ satisfying $\|(\tilde{D}, \tilde{\alpha}, \tilde{A}, \tilde{b}) - (\bar{D}, \bar{\alpha}, \bar{A}, \bar{b})\| \leq \|\tilde{\omega} - \bar{\omega}\| < \epsilon^1$.

By (5.15) and by the assumption that $S(\bar{\omega})$ is finite, we obtain that $\bar{Q}$ is nonsingular and x is a unique solution of (5.15). This gives $x = -\bar{Q}^{-1}\bar{c}$. Then, there exists $\epsilon^2 > 0$ such that $\tilde{x} = -\tilde{Q}^{-1}\tilde{c} \in V_x$ for every $(\tilde{Q}, \tilde{c})$ satisfying $\|(\tilde{Q}, \tilde{c}) - (\bar{Q}, \bar{c})\| \leq \|\tilde{\omega} - \bar{\omega}\| < \epsilon^2$.

Let $\epsilon = min\{\epsilon^1, \epsilon^2\}$ and let $\tilde{\omega}$ such that $\|\tilde{\omega} - \bar{\omega}\| < \epsilon$. Then, $\tilde{x} \in V_x \subset \mathcal{F}(\tilde{D}, \tilde{\alpha}, \tilde{A}, \tilde{b})$ and $(\lambda, \mu) = (0, 0) \in \mathbb{R} \times \mathbb{R}^m$ is the unique Lagrange multiplier corresponding to $\tilde{x}$. We have

$$\tilde{Q}\tilde{x} + \tilde{c} = 0,\ \tilde{x}^T\tilde{D}\tilde{x} - \tilde{\alpha} < 0,\ \tilde{A}\tilde{x} + \tilde{b} < 0.$$

Hence, (5.14) is satisfied for every $\tilde{\omega} \in \Omega$ satisfying $\|\tilde{\omega} - \bar{\omega}\| < \epsilon$.

Finally, we consider the case where (vi) holds, i. e., $x^T\bar{D}x < \bar{\alpha}$, $\bar{A}x + \bar{b} < 0$ and $\bar{Q}$ is nonsingular. Repeating the previous argument and using the assumption that $\bar{Q}$ is nonsingular leads to (5.14). The proof is complete. □

We conclude this section by a simple example showing that S is not lower semicontinuous at $\bar{\omega}$ if $S(\bar{\omega})$ is infinite.

Example 5.7. Consider $(ET(\bar{\omega}))$ with $n = 2, m = 1$,

$$\bar{Q} = -I, \quad \bar{c} = 0, \quad \bar{D} = I, \quad \bar{\alpha} = 1, \quad \bar{A}_1 = (-1, -1), \quad \bar{b}_1 = 1.$$

This problem has the following form

$$\min\left\{f(x;\bar{Q},\bar{c}) = -\frac{1}{2}(x_1^2+x_2^2) : x=(x_1,x_2)\in\mathcal{F}(\bar{D},\bar{\alpha},\bar{A},\bar{b})\right\},$$

where

$$\mathcal{F}(\bar{D},\bar{\alpha},\bar{A},\bar{b}) = \{(x_1,x_2)\in\mathbb{R}^2 : x_1^2+x_2^2\le 1; -x_1-x_2+1\le 0\}.$$

By the system (2.2)–(2.4), we obtain that $(x,1,0)$ is a KKT pair for every $x\in\{(x_1,x_2)\in\mathbb{R}^2 : x_1^2+x_2^2=1; x_1\ge 0; x_2\ge 0\}$. Hence, $S(\bar{\omega})$ is infinite. By Theorem 5.4, S is not lower semicontinuous at $\bar{\omega}$.

Remark 5.8. The obtained results are really based on the special structure of the extended trust region subproblem. The following examples show that Theorems 3.1 and 4.1 are not true if the constraint $x^TDx\le\alpha$ is omitted.

Example 5.9. Consider the quadratic problem as follows

$$\begin{array}{ll}\min\ f(x;\bar{Q},\bar{c}) := -\frac{1}{2}x_2^2 \\ \text{s.t. } x=(x_1,x_2)\in\mathbb{R}^2 : g_1(x;\bar{C},\bar{d}) := x_1^2-x_2\le 0,\ g_2(x;\bar{A},\bar{b}) := 0x_2-1\le 0,\end{array} \qquad (P_1(\bar{\omega}))$$

where

$$\bar{Q}=\begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix},\quad \bar{c}=\begin{pmatrix}0\\0\end{pmatrix},\quad \bar{C}=\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix},\quad \bar{d}=\begin{pmatrix}0\\-1\end{pmatrix},\quad \bar{A}=(0\ \ 0),\quad \bar{b}=1,$$

and $\bar{\omega} := (\bar{Q},\bar{c},\bar{C},\bar{d},\bar{A},\bar{b})$. It is easy to check that $(P_1(\bar{\omega}))$ satisfies Slater's condition and the global solution set is empty, $G_{P_1}(\bar{\omega})=\emptyset$. We consider the following perturbation problem of the problem $(P_1(\bar{\omega}))$ with $\omega^\epsilon := (\bar{Q},\bar{c},\bar{C},\bar{d},\bar{A}+\epsilon(0\ \ 1),\bar{b})$, where $\epsilon>0$,

$$\begin{array}{ll}\min\ f(x_1,x_2) := -x_2^2 \\ \text{s.t. } (x_1,x_2)\in\mathbb{R}^2 : x_1^2-x_2\le 0,\ \epsilon x_2-1\le 0.\end{array} \qquad (P_1(\omega^\epsilon))$$

We can check that $G_{P_1}(\omega^\epsilon) := \{(x_1,\frac{1}{\epsilon}) : x_1\in[-1/\sqrt{\epsilon},1/\sqrt{\epsilon}]\}$ is the global solution set of $(P_1(\omega^\epsilon))$. For the open set $U=\emptyset$ containing $G_{P_1}(\bar{\omega})$, we have $G_{P_1}(\omega^\epsilon)$ is not contained in U. Hence, the global solution map G_{P_1} is not upper semicontinuous at $\bar{\omega}$.

Example 5.10. We consider the following quadratic problem

$$\begin{array}{ll}\min\ f(x;\bar{Q},\bar{c}) := 0x^2+x \\ \text{s.t. } x\in\mathbb{R} : g(x;\bar{A},\bar{b}) := -x\le 0.\end{array} \qquad (P_2)$$

where

$$\bar{Q}=0,\quad \bar{c}=1,\quad \bar{A}=-1,\quad \bar{b}=0,$$

and $\bar{\omega} := (\bar{Q},\bar{c},\bar{A},\bar{b})$. We can check that $(P_2(\bar{\omega}))$ satisfies Slater's condition and the global solution set is singleton, $G_{P_2}(\bar{\omega})=\{0\}$. Then $\varphi_{P_2}(\bar{\omega})=0$. The following perturbation problem of the problem $(P_2(\bar{\omega}))$ is considered as follows

$$\begin{array}{ll}\min\ f(x;\bar{Q},\bar{c}) := -\epsilon x^2+x \\ \text{s.t. } x\in\mathbb{R} : -x\le 0,\end{array} \qquad (P_2(\omega^\epsilon))$$

with $\omega^\epsilon := (\bar{Q} - \epsilon, \bar{c}, \bar{A}, \bar{b})$, where $\epsilon > 0$. For each $\epsilon > 0$, since $f(x; \bar{Q}, \bar{c}) \to -\infty$ as $x \to +\infty$, we have $G_{P_2}(\omega^\epsilon) = \emptyset$ and $\varphi_{P_2}(\omega^\epsilon) = -\infty$. So, the global solution map G_{P_2} is not lower semicontinuous at $\bar{\omega}$ and the optimal value function φ_{P_2} is not continuous at $\bar{\omega}$.

6 Conclusions

In this paper, we have investigated stability of parametric ETRSs. We have obtained conditions for continuity of global solution map (Theorem 3.1); continuity of the local solution map (Theorem 3.3); continuity of the optimal value function (Theorem 4.1) and continuity of the stationary solution map (Theorems 5.1, 5.4, and 5.6). Our results develop and complement the published ones in [18,19]. The obtained results on the continuity of the solution maps and the continuity of the optimal value function is a powerful information in devising solution methods.

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