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A STRONG CONVERGENCE THEOREM FOR SOLVING THE SPLIT COMMON FIXED POINT PROBLEM IN TWO BANACH SPACES AND APPLICATIONS

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Abstract. In this paper, we deal with the split common fixed point problem in two Banach spaces. We prove a strong convergence theorem of Halpern's type iteration for finding a solution of the split common fixed point problem in two Banach spaces. It seems that such a theorem of Halpern's type iteration is first outside Hilbert spaces. Using this result, we obtain well-known and new strong convergence theorems which are connected with the feasibility problem, the split common null point problem and the split common fixed point problem in Hilbert spaces and in Banach spaces.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [8] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Byrne, Censor, Gibali and Reich [7] considered the following problem: Given set-valued mappings $B : H_1 \to 2^{H_1}$ and $G : H_2 \to 2^{H_2}$, respectively, and a bounded linear operator $A : H_1 \to H_2$, the *split common null point problem* [7] is to find a point $z \in H_1$ such that

$$z \in B^{-1}0 \cap A^{-1}(G^{-1}0),$$

where $B^{-1}0$ and $G^{-1}0$ are null point sets of B and G, respectively. Given nonlinear mappings $T : H_1 \to H_1$ and $U : H_2 \to H_2$, respectively, and a bounded linear operator $A : H_1 \to H_2$, the *split common fixed point problem* [9, 24] is to find a point $z \in H_1$ such that $z \in F(T) \cap A^{-1}F(U)$, where $F(T)$ and $F(U)$ are fixed point sets of T and U, respectively. If $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1.1}$$

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where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Furthermore, if $B^{-1}0 \cap A^{-1}(G^{-1}0)$ is nonempty, then $z \in B^{-1}0 \cap A^{-1}(G^{-1}0)$ is equivalent to

$$z = J_\lambda(I - \gamma A^*(I - Q_\mu)A)z, \tag{1.2}$$

where $\lambda, \mu > 0$ and $\gamma > 0$, and J_λ and Q_μ are the resolvents of B and G, respectively. Using such results regarding nonlinear operators and fixed points, many authors have studied the feasibility peoblem and generalized feasibility peoblems including the split common null point problem in Hilbert spaces; see, for instance, [2, 7, 9, 24, 42]. However, it is difficult to solve such problems outside Hilbert spaces. Takahashi [32, 33, 34] and Hojo and Takahashi [11] extended the results of (1.1) and (1.2) in Hilbert spaces to Banach spaces. By using the hybrid method of [25, 26, 27], Takahashi [36] also proved a strong convergence theorem for solving the split common fixed point problem in two Banach spaces. Furthermore, by using the shrinking projection method [40], Takahashi [37] proved a strong convergence theorem for solving such a problem in two Banach spaces.

On the other hand, we know the following iteration process introduced by Mann [21] in 1953: Let C be a nonempty, closed and convex subset of a Banach space E and let $T : C \to C$ be a nonexpansivemapping, that is, $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. For an initial guess $x_1 \in C$, an iteration process $\{x_n\}$ is defined recursively by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. Furthermore, in 1967, Halpern [10] gave the following iteration process: Take $x_0, x_1 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. There are many investigations for these two iterative processes in Hilbert spaces and in Banach spaces. However, we can not find the results under these two processes for solving the split common fixed point problem in two Banach spaces. Very recently, Takahashi [39] partially proved a weak convergence theorem of Mann's type iteration for solving the split common fixed point problem in two Banach space; see also [38]. It is natural to consider the strong convergence of Halpern's type iteration for solving the split common fixed point problem in two Banach spaces

In this paper, we prove a strong convergence theorem of Halpern's type iteration for finding a solution of the split common fixed point problem in two Banach spaces. It seems that such a theorem of Halpern's type iteration is first outside Hilbert spaces. Using this result, we obtain well-known and new strong convergence theorems which are connected with the feasibility problem, the split common null point problem and the split common fixed point problem in Hilbert spaces and in Banach spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty, closed and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle \tag{2.1}$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x \rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [30].

Let E be a real Banach space with norm $\| \cdot \|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ_E of convexity of E is defined by

$$\delta_E(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$ and.the smoothness ρ_E of E is defined by

$$\rho_E(t) = \inf \left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| = 1,\ \|y\| = t \right\}$$

for every $t > 0$. A Banach space E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let $p, q > 1$ be real numbers. A Banach space E is said to be p-uniformly convex if there is a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be q-uniformly smooth if there is a constant $c > 0$ such that $\rho_E(t) \leq ct^q$ for every $t > 0$. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. The norm of E is said to be Fréchet differentiable if for each $x \in U$, the limit (2.2) is attained uniformly for $y \in U$. The norm of E is said to be uniformly smooth if the limit (2.2) is attained uniformly for $x, y \in U$. If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [28, 29]. The following result is in Xu [44].

Lemma 2.1 ([44]). *Let E be a smooth Banach space. Then the following statements are equivalent:*

(1) *E is 2-uniformly smooth;*
(2) *there is a constant $c > 0$ such that for every $x, y \in E$ there holds the following equality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + c\|y\|^2.$$

A Hilbert space H is 2-uniformly smooth and L^p for $p > 1$ is 2-uniformly smooth; see [44]. We know the following result.

Lemma 2.2 ([28]). *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x-y, Jx-Jy\rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x-y, Jx-Jy\rangle = 0$, then $x = y$.*

Let E be a smooth Banach space. The function $\phi\colon E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{2.3}$$

for $x, y \in E$, where J is the duality mapping of E; see [1, 14]. We have from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle \tag{2.4}$$

for all $x, y, z \in E$. From $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. Furthermore, we can obtain the following equality:

$$2\langle x - y, Jz - Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{2.5}$$

for $x, y, z, w \in E$. If E is additionally assumed to be strictly convex, then from Lemma 2.2 we have

$$\phi(x, y) = 0 \iff x = y. \tag{2.6}$$

The following lemma which was by Kamimura and Takahashi [14] is well-known.

Lemma 2.3 ([14]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

The following lemmas are in Xu [45] and Kamimura and Takahashi [14].

Lemma 2.4 ([45]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and λ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.5 ([14]). *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the metric projection of E onto C.

Lemma 2.6 ([28]). *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = P_C x$;
(2) $\langle z - y, J(x - z)\rangle \geq 0, \quad \forall y \in C$.

For any $x \in E$, we also know that there exists a unique element $z \in C$ such that

$$\phi(z, x) = \min_{y \in C} \phi(y, x).$$

The mapping $\Pi_C : E \to C$ defined by $z = \Pi_C x$ is called the generalized projection of E onto C. We know the following result.

Lemma 2.7 ([1, 14]). *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = \Pi_C x$;
(2) $\langle z - y, Jx - Jz\rangle \geq 0, \quad \forall y \in C$.

Let E be a Banach space and let B be a mapping of of E into 2^{E^*}. A multi-valued mapping B on E is said to be monotone if $\langle x - y, u^* - v^*\rangle \geq 0$ for all $u^* \in Bx$, and $v^* \in By$. A monotone operator B on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [5]; see also [29, Theorem 3.5.4].

Theorem 2.8 ([5]). *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let B be a monotone operator of E into 2^{E^*}. Then B is maximal if and only if for any $r > 0$,*

$$R(J + rB) = E^*,$$

where $R(J + rB)$ is the range of $J + rB$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let B be a maximal monotone operator of E into 2^{E^*}. The set of null points of B is defined by $B^{-1}0 = \{z \in E : 0 \in Bz\}$. We know that $B^{-1}0$ is closed and convex; see [29]. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rBx_r.$$

This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such $J_r, r > 0$ is called the metric resolvent of B. For $r > 0$, the Yosida approximation $A_r : E \to E^*$ is defined by

$$A_r x = \frac{J(x - J_r x)}{r}, \quad \forall x \in E.$$

Lemma 2.9 ([29])**.** *Let E be a uniformly convex and smooth Banach space and let $B \subset E \times E^*$ be a maximal monotone operator. Let $r > 0$ and let J_r and A_r be the metric resolvent and the Yosida approximation of B, respectively. Then, the following hold:*

(1) $\langle J_r x - u, J(x - J_r x)\rangle \geq 0, \quad \forall x \in E,\ u \in B^{-1}0;$
(2) $(J_r x, A_r x) \in B, \quad \forall x \in E;$
(3) $F(J_r) = B^{-1}0.$

For all $x \in E$ and $r > 0$, we also consider the following equation

$$Jx \in Jx_r + rBx_r.$$

This equation has a unique solution x_r; see [18]. We define Q_r by $x_r = Q_r x$. Such a Q_r is called the generalized resolvent of B. For $r > 0$, the Yosida approximation $B_r : E \to E^*$ is defined by

$$B_r x = \frac{Jx - JQ_r x}{r}, \quad \forall x \in E.$$

When the Banach space is a Hilbert space, we have that $J_r = Q_r$ for all $r > 0$. Such a J_r is called the resolvent of B simply. We also know the following result.

Lemma 2.10 ([18])**.** *Let E be a uniformly convex and smooth Banach space and let $B \subset E \times E^*$ be a maximal monotone operator. Let $r > 0$ and let Q_r and B_r be the generalized resolvent and the Yosida approximation of B, respectively. Then, the following hold:*

(1) $\phi(u, Q_r x) + \phi(Q_r x, x) \leq \phi(u, x), \quad \forall x \in E,\ u \in B^{-1}0;$
(2) $(Q_r x, B_r x) \in B, \quad \forall x \in E;$
(3) $F(Q_r) = B^{-1}0.$

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let η be a real number with $\eta \in (-\infty, 1)$. Then a mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called η-demimetric [37] if, for any $x \in C$ and $q \in F(U)$,

$$\langle x - q, J(x - Ux)\rangle \geq \frac{1 - \eta}{2}\|x - Ux\|^2,$$

where $F(U)$ is the set of fixed points of U.

Examples We know examples of η-demimetric mappings from [37, 36].

(1) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \leq k < 1$. A mapping $U : C \to H$ is called a

k-strict pseudo-contraction [6] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. If U is a k-strict pseudo-contraction and $F(U) \neq \emptyset$, then U is k-demimetric; see [37].

(2) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called generalized hybrid [15] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1 - \alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. Such a mapping U is called (α, β)-generalized hybrid. Notice that the class of (α, β)-generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is nonspreading [18, 19] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Ux - Uy\|^2 \leq \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [31] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Ux - Uy\|^2 \leq \|x - y\|^2 + \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [13]. If U is generalized hybrid and $F(U) \neq \emptyset$, then U is 0-demimetric; see [37].

(3) Let E be a strictly convex, reflexive and smooth Banach space and let C be a nonempty, closed and convex subset of E. Let P_C be the metric projection of E onto C. Then P_C is (-1)-demimetric; see [37].

(4) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let $\lambda > 0$. Then the metric resolvent J_λ is (-1)-demimetric; see [37].

The following lemma was proved by Takahashi [37].

Lemma 2.11 ([37])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let η be a real number with $\eta \in (-\infty, 1)$. Let U be an η-demimetric mapping of C into E. Then $F(U)$ is closed and convex.*

We also know the following lemmas:

Lemma 2.12 ([3], [45])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence in $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.13 ([20]). *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n\geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ satisfies $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0 + 1) \leq \cdots$ *and* $\tau(n) \to \infty$;

(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}, \ \forall n \geq n_0$.

Let E be a smooth, strictly convex and reflexive Banach space. We make use of the following mapping V studied in Alber [1], Ibaraki and Takahashi [12] and Kohsaka and Takahashi [16, 17]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \tag{2.7}$$

for all $x \in E$ and $x^* \in E^*$. Kohsaka and Takahashi [17] proved the following lemma by using this mapping V. For the sake of completeness, we give the proof.

Lemma 2.14 ([17]). *Let E be a smooth, strictly convex and reflexive Banach space and let V be as in (2.7). Then*

$$V(x, x^*) - 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* - y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Proof. We have that

$$\begin{aligned} V(x, &x^* - y^*) - V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \\ &= \|x\|^2 - 2\langle x, x^* - y^*\rangle + \|x^* - y^*\|^2 \\ &\quad - \|x\|^2 + 2\langle x, x^*\rangle - \|x^*\|^2 + 2\langle J^{-1}x^*, y^*\rangle \\ &= \|x^* - y^*\|^2 - \|x^*\|^2 + 2\langle J^{-1}x^*, y^*\rangle \\ &\geq 2\langle J^{-1}x^*, -y^*\rangle + 2\langle J^{-1}x^*, y^*\rangle \\ &= 0. \end{aligned}$$

This completes the proof. □

3. Strong convergence theorem

In this section, we prove a strong convergence theorem of Halpern's type iteration for solving the split common fixed point problem in two Banach spaces. Let E be a Banach space and let D be a nonempty, closed and convex subset of E. A mapping $U : D \to E$ is called demiclosed if for a sequence $\{x_n\}$ in D such that $x_n \rightharpoonup p$ and $x_n - Ux_n \to 0$, $p = Up$ holds. The following lemma was proved by Matsushita and Takahashi [23].

Lemma 3.1 ([23]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let $T : C \to E$ be a mapping satisfying the following;*

$$\phi(z, Tx) \leq \phi(z, x), \quad \forall x \in C, \ z \in F(T).$$

Then $F(T)$ is closed and convex.

Let C be a nonempty, closed and convex subset of a smooth Banach space E. A mapping $T : C \to E$ is called relatively nonexpansive [23] if $F(T) \neq \emptyset$, T is demiclosed and it satisfies the following:

$$\phi(z, Tx) \leq \phi(z, x), \quad \forall x \in C, \ z \in F(T).$$

The following is our main result.

Theorem 3.2. *Let E be a uniformly convex and uniformly smooth Banach space which E^* is 2-uniformly smooth and it has the best smoothness number $c > 0$. Let F be a smooth, strictly convex and reflexive Banach space. Let J_E and J_F be the duality mappings on E and F, respectively and let η be a real number with $\eta \in (-\infty, 1)$. Let $T : E \to E$ be a relatively nonexpansive mapping and let $U : F \to F$ be an η-demimetric and demiclosed mapping with $F(U) \neq \emptyset$. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that*

$$F(T) \cap A^{-1}F(U) \neq \emptyset.$$

For $u, x_1 = x \in E$, let $\{x_n\} \subset E$ be a sequence generated by

$$\begin{cases} y_n = J_E^{-1}\big(J_E x_n - r_n A^* J_F(Ax_n - UAx_n)\big), \\ z_n = J_E^{-1}(\alpha_n J_E u + (1-\alpha_n) J_E T y_n), \\ x_{n+1} = J_E^{-1}(\beta_n J_E x_n + (1-\beta_n) J_E z_n), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, \delta, \gamma \in \mathbb{R}$, $\{r_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy the following:

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$0 < \delta \leq r_n \leq \gamma < \frac{1-\eta}{c\|A\|^2} \quad \text{and} \quad 0 < a \leq \beta_n \leq b < 1, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in F(T) \cap A^{-1}F(U)$, where $z_0 = \Pi_{F(T)\cap A^{-1}F(U)} u$.

Proof. Since T is relatively nonexpansive, $F(T)$ is closed and convex. We also have from Lemma 2.11 that $F(U)$ is closed and convex. Let $z \in F(T) \cap A^{-1}F(U)$. Then $z = Tz$ and $Az - UAz = 0$. Put

$$y_n = J_E^{-1}\big(J_E x_n - r_n A^* J_F(Ax_n - UAx_n)\big)$$

for all $n \in \mathbb{N}$. We have that

$$\begin{aligned} \phi(z, y_n) &= \phi(z, J_E^{-1}\big(J_E x_n - r_n A^* J_F(Ax_n - UAx_n)\big)) \\ &= \|z\|^2 - 2\langle z, J_E x_n - r_n A^* J_F(Ax_n - UAx_n)\rangle \\ &\quad + \|J_E x_n - r_n A^* J_F(Ax_n - UAx_n)\|^2 \\ &\leq \|z\|^2 - 2\langle z, J_E x_n\rangle + 2r_n\langle z, A^* J_F(Ax_n - UAx_n)\rangle \\ &\quad + \|x_n\|^2 - 2r_n\langle x_n, A^* J_F(Ax_n - UAx_n)\rangle \end{aligned}$$

$$
\begin{aligned}
&+ c\|r_n A^* J_F(Ax_n - UAx_n)\|^2 \\
&\leq \|z\|^2 - 2\langle z, J_E x_n\rangle + 2r_n\langle z, A^* J_F(Ax_n - UAx_n)\rangle \\
&+ \|x_n\|^2 - 2r_n\langle x_n, A^* J_F(Ax_n - UAx_n)\rangle \\
&+ c(r_n\|A\|)^2\|Ax_n - UAx_n\|^2 \qquad (3.1)\\
&= \phi(z, x_n) + 2r_n\langle Az, J_F(Ax_n - UAx_n)\rangle \\
&- 2r_n\langle Ax_n, J_F(Ax_n - UAx_n)\rangle \\
&+ c(r_n\|A\|)^2\|Ax_n - UAx_n\|^2 \\
&= \phi(z, x_n) - 2r_n\langle Ax_n - Az, J_F(Ax_n - UAx_n)\rangle \\
&+ c(r_n\|A\|)^2\|Ax_n - UAx_n\|^2 \\
&= \phi(z, x_n) - r_n(1-\eta)\|Ax_n - UAx_n\|^2 \\
&+ c(r_n\|A\|)^2\|Ax_n - UAx_n\|^2 \\
&= \phi(z, x_n) + r_n(cr_n\|A\|^2 - (1-\eta))\|Ax_n - UAx_n\|^2
\end{aligned}
$$

From $cr_n\|A\|^2 - (1-\eta) \leq 0$, we have that

$$\phi(z, y_n) \leq \phi(z, x_n), \quad \forall n \in \mathbb{N}. \qquad (3.2)$$

Put $z_n = J_E^{-1}(\alpha_n J_E u + (1-\alpha_n)J_E T y_n)$. We have that

$$
\begin{aligned}
\phi(z, z_n) &= \phi(z, J_E^{-1}(\alpha_n J_E u + (1-\alpha_n)J_E T y_n)) \\
&= \|z\|^2 - 2\langle z, \alpha_n J_E u + (1-\alpha_n)J_E T y_n\rangle \\
&\quad + \|\alpha_n J_E u + (1-\alpha_n)J_E T y_n\|^2 \\
&= \|z\|^2 - 2\alpha_n\langle z, J_E u\rangle - 2(1-\alpha_n)\langle z, J_E T y_n\rangle \\
&\quad + \alpha_n\|u\|^2 + (1-\alpha_n)\|Ty_n\|^2 \\
&= \alpha_n\phi(z, u) + (1-\alpha_n)\phi(z, Ty_n) \\
&\leq \alpha_n\phi(z, u) + (1-\alpha_n)\phi(z, y_n) \\
&\leq \alpha_n\phi(z, u) + (1-\alpha_n)\phi(z, x_n).
\end{aligned}
$$

Using this, we get that

$$
\begin{aligned}
\phi(z, x_{n+1}) &= \phi(z, J_E^{-1}(\beta_n J_E x_n + (1-\beta_n)J_E z_n)) \\
&= \|z\|^2 - 2\langle z, \beta_n J_E x_n + (1-\beta_n)J_E z_n\rangle \\
&\quad + \|\beta_n J_E x_n + (1-\beta_n)J_E z_n\|^2 \\
&= \|z\|^2 - 2\beta_n\langle z, J_E x_n\rangle - 2(1-\beta_n)\langle z, J_E z_n\rangle \\
&\quad + \beta_n\|x_n\|^2 + (1-\beta_n)\|z_n\|^2 \\
&= \beta_n\phi(z, x_n) + (1-\beta_n)\phi(z, z_n) \\
&\leq \beta_n\phi(z, x_n) + (1-\beta_n)(\alpha_n\phi(z, u) + (1-\alpha_n)\phi(z, x_n)) \\
&= (1-\alpha_n(1-\beta_n))\phi(z, x_n) + \alpha_n(1-\beta_n)\phi(z, u).
\end{aligned}
$$

Putting $K = \max\{\phi(z, x_1), \phi(z, u)\}$, we have that $\phi(z, x_n) \le K$ for all $n \in \mathbb{N}$. In fact, it is obvious that $\phi(z, x_1) \le K$. Suppose that $\phi(z, x_k) \le K$ for some $k \in \mathbb{N}$. Then we have that

$$\begin{aligned}\phi(z, x_{k+1}) &\le (1 - \alpha_k(1 - \beta_k))\phi(z, x_k) + \alpha_k(1 - \beta_k)\phi(z, u) \\ &\le (1 - \alpha_k(1 - \beta_k))K + \alpha_k(1 - \beta_k)K = K.\end{aligned}$$

By induction, we obtain that $\phi(z, x_n) \le K$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is bounded. Furthermore, $\{Ax_n\}$, $\{z_n\}$ and $\{y_n\}$ are bounded. Take $z_0 = \Pi_{F(T)\cap A^{-1}F(U)}u$. Since $z_n = J_E^{-1}(\alpha_n J_E u + (1 - \alpha_n)J_E T y_n)$, we have that

$$\begin{aligned}J_E x_{n+1} - J_E x_n &= \beta_n J_E x_n + (1 - \beta_n)J_E z_n - J_E x_n \\ &= (1 - \beta_n)(J_E z_n - J_E x_x) \\ &= (1 - \beta_n)\{\alpha_n J_E u + (1 - \alpha_n)J_E T y_n - J_E x_n\} \\ &= (1 - \beta_n)\{\alpha_n(J_E u - J_E T y_n) + J_E T y_n - J_E x_n\}.\end{aligned} \tag{3.3}$$

From (2.5) and (3.2), we have that

$$\begin{aligned}2\langle z_0 - x_n, J_E T y_n - J_E x_n\rangle &= \phi(z_0, x_n) + \phi(x_n, T y_n) - \phi(z_0, T y_n) \\ &\ge \phi(z_0, x_n) + \phi(x_n, T y_n) - \phi(z_0, y_n) \\ &\ge \phi(z_0, x_n) + \phi(x_n, T y_n) - \phi(z_0, x_n) \\ &= \phi(x_n, T y_n).\end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we have that

$$\begin{aligned}2\langle z_0 - x_n, J_E x_{n+1} - J_E x_n\rangle &= 2(1 - \beta_n)\alpha_n\langle z_0 - x_n, J_E u - J_E T y_n\rangle \\ &\quad + 2(1 - \beta_n)\langle z_0 - x_n, J_E T y_n - J_E x_n\rangle \\ &\ge 2(1 - \beta_n)\alpha_n\langle z_0 - x_n, J_E u - J_E T y_n\rangle \\ &\quad + (1 - \beta_n)\phi(x_n, T y_n).\end{aligned} \tag{3.5}$$

Furthermore, using (2.5) and (3.5), we have that

$$\begin{aligned}\phi(z_0, x_n) + \phi(x_n, x_{n+1}) - \phi(z_0, x_{n*1}) &\ge 2(1 - \beta_n)\alpha_n\langle z_0 - x_n, J_E u - J_E T y_n\rangle \\ &\quad + (1 - \beta_n)\phi(x_n, T y_n).\end{aligned}$$

Setting $\mathit{\Gamma}_n = \phi(z_0, x_n)$, we have that

$$\begin{aligned}\mathit{\Gamma}_n - \mathit{\Gamma}_{n+1} + \phi(x_n, x_{n+1}) &\ge 2(1 - \beta_n)\alpha_n\langle z_0 - x_n, J_E u - J_E T y_n\rangle \\ &\quad + (1 - \beta_n)\phi(x_n, T y_n)\end{aligned} \tag{3.6}$$

and hence

$$\begin{aligned}\mathit{\Gamma}_{n+1} - \mathit{\Gamma}_n \le \phi(x_n, x_{n+1}) &\quad - 2(1 - \beta_n)\alpha_n\langle z_0 - x_n, J_E u - J_E T y_n\rangle \\ &\quad - (1 - \beta_n)\phi(x_n, T y_n).\end{aligned} \tag{3.7}$$

Putting

$$r = \max\Big\{\sup_{n\in\mathbb{N}}\|x_n\|, \sup_{n\in\mathbb{N}}\|z_n\|\Big\},$$

we have from Lemma 2.4 that there exists a strictly increasing, continuous and convex function $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$ and

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$$

for all $x, y \in B_r$ and λ with $0 \le \lambda \le 1$, where $B_r = \{z \in E^* : \|z\| \le r\}$. Using this, we have that

$$\begin{aligned}
\phi(x_n, x_{n+1}) &= \|x_n\|^2 - 2\langle x_n, \beta_n J_E x_n + (1-\beta_n) J_E z_n\rangle \\
&\quad + \|\beta_n J_E x_n + (1-\beta_n) J_E z_n\|^2 \\
&\le \|x_n\|^2 - 2\langle x_n, \beta_n J_E x_n + (1-\beta_n) J_E z_n\rangle \\
&\quad + \beta_n\|x_n\|^2 + (1-\beta_n)\|z_n\|^2 - \beta_n(1-\beta_n)g(\|J_E x_n - J_E z_n\|) \\
&= \beta_n \phi(x_n, x_n) + (1-\beta_n)\phi(x_n, z_n) \\
&\quad - \beta_n(1-\beta_n)g(\|J_E x_n - J_E z_n\|) \\
&= (1-\beta_n)\phi(x_n, z_n) - \beta_n(1-\beta_n)g(\|J_E x_n - J_E z_n\|) \\
&= (1-\beta_n)\{\|x_n\|^2 - 2\langle x_n, \alpha_n J_E u + (1-\alpha_n) J_E T y_n\rangle \\
&\quad + \|\alpha_n J_E u + (1-\alpha_n) J_E T y_n\|^2\} \\
&\quad - \beta_n(1-\beta_n)g(\|J_E x_n - J_E z_n\|) \\
&\le (1-\beta_n)\{\alpha_n \phi(x_n, u) + (1-\alpha_n)\phi(x_n, T y_n)\} \\
&\quad - \beta_n(1-\beta_n)g(\|J_E x_n - J_E z_n\|).
\end{aligned} \tag{3.8}$$

We have from (3.7) and (3.8) that

$$\begin{aligned}
\Gamma_{n+1} - \Gamma_n &\le \phi(x_n, x_{n+1}) - 2(1-\beta_n)\alpha_n\langle z_0 - x_n, J_E u - J_E T y_n\rangle \\
&\quad - (1-\beta_n)\phi(x_n, T y_n) \\
&\le (1-\beta_n)\{\alpha_n \phi(x_n, u) + (1-\alpha_n)\phi(x_n, T y_n)\} \\
&\quad - \beta_n(1-\beta_n)g(\|J_E x_n - J_E z_n\|) \\
&\quad - 2(1-\beta_n)\alpha_n\langle z_0 - x_n, J_E u - J_E T y_n\rangle \\
&\quad - (1-\beta_n)\phi(x_n, T y_n). \\
&= (1-\beta_n)\{\alpha_n \phi(x_n, u) + (1-\alpha_n)\phi(x_n, T y_n)\} \\
&\quad - \beta_n(1-\beta_n)g(\|J_E x_n - J_E z_n\|) \\
&\quad - (1-\beta_n)\alpha_n\{\phi(z_0, T y_n) + \phi(x_n, u) - \phi(z_0, u) - \phi(x_n, T y_n)\} \\
&\quad - (1-\beta_n)\phi(x_n, T y_n) \\
&= (1-\beta_n)\alpha_n \phi(x_n, u) + (1-\beta_n)(1-\alpha_n)\phi(x_n, T y_n) \\
&\quad - \beta_n(1-\beta_n)g(\|J_E x_n - J_E z_n\|) \\
&\quad - (1-\beta_n)\alpha_n\{\phi(z_0, T y_n) + \phi(x_n, u) - \phi(z_0, u)\} \\
&\quad + (1-\beta_n)\alpha_n \phi(x_n, T y_n) - (1-\beta_n)\phi(x_n, T y_n) \\
&= (1-\beta_n)\alpha_n \phi(x_n, u) + (1-\beta_n)\phi(x_n, T y_n) \\
&\quad - \beta_n(1-\beta_n)g(\|J_E x_n - J_E z_n\|)
\end{aligned}$$

$$\begin{aligned}
&- (1-\beta_n)\alpha_n\{\phi(z_0, Ty_n) + \phi(x_n, u) - \phi(z_0, u)\} \\
&- (1-\beta_n)\phi(x_n, Ty_n) \\
= &(1-\beta_n)\alpha_n\phi(x_n, u) - \beta_n(1-\beta_n)g(\|J_E x_n - J_E z_n\|) \\
&- (1-\beta_n)\alpha_n\{\phi(z_0, Ty_n) + \phi(x_n, u) - \phi(z_0, u)\}
\end{aligned}$$

and hence

$$\begin{aligned}
\Gamma_{n+1} - &\Gamma_n + \beta_n(1-\beta_n)g(\|J_E x_n - J_E z_n\|) \\
&\leq (1-\beta_n)\alpha_n\phi(x_n, u) \\
&\quad - (1-\beta_n)\alpha_n\{\phi(z_0, Ty_n) + \phi(x_n, u) - \phi(z_0, u)\}.
\end{aligned} \tag{3.9}$$

We will divide the proof into two cases.

Case 1: Suppose that there is a natural number N such that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \geq N$. In this case, $\lim_{n\to\infty} \Gamma_n$ exists and then $\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0$. Using $\lim_{n\to\infty} \alpha_n = 0$ and $0 < a \leq \beta_n \leq b < 1$, we have from (3.9) that

$$\lim_{n\to\infty} \|J_E z_n - J_E x_n\| = 0. \tag{3.10}$$

We also have that

$$\begin{aligned}
\|J_E z_n - J_E T y_n\| &= \|\alpha_n J_E u + (1-\alpha_n) J_E T y_n - J_E T y_n\| \\
&= \alpha_n \|J_E u - J_E T y_n\| \to 0.
\end{aligned} \tag{3.11}$$

Furthermore, from $\|J_E T y_n - J_E x_n\| \leq \|J_E T y_n - J_E z_n\| + \|J_E z_n - J_E x_n\|$, we have that

$$\lim_{n\to\infty} \|J_E T y_n - J_E x_n\| = 0. \tag{3.12}$$

From (3.3) we have that

$$\lim_{n\to\infty} \|J_E x_{n+1} - J_E x_n\| = 0. \tag{3.13}$$

We have from (3.1) that

$$\begin{aligned}
&r_n((1-\eta) - c r_n \|A\|^2)\|Ax_n - UAx_n\|^2 \\
&\leq \phi(z, x_n) - \phi(z, y_n) \\
&\leq \phi(z, x_n) - \phi(z, Ty_n) \\
&= 2\langle z, J_E T y_n - J_E x_n\rangle + \|J_E x_n\|2 - \|J_E T y_n\|2 \\
&= 2\langle z, J_E T y_n - J_E x_n\rangle \\
&\quad + (\|J_E x_n\| - \|J_E T y_n\|)(\|J_E x_n\| + \|J_E T y_n\|) \\
&\leq 2\|z\|\|J_E T y_n - J_E x_n\| \\
&\quad + \|J_E x_n - J_E T y_n\|(\|J_E x_n\| + \|J_E T y_n\|).
\end{aligned}$$

Since $0 < \delta \leq r_n \leq \gamma < \frac{1-\eta}{c\|A\|^2}$ and $\|J_E T y_n - J_E x_n\| \to 0$ from (3.12), we have that

$$\lim_{n\to\infty} \|Ax_n - UAx_n\|^2 = 0. \tag{3.14}$$

Furthermore, since

$$J_E y_n - J_E x_n = -r_n A^* J_F (Ax_n - UAx_n),$$

we have (3.14) that

$$\lim_{n\to\infty} \|J_E x_n - J_E y_n\| = 0. \tag{3.15}$$

We show that

$$\limsup_{n\to\infty} \langle x_n - z_0, J_E u - J_E z_0\rangle \le 0, \tag{3.16}$$

where $z_0 = \Pi_{F(T)\cap A^{-1}F(U)} u$. Put $l = \limsup_{n\to\infty} \langle x_n - z_0, J_E u - J_E z_0\rangle$. Then without loss of generality, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$l = \lim_{i\to\infty} \langle x_{n_i} - z_0, J_E u - J_E z_0\rangle$$

and $\{x_{n_i}\}$ converges weakly to some point $w \in E$. Since $\|J_E x_n - J_E y_n\| \to 0$ from (3.15) and $\lim_{n\to\infty} \|J_E T y_n - J_E x_n\| = 0$ from(3.12), we have

$$\lim_{n\to\infty} \|J_E T y_n - J_E y_n\| = 0.$$

Since E^* is uniformly smooth, we have $\lim_{n\to\infty} \|T y_n - y_n\| = 0$. Since $\|J_E x_n - J_E y_n\| \to 0$ and hence $\|x_n - y_n\| \to 0$ from the uniform smoothness of E^*, we have that $\{y_{n_i}\}$ converges weakly to some point $w \in E$. Since T is relatively nonexpansive, we get that $w \in F(T)$. On the other hand, from (3.14) we have that

$$\lim_{n\to\infty} \|Ax_n - UAx_n\| = 0.$$

Since $\{x_{n_i}\}$ converges weakly to $w \in E$ and A is bounded and linear, we also have that $\{Ax_{n_i}\}$ converges weakly to Aw. Using the demiclosedness of U, we have that $Aw = UAw$. Therefore, $w \in F(T) \cap A^{-1}F(U)$. Since $\{x_{n_i}\}$ converges weakly to $w \in F(T) \cap A^{-1}F(U)$, we have that

$$l = \lim_{i\to\infty} \langle x_{n_i} - z_0, J_E u - J_E z_0\rangle = \langle w - z_0, J_E u - J_E z_0\rangle \le 0.$$

Since $z_n = J_E^{-1}(\alpha_n J_E u + (1-\alpha_n) J_E T y_n)$, we have from Lemma 2.14 that

$$\begin{aligned}
\phi(z_0, z_n) &= \phi(z_0, J_E^{-1}(\alpha_n J_E u + (1-\alpha_n) J_E T y_n)) \\
&= \|z_0\|^2 - 2\langle z_0, \alpha_n J_E u + (1-\alpha_n) J_E T y_n\rangle \\
&\quad + \|\alpha_n J_E u + (1-\alpha_n) J_E T y_n\|^2 \\
&= V(z_0, \alpha_n J_E u + (1-\alpha_n) J_E T y_n) \\
&\le V(z_0, \alpha_n J_E u + (1-\alpha_n) J_E T y_n - \alpha_n (J_E u - J_E z_0)) \\
&\quad + 2\alpha_n \langle J_E^{-1}(\alpha_n J_E u + (1-\alpha_n) J_E T y_n) - z_0, J_E u - J_E z_0\rangle \\
&= V(z_0, \alpha_n J_E z_0 + (1-\alpha_n) J_E T y_n) \\
&\quad + 2\alpha_n \langle z_n - z_0, J_E u - J_E z_0\rangle \\
&= \|z_0\|^2 - 2\langle z_0, \alpha_n J_E z_0 + (1-\alpha_n) J_E T y_n\rangle \\
&\quad + \|\alpha_n J_E z_0 + (1-\alpha_n) J_E T y_n\|^2 \\
&\quad + 2\alpha_n \langle z_n - z_0, J_E u - J_E z_0\rangle
\end{aligned}$$

$$\leq (1-\alpha_n)\phi(z_0, Ty_n) + 2\alpha_n\langle z_n - z_0, J_E u - J_E z_0\rangle$$
$$\leq (1-\alpha_n)\phi(z_0, x_n) + 2\alpha_n\langle z_n - z_0, J_E u - J_E z_0\rangle$$

Thus we have that

$$\begin{aligned}\phi(z_0, x_{n+1}) &\leq \beta_n\phi(z_0, x_n) + (1-\beta_n)\phi(z_0, z_n)\\ &\leq \beta_n\phi(z_0, x_n)\\ &\quad + (1-\beta_n)\left((1-\alpha_n)\phi(z_0, x_n) + 2\alpha_n\langle z_n - z_0, J_E u - J_E z_0\rangle\right)\\ &= \left(\beta_n + (1-\beta_n)(1-\alpha_n)\right)\phi(z_0, x_n)\\ &\quad + 2(1-\beta_n)\alpha_n\langle z_n - z_0, J_E u - J_E z_0\rangle\\ &= (1-(1-\beta_n)\alpha_n)\phi(z_0, x_n)\\ &\quad + 2(1-\beta_n)\alpha_n(\langle z_n - x_n, J_E u - J_E z_0\rangle + \langle x_n - z_0, J_E u - J_E z_0\rangle).\end{aligned}$$

Since $\sum_{n=1}^{\infty}(1-\beta_n)\alpha_n = \infty$, by Lemma 2.12, (3.16) and $x_n - z_n \to 0$, we obtain that $x_n \to z_0$.

Case 2: Suppose that there exists a subsequence $\{\mathit{\Gamma}_{n_i}\}$ of the sequence $\{\mathit{\Gamma}_n\}$ such that $\mathit{\Gamma}_{n_i} < \mathit{\Gamma}_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \leq n : \mathit{\Gamma}_k < \mathit{\Gamma}_{k+1}\}.$$

Then we have from Lemma 2.13 that $\mathit{\Gamma}_{\tau(n)} \leq \mathit{\Gamma}_{\tau(n)+1}$. Thus we have from (3.9) that for all $n \in \mathbb{N}$,

$$\begin{aligned}&\beta_{\tau(n)}(1-\beta_{\tau(n)})g(\|J_E z_{\tau(n)} - J_E x_{\tau(n)}\|)\\ &\qquad \leq (1-\beta_{\tau(n)})\alpha_{\tau(n)}\phi(x_{\tau(n)}, u)\\ &\qquad\quad - (1-\beta_{\tau(n)})\alpha_{\tau(n)}\{\phi(z_0, Ty_{\tau(n)}) + \phi(x_{\tau(n)}, u) - \phi(z_0, u)\}.\end{aligned}$$

Using $\lim_{n\to\infty}\alpha_n = 0$ and $0 < a \leq \beta_n \leq b < 1$, as in the proof of Case 1 we have that

$$\lim_{n\to\infty}\|J_E z_{\tau(n)} - J_E x_{\tau(n)}\| = 0. \tag{3.17}$$

As in the proof of Case 1 we also have that

$$\lim_{n\to\infty}\|J_E T_{\tau(n)} y_{\tau(n)} - J_E z_{\tau(n)}\| = 0. \tag{3.18}$$

Since

$$\begin{aligned}&\|J_E T_{\tau(n)} y_{\tau(n)} - J_E x_{\tau(n)}\|\\ &\qquad \leq \|J_E T_{\tau(n)} y_{\tau(n)} - J_E z_{\tau(n)}\| + \|J_E z_{\tau(n)} - J_E x_{\tau(n)}\|,\end{aligned}$$

we have that

$$\lim_{n\to\infty}\|J_E T_{\tau(n)} y_{\tau(n)} - J_E x_{\tau(n)}\| = 0. \tag{3.19}$$

As in the proof of Case 1 we also have that

$$\lim_{n\to\infty}\|J_E x_{\tau(n)+1} - J_E x_{\tau(n)}\| = 0. \tag{3.20}$$

Furthermore, as in the proof of Case 1 we have that

$$\lim_{n\to\infty} \|Ax_{\tau(n)} - UAx_{\tau(n)}\|^2 = 0. \tag{3.21}$$

and

$$\lim_{n\to\infty} \|J_E x_{\tau(n)} - J_E y_{\tau(n)}\| = 0. \tag{3.22}$$

Since E^* is uniformly smooth, we have from (3.17), (3.18), (3.20) and (3.22) that $\|z_{\tau(n)} - x_{\tau(n)}\| \to 0,$. $\|Ty_{\tau(n)} - z_{\tau(n)}\| \to 0$, $\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0,$. and $\|x_{\tau(n)} - y_{\tau(n)}\| \to 0$, respectively.

For $z_0 = \Pi_{F(T)\cap A^{-1}F(U)}u$, let us show that

$$\limsup_{n\to\infty} \langle x_{\tau(n)} - z_0, J_E z_0 - J_E u\rangle \geq 0.$$

Put $l = \limsup_{n\to\infty} \langle x_{\tau(n)} - z_0, J_E z_0 - J_E u\rangle$. Without loss of generality, there exists a subsequence $\{x_{\tau(n_i)}\}$ of $\{x_{\tau(n)}\}$ such that

$$l = \lim_{i\to\infty} \langle x_{\tau(n_i)} - z_0, J_E z_0 - J_E u\rangle$$

and $\{x_{\tau(n_i)}\}$ converges weakly to some point $w \in E$. From $\|y_{\tau(n)} - x_{\tau(n)}\| \to 0$, $\{y_{\tau(n_i)}\}$ converges weakly to $w \in E$. Furthermore, since $\|z_{\tau(n)} - x_{\tau(n)}\| \to 0$, we also have that $\{z_{\tau(n_i)}\}$ converges weakly to $w \in E$. As in the proof of Case 1 we have that $w \in F(T) \cap A^{-1}F(U)$. Then we have

$$l = \lim_{i\to\infty} \langle x_{\tau(n_i)} - z_0, J_E z_0 - J_E u\rangle = \langle w - z_0, J_E z_0 - J_E u\rangle \geq 0.$$

As in the proof of Case 1, we also have that

$$\phi(z_0, z_{\tau(n)}) \leq (1 - \alpha_{\tau(n)})\phi(z_0, Ty_{\tau(n)}) + 2\alpha_{\tau(n)}\langle z_{\tau(n)} - z_0, J_E u - J_E z_0\rangle$$

and then

$$\begin{aligned}\phi(z_0, x_{\tau(n)+1}) \leq \big(\beta_{\tau(n)} + (1 - \beta_{\tau(n)})(1 - \alpha_{\tau(n)})\big)\, \phi(z_0, x_{\tau(n)}) \\ + 2(1 - \beta_{\tau(n)})\alpha_{\tau(n)}\langle z_{\tau(n)} - z_0, J_E u - J_E z_0\rangle.\end{aligned}$$

From $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, we have that

$$(1 - \beta_{\tau(n)})\alpha_{\tau(n)}\phi(z_0, x_{\tau(n)}) \leq 2(1 - \beta_{\tau(n)})\alpha_{\tau(n)}\langle z_{\tau(n)} - z_0, J_E u - J_E z_0\rangle.$$

Since $(1 - \beta_{\tau(n)})\alpha_{\tau(n)} > 0$, we have that

$$\begin{aligned}\phi(z_0, x_{\tau(n)}) &\leq 2\langle z_{\tau(n)} - z_0, J_E u - J_E z_0\rangle \\ &= 2\langle z_{\tau(n)} - x_{\tau(n)}, J_E u - J_E z_0\rangle + 2\langle x_{\tau(n)} - z_0, J_E u - J_E z_0\rangle.\end{aligned}$$

Thus we have that

$$\limsup_{n\to\infty} \phi(x_{z_0,\tau(n)}) \leq 0$$

and hence $\phi(z_0, x_{\tau(n)}) \to 0$. From (3.20), we have also that $J_E x_{\tau(n)} - J_E x_{\tau(n)+1} \to 0$ and hence $\phi(x_{\tau(n)}, x_{\tau(n)+1}) \to 0$ as $n \to 0$. Using these results, we have that $\phi(z_{0)}, x_{\tau(n)+1}) \to 0$. Using Lemma 2.13 again, we obtain that

$$\phi(z_0, x_n) \leq \phi(z_0, x_{\tau(n)+1}) \to 0$$

as $n \to \infty$. This implies that $x_n \to z_0$. This completes the proof. □

Problem. Can we remove the condition "E^* is a 2-uniforly smooth Banach space" in Theorem 3.2?

4. Applications

In this section, using Theorem 3.2, we first get well-known and new strong convergence theorems which are connected with the feasibility problem, the split common null point problem and the split common fixed point problem in Hilbert spaces and in Banach spaces. We know the following result obtained by Marino and Xu [22]; see also [41].

Lemma 4.1 ([22, 41]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let k be a real number with $0 \leq k < 1$. Let $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

We also know the following result from Kocourek, Takahashi and Yao [15]; see also [43].

Lemma 4.2 ([15, 43]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $U : C \to H$ be generalized hybrid. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

The following theorem was prove by Takahashi [36].

Theorem 4.3. *Let H be a Hilbert space and let F be a smooth, strictly convex and reflexive Banach space. Let J_F be the duality mapping on F and let η be a real number with $\eta \in (-\infty, 1)$. Let $T : H \to H$ be a nonexpansive mapping and let $U : F \to F$ be an η-demimetric and demiclosed mapping with $F(U) \neq \emptyset$. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \neq \emptyset$. For $u, x_1 = x \in H$, let $\{x_n\} \subset H$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)\big(\alpha_n u + (1-\alpha_n)T(x_n - r_n A^* J_F(I-U)Ax_n)\big)$$

for all $n \in \mathbb{N}$, where $a, b, \delta, \gamma \in \mathbb{R}$, $\{r_n\} \subset (0,\infty)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

$$0 < \delta \leq r_n \leq \gamma < \frac{1-\eta}{\|A\|^2}, \quad 0 < a \leq \beta_n \leq b < 1, \quad \forall n \in \mathbb{N},$$

$$\lim_{n\to\infty} \alpha_n = 0 \quad and \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in F(T) \cap A^{-1}F(U)$, where $z_0 = P_{F(T)\cap A^{-1}F(U)}u$.

Proof. A Hilbert space H is a 2-uniformly smooth Banach space which has the best smoothness number $1 > 0$. Since T is a nonexpansive mapping of H_1 into H_1 such that $F(T) \neq \emptyset$, it is relatively nonexpansive. Since $F(T) \cap A^{-1}F(U)$ is nonempty, closed and convex. there exists the metric profection $P_{F(T)\cap A^{-1}F(U)}$ of H onto $F(T) \cap A^{-1}F(U)$. From Theorem 3.2, we have the desired result. □

The following are two strong convergence theorems for solving the split common fixed point problem in two Hilbert spaces.

Theorem 4.4. *Let H_1 and H_2 be Hilbert spaces. Let k be a real number with $k \in [0,1)$. Let $T : H_1 \to H_1$ be a nonspreading mapping with $F(T) \neq \emptyset$ and let $U : H_2 \to H_2$ be a k-strict pseudo-contraction with $F(U) \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \neq \emptyset$. For $u, x_1 = x \in H_1$, let $\{x_n\} \subset H_1$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)\big(\alpha_n u + (1-\alpha_n)T(x_n - r_n A^*(I-U)Ax_n)\big)$$

for all $n \in \mathbb{N}$, where $a, b, \delta, \gamma \in \mathbb{R}$, $\{r_n\} \subset (0,\infty)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

$$0 < \delta \le r_n \le \gamma < \frac{1-k}{\|A\|^2}, \quad 0 < a \le \beta_n \le b < 1, \quad \forall n \in \mathbb{N},$$

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to a point $z_0 \in F(T) \cap A^{-1}F(U)$, where $z_0 = P_{F(T)\cap A^{-1}F(U)}u$.

Proof. Since T is nonspreading of H_1 into H_1, from (2) in Examples, it satisfies the following:

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in H_1.$$

Putting $y = p$ for $p \in F(T)$, we have that

$$2\|Tx - y\|^2 \le \|Tx - y\|^2 + \|y - x\|^2, \quad \forall x \in H_1$$

and hence

$$\|Tx - y\|^2 \le \|y - x\|^2, \quad \forall x \in H_1.$$

This implies that T is quasi-nonexpansive. Furthermore, we have from Lemma 4.2 that T is demiclosed. On the other hand, since U is a k-strict pseudo-contraction of H_2 into H_2 such that $F(U) \neq \emptyset$, from (1) in Examples, U is k-demimetric. Furthermore, from Lemma 4.1, U is demiclosed. Therefore, we have the desired result from Theorem 3.2. $\square$

Theorem 4.5. *Let H_1 and H_2 be Hilbert spaces. Let $T : H_1 \to H_1$ be a hybrid mapping with $F(T) \neq \emptyset$ and let $U : H_2 \to H_2$ be a generalized hybrid mapping with $F(U) \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \neq \emptyset$. For $u, x_1 = x \in H_1$, let $\{x_n\} \subset H$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)\big(\alpha_n u + (1-\alpha_n)T(x_n - r_n A^*(I-U)Ax_n)\big)$$

for all $n \in \mathbb{N}$, where $a, b, \delta, \gamma \in \mathbb{R}$, $\{r_n\} \subset (0,\infty)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

$$0 < \delta \le r_n \le \gamma < \frac{1}{\|A\|^2}, \quad 0 < a \le \beta_n \le b < 1, \quad \forall n \in \mathbb{N},$$

$$\lim_{n\to\infty}\alpha_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty}\alpha_n = \infty.$$

Then $\{x_n\}$ *converges strongly to a point* $z_0 \in F(T) \cap A^{-1}F(U)$, *where* $z_0 = P_{F(T)\cap A^{-1}F(U)}u$.

Proof. Since T is a hybrid mapping of H_1 into H_1 such that $F(T) \neq \emptyset$, from (2) in Examples, it satisfies the following:

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in H_1.$$

Putting $y = p$ for $p \in F(T)$, we have that

$$3\|Tx - y\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|y - x\|^2, \quad \forall x \in H_1$$

and hence

$$\|Tx - y\|^2 \le \|y - x\|^2, \quad \forall x \in H_1.$$

This implies that T is quasi-nonexpansive. Furthermore, we have from Lemma 4.2 that T is demiclosed. Since U is a generalized hybrid mapping of H_2 into H_2 such that $F(U) \neq \emptyset$, from (2) in Examples, U is 0-demimetric. Furthermore, from Lemma 4.2, U is demiclosed. Therefore, we have the desired result from Theorem 3.2. □

The following theorem is a strong convergence theorems for solving the feasibility problem in two Banach spaces.

Theorem 4.6. *Let* E *be a uniformly convex and uniformly smooth Banach space which* E^* *is 2-uniformly smooth and it has the best smoothness number* $c > 0$. *Let* F *be a smooth, strictly convex and reflexive Banach space. Let* J_E *and* J_F *be the duality mappings on* E *and* F, *respectively. Let* C *and* D *be nonempty, closed and convex subsets of* E *and* F, *respectively. Let* Π_C *and* P_D *be the generalized projection of* E *onto* C *and the metric projection of* F *onto* D, *respectively. Let* $A : H \to F$ *be a bounded linear operator such that* $A \neq 0$ *and let* A^* *be the adjoint operator of* A. *Suppose that* $C \cap A^{-1}D \neq \emptyset$. *For* $u, x_1 = x \in E$, *let* $\{x_n\} \subset E$ *be a sequence generated by*

$$\begin{cases} y_n = J_E^{-1}\big(J_E x_n - r_n A^* J_F(Ax_n - P_D A x_n)\big), \\ z_n = J_E^{-1}(\alpha_n J_E u + (1-\alpha_n) J_E \Pi_C y_n), \\ x_{n+1} = J_E^{-1}(\beta_n J_E x_n + (1-\beta_n) J_E z_n), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, \delta, \gamma \in \mathbb{R}$, $\{r_n\} \subset (0,\infty)$, $\{\alpha_n\} \subset (0,1)$ *and* $\{\beta_n\} \subset (0,1)$ *satisfy the following:*

$$\lim_{n\to\infty}\alpha_n = 0, \quad \sum_{n=1}^{\infty}\alpha_n = \infty,$$

$$0 < \delta \le r_n \le \gamma < \frac{2}{c\|A\|^2} \quad \textit{and} \quad 0 < a \le \beta_n \le b < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ *converges strongly to a point* $z_0 \in C \cap A^{-1}D$, *where* $z_0 = \Pi_{C\cap A^{-1}D}u$.

Proof. Since Π_C is the genralized projection of E onto C, we have from Lemma 2.7 that

$$\phi(z, \Pi_C x) \leq \phi(z, x), \quad \forall x \in E, \ z \in C.$$

We show that Π_C is demiclosed. In fact, assume that $x_n \rightharpoonup p$ and $x_n - \Pi_C x_n \to 0$. It is clear that $\Pi_C x_n \rightharpoonup p$. Since E is uniformly smooth, we have that $\|J_E x_n - J_E \Pi_C x_n\| \to 0$. Since Π_C is the generalized projection of E onto C, we have that

$$\langle \Pi_C x_n - \Pi_C p, J_E x_n - J_E \Pi_C x_n - (J_E p - J_E \Pi_C p)\rangle \geq 0.$$

Therefore, $\langle p - \Pi_C p, -(J_E p - J_E \Pi_C p)\rangle \geq 0$ and hence $\phi(p, \Pi_C p) + \phi(\Pi_C p, p) \leq 0$. This implies that $p = \Pi_C p$ and hence Π_C is demiclosed. On the other hand, since P_D is the metric projection of F onto D, from (3) in Examples, P_D is (-1)-demimetric. We also have that if $\{x_n\}$ is a sequence in F such that $x_n \rightharpoonup p$ and $x_n - P_D x_n \to 0$, then $p = P_D p$. In fact, assume that $x_n \rightharpoonup p$ and $x_n - P_D x_n \to 0$. It is clear that $P_D x_n \rightharpoonup p$ and $\|J_F(x_n - P_D x_n)\| = \|x_n - P_D x_n\| \to 0$. Since P_D is the metric projection of F onto D, we have that

$$\langle P_D x_n - P_D p, J_F(x_n - P_D x_n) - J_F(p - P_D p)\rangle \geq 0.$$

Therefore, $-\|p - P_D p\|^2 = \langle p - P_D p, -J_F(p - P_D p)\rangle \geq 0$ and hence $p = P_D p$. This implies that P_D is demiclosed. Therefore, we have the desired result from Theorem 3.2. □

The following theorem is a strong convergence theorems for solving the split null point problem in two Banach spaces.

Theorem 4.7. *Let E be a uniformly convex and uniformly smooth Banach space which E^* is 2-uniformly smooth and it has the best smoothness number $c > 0$. Let F be a smooth, strictly convex and reflexive Banach space. Let J_E and J_F be the duality mappings on E and F, respectively. Let B and G be maximal monotone operators of E into E^* and F into F^*, respectively. Let Q_μ be the generalized resolvent of B for $\mu > 0$ and let J_λ be the metric resolvent of G for $\lambda > 0$, respectively. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $B^{-1}0 \cap A^{-1}(G^{-1}0) \neq \emptyset$. For $u, x_1 = x \in E$, let $\{x_n\} \subset E$ be a sequence generated by*

$$\begin{cases} y_n = J_E^{-1}\big(J_E x_n - r_n A^* J_F(Ax_n - J_\lambda A x_n)\big), \\ z_n = J_E^{-1}(\alpha_n J_E u + (1-\alpha_n) J_E Q_\mu y_n), \\ x_{n+1} = J_E^{-1}(\beta_n J_E x_n + (1-\beta_n) J_E z_n), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, \delta, \gamma \in \mathbb{R}$, $\{r_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy the following:

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$0 < \delta \leq r_n \leq \gamma < \frac{2}{c\|A\|^2} \quad \text{and} \quad 0 < a \leq \beta_n \leq b < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ *converges strongly to a point* $z_0 \in B^{-1}0 \cap A^{-1}(G^{-1}0)$*, where* $z_0 = \Pi_{B^{-1}0 \cap A^{-1}(G^{-1}0)} u$*.*

Proof. Since Q_μ is the generalized resolvent of B on E, we have from Lemma 2.10 that

$$\phi(z, Q_\mu x) \leq \phi(z, x), \quad \forall x \in E,\ z \in B^{-1}0.$$

Next, we show that Π_C is demiclosed. In fact, assume that $x_n \rightharpoonup p$ and $x_n - Q_\mu x_n \to 0$. It is clear that $Q_\mu x_n \rightharpoonup p$. Since E is unifrmly smooth, we have that $\|J_E x_n - J_E Q_\mu x_n)\| \to 0$. Since Q_μ is the generalized resolvent of B, we have from [4] that

$$\langle Q_\mu x_n - Q_\mu p, J_E x_n - J_E Q_\mu x_n - (J_E p - J_E Q_\mu p)\rangle \geq 0.$$

Therefore, $\langle p - Q_\mu p, -(J_E p - J_E Q_\mu p)\rangle \geq 0$ and hence $\phi(p, Q_\mu p) + \phi(Q_\mu p, p) \leq 0$. This implies that $p = Q_\mu p$ and hence Q_μ is demiclosed. On the other hand, since J_λ is the metric resolvent of G for $\lambda > 0$, from (4) in Examples, J_λ is (-1)-demimetric. We also have that if $\{x_n\}$ is a sequence in F such that $x_n \rightharpoonup p$ and $x_n - J_\lambda x_n \to 0$, then $p = J_\lambda p$. In fact, assume that $x_n \rightharpoonup p$ and $x_n - J_\lambda x_n \to 0$. It is clear that $J_\lambda x_n \rightharpoonup p$ and $\|J_F(x_n - J_\lambda x_n)\| = \|x_n - J_\lambda x_n\| \to 0$. Since J_λ is the metric resolvent of G, we have from [4] that

$$\langle J_\lambda x_n - J_\lambda p, J_F(x_n - J_\lambda x_n) - J_F(p - J_\lambda p)\rangle \geq 0.$$

Therefore, $-\|p - J_\lambda p\|^2 = \langle p - J_\lambda p, -J_F(p - J_\lambda p)\rangle \geq 0$ and hence $p = J_\lambda p$. This implies that J_λ is demiclosed. Therefore, we have the desired result from Theorem 3.2. □

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