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APPROXIMATION OF COMMON FIXED POINTS OF A FINITE FAMILY OF MULTIVALUED MAPPINGS

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Abstract: Let K be a nonempty closed and convex subset of a uniformly convex real Banach space E and let $T_1, \ldots, Tm : K \to 2^K$ be m multivalued quasi-nonexpansive mappings. A new iterative algorithm is constructed and the corresponding sequence $\{x_n\}$ is proved to be an approximating fixed point sequence of each T_i, i.e., $\lim d(x_n; Tx_n) = 0$. Then, convergence theorems are proved under appropriate additional conditions. Our results extend and improve some important recent results (e.g. Abbas et. al., Common fixed points of two multivalued nonexpansive mappings by one-step itrative scheme, Appl. Math. Letters 24 (2011), 97-102).

Key words: *quasi-nonexpansive mappings, multivalued mappings, uniformly convex Banach spaces*

Mathematics Subject Classification: *47H04, 47H06, 47H15, 47H17, 47J25*

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1 Introduction

Let (X, d) be a metric space, K be a nonempty subset of X and $T : K \to 2^K$ be a multivalued mapping. An element $x \in K$ is called a fixed point of T if $x \in Tx$. For single valued mapping, this reduces to $Tx = x$. The fixed point set of T is denoted by $F(T) := \{x \in D(T) : x \in Tx\}$, where $D(T)$ is the domain of T.

For several years, the study of fixed point theory for *multi-valued nonlinear mappings* has attracted, and continues to attract, the interest of several well known mathematicians (see, for example, Brouwer [2], Kakutani [10], Nash [13, 14], Geanakoplos [9], Nadla [12], Downing and Kirk [7]).

Interest in the study of fixed point theory for multi-valued nonlinear mappings stems, perhaps, mainly from its usefulness in real-world applications such as *Game Theory* and *Non-Smooth Differential Equations*.

Game Theory. Nash showed the existence of equilibria for non-cooperative *static* games as a direct consequence of *multi-valued* Brouwer or Kakutani fixed point theorem. More precisely, under some regularity conditions, given a game, there always exists a *multi-valued mapping* whose fixed points coincide with the equilibrium points of the game. This, among other things, made Nash a recipient of Nobel Prize in Economic Sciences in 1994. However, it has been remarked that the applications of this theory to equilibrium are mostly *static*: they enhance understanding conditions under which equilibrium may be achieved but do not

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indicate how to construct a process starting from a non-equilibrium point and convergent to equilibrium solution. This is part of the problem that is being addressed by *iterative methods for fixed point of multi-valued mappings.*

Nonsmooth differential equations. A large number of problems from mechanics and electrical engineering leads to differential inclusions and differential equations with discontinuous right-hand sides, for example, a dry friction force of some electronic devices. Below are two models.

$$\frac{du}{dt} = f(t, u),\ a.e.\, t \in I := [-a, a],\ u(0) = u_0, \tag{1.1}$$

a, u_0 fixed in $\mathbb{R}$. These types of differential equations do not have solutions in the classical sense. A generalized notion of solution is what is called a solution in the sense of Fillipov. Consider the following *multi-valued* initial value problem.

$$\begin{cases} -\dfrac{d^2u}{dt^2} & \in & u - \dfrac{1}{4} - \dfrac{1}{4}\text{sign}(u-1) \text{ on } \Omega = (0, \pi); \\ u(0) & = & 0; \\ u(\pi) & = & 0. \end{cases} \tag{1.2}$$

Under some conditions, the solutions set of equations (1.1) and (1.2) coincides with the fixed point set of some multi-valued mappings.

Let D be a nonempty subset of a normed space E. The set D is called *proximinal* (see, *e.g.*, [15, 17, 18]) if for each $x \in E$, there exists $u \in D$ such that

$$d(x, u) = \inf\{\|x - y\| : y \in D\} = d(x, D),$$

where $d(x, y) = \|x - y\|$ for all $x, y \in E$. Every nonempty, closed and convex subset of a real Hilbert space is proximinal. Let $CB(D)$, $K(D)$ and $C(D)$ denote the families of nonempty, closed and bounded subsets, nonempty, compact subsets and nonempty, compact convex subsets of D, respectively. The *Hausdorff metric* on $CB(K)$ is defined by:

$$H(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\}$$

for all $A, B \in CB(K)$. A multi-valued mapping $T : D(T) \subseteq E \to CB(E)$ is called *L-Lipschitzian* if there exists $L > 0$ such that

$$H(Tx, Ty) \le L\|x - y\|\ \ \forall x, y \in D(T). \tag{1.3}$$

When $L \in (0, 1)$ in (1.3), we say that T is a *contraction*, and T is called *nonexpansive* if $L = 1$. Finally, A multivalued mapping $T : K \to CB(K)$ is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and $H(Tx, Tp) \le \|x - p\|$ for all $x \in K$ and $p \in F(T)$.

Several papers deal with the problem of *approximating* fixed points of *multi-valued nonexpansive* mappings (see, *e.g.*, [1, 11, 15, 17, 18], , and the references therein) and their generalizations (see, *e.g.*, [5, 6, 8]).

On the other hand, Abbas et al. [1] introduced a new one-step iterative process for approximating a common fixed point of two multivalued *nonexpansive mappings* in a real uniformly convex Banach space and established weak and strong convergence theorems for the proposed process under some basic boundary conditions. Let $S, T : K \to CB(K)$ be two multivalued nonexpansive mappings. They introduced the following iterative scheme:

$$\begin{cases} x_1 \in K \\ x_{n+1} = a_n x_n + b_n y_n + c_n z_n \end{cases} \tag{1.4}$$

where $y_n \in Tx_n$ and $z_n \in Sx_n$ are such that

$$\begin{cases} \|y_n - p\| & \leq & d(p, Sx_n); \\ \|z_n - p\| & \leq & d(p, Tx_n), \end{cases} \tag{1.5}$$

and $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are real sequences in $(0,1)$ satisfying $a_n + b_n + c_n = 1$.

The following lemma is a consequence of the definition of Hausdorff metric, as remarked by Nadler [12].

Lemma 1.1. *Let $A, B \in CB(X)$ and $a \in A$. For every $\gamma > 0$, there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \gamma. \tag{1.6}$$

Following the work of Abbas et. al. [1], Rashwan and Altwqi [16] introduced a new scheme for approximation a common fixed point of three multivalued nonexpansive mappings in uniformly convex Banach space. Let $T, S, R : K \to CB(K)$ be three multivalued nonexpansive mappings. They employed the following iterative process:

$$\begin{cases} x_1 \in K \\ x_{n+1} = a_n y_n + b_n z_n + c_n w_n, \ n \geq 1 \end{cases} \tag{1.7}$$

where $y_n \in Tx_n$, $z_n \in Sx_n$ and $w_n \in Rx_n$ are such that:

$$\begin{cases} \|y_n - y_{n+1}\| & \leq & H(Tx_n, Tx_{n+1}) + \eta_n; \\ \|z_n - z_{n+1}\| & \leq & H(Sx_n, Sx_{n+1}) + \eta_n; \\ \|w_n - w_{n+1}\| & \leq & H(Rx_n, Rx_{n+1}) + \eta_n \end{cases} \tag{1.8}$$

and $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are real sequences in $(0,1)$ satisfying $a_n + b_n + c_n = 1$.

Remark 1.2. Noting that, if y_n, z_n and w_n are known, then the existence of y_{n+1}, z_{n+1} and w_{n+1} satisfying (1.8) is guaranted by Lemma 1.1.

Before we state the result of Rashwan and Altwqi [16], we need the following definition .

Definition 1.3. The mappings $T, S, R : K \to CB(K)$ are said to satisfy condition (C) if $d(x, y) \leq d(z, y)$, for $z \in Tx$ and $y \in Sx$ or $d(x, y) \leq d(z, y)$, for $z \in Tx$ and $y \in Rx$, or $d(x, y) \leq d(z, y)$, for $z \in Rx$ and $y \in Sx$.

Let $F = F(T) \cap F(S) \cap F(R)$ be the set of all common fixed points of the mappings T, S and R.

Theorem 1.4 (Rashwan and Altwqi [16])**.** *Let E be a uniformly convex Banach space and K be a nonempty closed and convex subset of E. Let $T, S, R : K \to CB(K)$ be multivalued nonexpansive mappings satisfying condition (C) and $\{x_n\}$ be the sequence defined by* (1.7) *and* (1.8)*. If $F \neq \emptyset$ and $Tp = Sp = Rp = \{p\}$ for any $p \in F$, then*

$$\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, Sx_n) = 0 = \lim_{n\to\infty} d(x_n, Rx_n).$$

Recently, A. Bunyawat and S. Suantai [3] introduced a one-step iterative scheme for finding a common fixed point of a finite family of multivalued quasi-nonexpansive mappings in a uniformly convex real Banach space. They proposed the following algorithm: let K be a nonempty closed and convex subset of a uniformly convex real Banach space E and

$T_1, T_2, \dots, T_m : K \to CB(K)$ be multivalued quasi-nonexpansive mappings. For $x_1 \in K$, let $\{x_n\}$ be the sequence defined iteratively from x_1 by:

$$x_{n+1} = a_{n,0}x_{n,0} + a_{n,1}x_{n,1} + \cdots + a_{n,m}x_{n,m},\ n \geq 1 \tag{1.9}$$

where the sequence $\{a_{n,i}\} \subset [0,1)$ satisfies $\sum_{i=0}^{m} a_{n,i} = 1\ \forall n \geq 1$ and $x_{n,i} \in T_i x_n$ with $d(p, x_{n,i}) = d(p, T_i x_n)$.

Then, they proved the following theorem.

Theorem 1.5 (Bunyawat and Suantai [3]). *Let E be a real Banach space and K be a closed convex sub- set of X. Let $\{Ti : i = 1, 2, \dots, m\}$ be a finite family of multivalued quasi-nonexpansive mappings from K into $C(K)$ with $F := \cap_{i=1}^{m} F(Ti) \neq \emptyset$. Then the sequence x_n defined by* (1.9) *converges strongly to a common fixed point of $\{Ti : i = 1, 2, \dots, m\}$ if and only if* $\liminf d(x_n, F) = 0$.

It is our purpose in this paper to construct a new iterative algorithm and prove strong convergence theorems for approximating a common fixed point of a finite family of mutivalued quasi-nonexpansive mappings in uniformly convex real Banach spaces. The class of mappings using in our theorems is much more larger than that of mutivalued nonexpansive mappings. Our theorems generalize and extend those of Abbas et. al. [1], Rashwan and Altwqi [16], Bunyawat and Suantai [3] and many other important results.

2 Preliminaries

Lemma 2.1 (Kim et al. [4]). *Let E be uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) = \{x \in E\ :\ \|x\| \leq r\}$. Then, for any given subset $\{u_1, u_2, \dots, u_N\} \subset B_r(0)$ and for any positive numbers $\lambda_1, \lambda_2, \dots, \lambda_N$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_N = 1$, there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \longrightarrow [0, +\infty)$ with $g(0) = 0$ such that, for any $i, j \in \{1, 2, \dots, N\}$ with $i < j$,*

$$\|\sum_{k=1}^{N} \lambda_k u_k\|^2 \leq \sum_{k=1}^{N} \lambda_k \|u_k\|^2 - \lambda_i \lambda_j g(\|u_i - u_j\|). \tag{2.1}$$

3 Main results

We use the following iterative algorithm.

Let $m \geq 1$, K a nonempty closed convex subset of a uniformly convex real Banach space E. Let $T_1, \dots, T_m : K \to CB(K)$ be multivalued quasi-nonexpansive mappings. Let $\{x_n\}$ be a sequence defined iteratively as follows:

$$\begin{cases} x_1 \in K \\ x_{n+1} = \lambda_0 x_n + \lambda_1 u_n^1 + \cdots + \lambda_m u_n^m, \end{cases} \tag{3.1}$$

where $u_n^i \in T_i x_n,\ i = 1, \dots, m$, $\lambda_i \in (0,1),\ i = 0, 1, \dots, m$ with $\lambda_0 + \lambda_1 + \cdots \lambda_m = 1$. In the sequel, we will write $F := \cap_{i=1}^{m} F(T_i)$ for the set of common fixed points of the mappings $T_i, i = 1, \dots, m$.

Lemma 3.1. *Let K be a nonempty, closed and convex subset of a real Banach space E. For $m \geq 1$, Let $\{T_i : i = 1, 2, \ldots, m\}$ be a finite family of multivalued quasi-nonexpansive mappings from K into $CB(K)$ with $F := \cap_{i=1}^m F(T_i) \neq \emptyset$ and such that $T_p = \{p\}$ $\forall p \in F$ and $i = 1, 2, \ldots, m$. Let $\{x_n\}$ be the sequence defined by* (3.1)*. Then, for all $p \in F$, $\lim_{n\to\infty} \|x_n - p\|$ exits.*

Proof. Let $p \in F$. By (3.1) and the quasi-nonexpansiveness of T_i we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\lambda_0(x_n - p) + \sum_{i=1}^m \lambda_i(u_i^n - p)\| \\ &\leq \|\lambda_0(x_n - p)\| + \sum_{i=1}^m \lambda_i \|u_i^n - p\| \\ &\leq \lambda_0 \|x_n - p\| + \sum_{i=1}^m \lambda_i H(T_i x_n, T_i p) \\ &\leq \lambda_0 \|x_n - p\| + \sum_{i=1}^m \lambda_i \|x_n - p\| \\ &= \|x_n - p\| \qquad (3.2) \end{aligned}$$

The equation (3.2) implies that the sequence is not increasing. Since it is bounded from below by 0, we conclude that $\lim_{n\to\infty} \|x_n - p\|$ exists. □

Next, we prove the following theorem

Theorem 3.2. *Let K be a nonempty, closed and convex subset of a real uniformly convex Banach space E. For $i = 1, \ldots, m$ let $T_i : K \to CB(K)$ be a multivalued quasi-nonexpansive mapping. Suppose that $F \neq \emptyset$ and that $T_i p = \{p\}$ for all $p \in F$. Let $\{x_n\}$ be the sequence defined by* (3.1)*. Then, for all $i = 1, \ldots, m$,*

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0.$$

Proof. Using Lemma 3.1 and the quasi-nonexpansiveness of T_i, there exists some positive real r such that

$$\|u_i^n - p\| \leq \|x_n - p\| \leq r. \qquad (3.3)$$

This implies that for each i, $1 \leq i \leq m$,

$$\|u_i^n - x_n\| \leq 2r.$$

Next, let $1 \leq i \leq m$, using inequality (3.3) and Lemma 2.1, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\lambda_0(x_n - p) + \sum_{k=1}^m \lambda_k(u_k^n - p)\|^2 \\ &= \lambda_0 \|x_n - p\|^2 + \sum_{k=1}^m \lambda_k \|u_k^n - p\|^2 - \lambda_0 \lambda_i g(\|u_i^n - x_n\|) \\ &\leq \lambda_0 \|x_n - p\|^2 + \sum_{k=1}^m \lambda_k \|x_n - p\|^2 - \lambda_0 \lambda_i g(\|u_i^n - x_n\|) \\ &= \|x_n - p\|^2 - \lambda_0 \lambda_i g(\|u_i^n - x_n\|) \end{aligned}$$

Therefore

$$\lambda_0\lambda_i g(\|u_i^n - x_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

This implies that

$$\lim_{n\to\infty} g(\|u_i^n - x_n\|) = 0.$$

It follows that

$$\lim_{n\to\infty} \|u_i^n - x_n\| = 0.$$

Since $u_i^n \in T_i x_n$, then,

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0 \quad \text{for all} \quad i = 1, \ldots, m.$$

This completes the proof. □

We now approximate common fixed points of $\{T_1, T_2, \ldots, T_m\}$ through strong convergence of the sequence $\{x_n\}$ defined by (3.1). We start with the following definition.

Definition 3.3. A family $\{T_1, \ldots, T_m : K \to CB(K)\}$ is said to satisfy *Condition* (I^*), if there exists a strictly increasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ and i_0, $1 \leq i_0 \leq m$ such that

$$d(x, T_{i_0} x) \geq f(d(x, F)) \; \forall x \in K.$$

Theorem 3.4. *Let K be a nonempty, closed and convex subset of a uniformly convex real Banach space E. For $m \geq 1$, let $T_1, T_2, \ldots, T_m : K \to CB(K)$ be a multi-valued quasi-nonexpansive. Assume that $\{T_1, \ldots, T_m\}$ satisfies condition (I^*). If $F \neq \emptyset$ and that $T_i p = \{p\}$ for all $p \in F$, then, the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i,\, i = 1, \ldots, m\}$.*

Proof. From Theorem 3.2, we have $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$. Using the fact that $\{T_1, \ldots, T_m\}$ satisfies condition (I^*), it follows that there exists some i_0 such that $\lim_{n\to\infty} f(d(x_n, F(T_{i_0}))) = 0$. Thus there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{p_j\} \subset F(T_i)$ such that

$$\|x_{n_j} - p_j\| < \frac{1}{2^j} \; \forall\, j \in \mathbb{N}.$$

By setting $x^* = p_j$ and following the same arguments as in the proof of Lemma 3.1, we obtain from inequality (3.2) that

$$\|x_{n_{j+1}} - p_j\| \leq \|x_{n_j} - p_j\| < \frac{1}{2^j}.$$

We now show that $\{p_j\}$ is a Cauchy sequence in K. Notice that

$$\begin{aligned}\|p_{j+1} - p_j\| &\leq \|p_{j+1} - x_{n_{j+1}}\| + \|x_{n_{j+1}} - p_j\| \\ &< \frac{1}{2^{j+1}} + \frac{1}{2^j} < \frac{1}{2^{j-1}}.\end{aligned}$$

This shows that $\{p_j\}$ is a Cauchy sequence in K and thus converges strongly to some $p \in K$. Using the fact that T_i is quasi-nonexpansive and $p_j \to p$, we have

$$\begin{aligned}d(p_j, T_i p) &\leq H(T_i p_j, T_i p) \\ &\leq \|p_j - p\|,\end{aligned}$$

so that $d(p, T_i p) = 0$ and thus $p \in T_i p$. Therefore, $p \in F(T_i)$ and $\{x_{n_j}\}$ converges strongly to p. Setting $x^* = p$ in the proof of Lemma 3.2, it follows from inequality (3.2) that $\lim_{n\to\infty} \|x_n - p\|$ exists. So, $\{x_n\}$ converges strongly to p. This completes the proof. □

Definition 3.5. A mapping $T : K \to CB(K)$ is called *hemicompact* if, for any sequence $\{x_n\}$ in K such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in K$. We note that if K is compact, then every multi-valued mapping $T : K \to CB(K)$ is hemicompact.

Theorem 3.6. *Let K be a nonempty, closed and convex subset of a uniformly convex real Banach space E. For $i = 1, \ldots, m$ let $T_i : K \to CB(K)$ be a multi-valued continuous and quasi-nonexpansive mapping. Assume that T_{i_0} is hemicompact for some i_0. If $F \neq \emptyset$ and that $T_i p = \{p\}$ for all $p \in F$, then the sequence $\{x_n\}$ defined by* (3.1) *converges strongly to a common fixed point of $\{T_i, \, i = 1, \ldots, m\}$.*

Proof. From Theorem 3.2, we have that $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$. Since T_{i_0} is hemicompact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p$ for some $p \in K$. Since for each $i = 1, 2, \ldots, m$, T_i is continuous, we have $d(x_{n_j}, T_i x_{n_j}) \to d(p, T_i p)$. Therefore, $d(p, T_i p) = 0$ and so $p \in F(T_i)$. Setting $x^* = p$ in the proof of Theorem 3.2, it follows from inequality (3.2) that the sequence $\{\|x_n - p\|\}$ is decreasing and bounded from bellow. Therefore, $\lim_{n\to\infty} \|x_n - p\|$ exists. So, $\{x_n\}$ converges strongly to p. This completes the proof. $\square$

The following result gives a necessary and sufficient condition for strong convergence of the sequence in (3.1) to a common fixed point of afinite family of multi-valued nonexpansive maps T_i, $i = 1, 2, \ldots, m$.

Theorem 3.7. *Let K be a nonempty, closed and convex subset of a uniformly convexreal Banach space E. For $i = 1, \ldots, m$ let $T_i : K \to CB(K)$ be a multi-valued nonexpansive mapping. Assume that $F \neq \emptyset$ and that $T_i p = \{p\}$ for all $p \in F$. Then the sequence $\{x_n\}$ defined by* (3.1) *converges strongly to a common fixed point of $\{T_i, \, i = 1, \ldots, m\}$ if and only if* $\liminf d(x_n, F) = 0$.

Proof. The necessity is obvious. Conversely, suppose that $\liminf d(x_n, F) = 0$. Let $p \in F$. By (3.2), we have $\|x_{n+1} - p\| \leq \|x_n - p\|$. This gives $d(x_{n+1}, F) \leq d(x_n, F)$. Hence, $\lim d(x_n, F)$ exists. By hypothesis, $\liminf d(x_n, F) = 0$ so we must have $\lim d(x_n, F) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence in K. Let $\epsilon > 0$ be given and since $\lim d(x_n, F) = 0$, there $N \geq 0$ such that for all $n \geq N$, we have

$$d(x_n, F) < \frac{\epsilon}{4}.$$

In particular, $\inf\{\|x_N - p\| \, : \, p \in F\} < \frac{\epsilon}{4}$, so that there exist $p \in F$ such that

$$\|x_N - p\| < \frac{\epsilon}{2}.$$

Now for $m, n \geq N$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p\| + \|x_n - p\| \\ &\leq 2\|x_N - p\| \\ &< 2\Big(\frac{\epsilon}{2}\Big) = \epsilon. \end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in a closed subset K of a Banach space E and therefore, it converges in K. Let $\lim x_n = x^*$. Now, for each $i = 1, 2, 3, \ldots, m$, we have

$$\begin{aligned} d(x^*, T_i x^*) &\leq d(x^*, x_n) + d(x_n, T_i x_n) + H(T_i x_n, T_i x^*) \\ &\leq d(p, x_n) + d(x_n, T_i x_n) + d(x_n, p) \to 0, \text{ as } n \to \infty. \end{aligned}$$

This gieves that $d(x^*, T_i x^*) = 0, i = 1, 2, 3, \ldots, m$ which implies that $x^* \in T_i x^*$. Consequently, $x^* \in F$. □

Corollary 3.8 (Abbas et. al. [1]). *Let E be a real Banach space and D a nonempty, closed and convex subset of E. Let T, S be multi-valued nonexpansive mappings of D into $K(D)$ such that $F(T) \cap F(S) \neq \emptyset$. Let $\{a_n\}, \{b_n\}$and$\{c_n\}$ be sequences in $(0,1)$ satisfying $a_n + b_n + c_n = 1$. Let $\{x_n\}$ be a sequence defined iteratively by:*

$$\begin{cases} x_1 \in D \\ x_{n+1} = a_n x_n + b_n y_n + c_n z_n, n \geq 1 \end{cases} \tag{3.4}$$

where $y_n \in Tx_n, z_n \in Sx_n$ are such that $\|y_n - p\| \leq d(p, Sx_n)$ and $\|z_n - p\| \leq d(p, Tx_n)$ whenever p is a fixed point of any one of mappings T and S. Then, $\{x_n\}$ converges strongly to a common fixed point of $F(T) \cap F(S)$ if and only if $\liminf d(x_n, F) = 0$.

Corollary 3.9 (Bunyawat and Suantai [3]). . *Let E be a real Banach space and K be a closed convex subset of X. Let $\{Ti : i = 1, 2, \ldots, m\}$ be a finite family of multivalued quasi-nonexpansive mappings from K into $C(K)$ with $F := \cap_{i=1}^m F(Ti) \neq \emptyset$. Then the sequence x_n defined by* (1.9) *converges strongly to a common fixed point of $\{Ti : i = 1, 2, \ldots, m\}$ if and only if* $\liminf d(x_n, F) = 0$.

Remark 3.10. The recursion formula (3.1) used in our theorems is simpler to the recursion formula of Abbas et. al. (1.4), the one of Rashwan and Altwqiin (1.7) and the one of Bunyawat and Suantai (1.9) in the following sense: in our algorithm, $u_i^n \in T_i x_n$ for $i = 1, \ldots, m$ and do not have to satisfy the restrictive conditions: (1.5) in the recursion formula (1.4), (1.8) in the recursion formulas (1.7) and similar restrictive conditions in the recursion formula (1.9).

Remark 3.11. Our theorems in this papers are important generalizations of several important recent results. Our theorems extend results proved for multi-valued *nonexpansive* mappings in *uniformly convex real Banach spaces* (see *e.g.,* [1, 11, 15, 17, 18]) to the much more larger class of multi-valued *quasi-nonexpansive mappings.*

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