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ON DUALITY THEOREMS FOR LINEAR FRACTIONAL OPTIMIZATION PROBLEMS INVOLVING INTEGRAL FUNCTIONS

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Abstract. We consider a linear fractional optimization problem which consists of a linear fractional integral objective function, linear integral constraint functions and a constraint cone. We formulate a nonfractional dual problem for the problem, and then establish duality theorems (the weak duality theorem, the strong duality theorem and the converse duality theorem) which hold between the problem and its dual problem.

1. Introduction and preliminaries

Craven [1, 2] studied the linear fractional program:

$$\text{(LF)} \quad \text{maximize } \frac{c^T x + \alpha}{d^T x + \beta} \text{ subject to } x \geqq 0, Ax = b,$$

where $c, d \in \mathbb{R}^n,\ \alpha, \beta \in \mathbb{R},\ A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ are given, and made the following linear program (LF$^{\#}$) by using a trasformation: $x \to \left(\frac{1}{d^T x+\beta} x, \frac{1}{d^T x+\beta}\right)$:

$$\begin{array}{lll} (\text{LF}^{\#}) & \text{maximize} & c^T y + \alpha t \\ & \text{subject to} & y \geqq 0, t \geqq 0, d^T y + \beta t = 1,\ Ay - bt = 0. \end{array}$$

Here the variables are $y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, instead of $x \in \mathbb{R}^n$ in (LF). Craven proved in Theorem 2.2.3 of [1] the following theorem:

Theorem 1.1. *Assume that no point $(y, 0)$ with $y \geqq 0$ is feasible for (LF$^{\#}$) and that $[x \geqq 0, Ax = b] \Rightarrow [d^T x + \beta > 0]$. Then the linear fractional program (LF) is equivalent to the linear program (LF$^{\#}$).*

Moreover, Craven formulated the dual problem for (LF) by using the dual problem of (LF$^{\#}$) as follows:

$$\text{(LD)} \quad \text{Minimize } \gamma \text{ subject to } A^T s + d\gamma \geqq c, \quad \beta\gamma - b^T s \geqq \alpha.$$

Noticing that the linear program dual is a proper dual, Craven [1] gave the following theorem:

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Theorem 1.2. *The linear program (LD) is a proper dual to the linear fractional program (LF), that is, the weak duality, the strong duality and the converse duality hold between (LF) and (LD).*

Recently, we [5] studied duality theorems for a linear fractional semidefinite optimization problem. By using the Craven's approaches mentioned above, that is, the transformation and the equivalence in Theorem 1.1, we can prove duality theorems for the problem. But in the paper [5], we directly proved the duality theorems for the problem.

In this paper, we will directly prove the duality theorems for a linear fractional optimization problem which consists of a linear fractional integral objective function, linear integral constraint functions and a constraint cone.

Consider the following linear fractional optimization problem :

$$
\begin{aligned}
\text{(P)} \qquad & \text{Minimize} \quad && \frac{\displaystyle\int_0^1 c(t)^T x(t)dt + \alpha}{\displaystyle\int_0^1 d(t)^T x(t)dt + \beta} \\
& \text{subject to} \quad && x(\cdot) \in K, \\
& && \int_0^1 a_i(t)^T x(t)dt = b_i, \ i = 1, \ldots, m,
\end{aligned}
$$

where $c, d \in L_n^2[0,1]$, $a_i \in L_n^2[0,1]$, $b_i \in \mathbb{R}, i = 1, \ldots, m$, $\alpha, \beta \in \mathbb{R}$ are given, and K is a closed convex cone in $L_n^2[0,1]$ with $\text{int}K \neq \emptyset$. Let $K^* = \{z \in L_n^2[0,1] \mid \int_0^1 z(t)^T x(t)dt \geqq 0 \text{ for any } x \in K\}$ and assume that $\text{int}K^* \neq \emptyset$.

Let $\triangle = \{x \in K \mid \int_0^1 a_i(t)^T x(t)dt = b_i, \ i = 1, \ldots, m\}$.

Theorem 1.3 ([6])**.** *Let X and Y be Banach spaces. $f : X \to \mathbb{R}$, $g : X \to Y$ and C a convex subset of X. Let x_0 be an optimal solution of : $\inf\{f(x) \mid g(x) = 0, \quad x \in C\}$. Assume that f is Fréchet differentiable at x_0 and g is Fréchet differentiable in a neighborhood of x_0 and that the Fréchet differential g' is continuous at x_0. Furthermore assume that there exists $\widehat{x} \in intC$ such that $g'(x_0)(\widehat{x} - x_0) = 0$. Assume that $g'(x_0)(\cdot)$ is surjective. Then there exists $y^* \in Y^*$ such that*

$$\Big[f'(x_0) + y^* \circ g'(x_0)\Big](x - x_0) \geqq 0 \text{ for any } x \in C.$$

2. Duality theorems

We formulate the dual problem for (P) as follows:

$$
\begin{aligned}
\text{(D)} \qquad & \text{Maximize} \quad && \gamma \\
& \text{subject to} \quad && c - \sum_{i=1}^m y_i a_i - \gamma d \in K^*, \\
& && \beta\gamma - b^T y \leqq \alpha, \\
& && y \in \mathbb{R}^m, \ \gamma \in \mathbb{R},
\end{aligned}
$$

Theorem 2.1 (Weak duality)**.** *Assume that for any* $x \in \triangle$, $\int_0^1 d(t)^T x(t)dt + \beta > 0$. *Let* x *be feasible for (P) and let* (γ, y) *be feasible for (D). Then*

$$\frac{\displaystyle\int_0^1 c(t)^T x(t)dt + \alpha}{\displaystyle\int_0^1 d(t)^T x(t)dt + \beta} \geqq \gamma.$$

Proof. Since $c - \sum_{i=1}^m y_i a_i - \gamma d \in K^*$ and $x \in K$ we have

$$\int_0^1 \Big[c(t) - \sum_{i=1}^m y_i a_i(t) - \gamma d(t)\Big]^T x(t) \geqq 0.$$

For any $x \in \triangle$,

$$\begin{aligned}
&\int_0^1 c(t)^T x(t)dt + \alpha - \gamma\Big[\int_0^1 d(t)^T x(t)dt + \beta\Big] \\
&= \int_0^1 \Big[c(t) - \gamma d(t)\Big]^T x(t)dt + \alpha - \gamma\beta \\
&\geqq \int_0^1 \Big[\sum_{i=1}^m y_i a_i(t)\Big]^T x(t)dt + \alpha - \gamma\beta \\
&= \sum_{i=1}^m y_i \int_0^1 a_i(t)^T x(t)dt + \alpha - \gamma\beta \\
&= \sum_{i=1}^m y_i b_i + \alpha - \gamma\beta \\
&\geqq 0.
\end{aligned}$$

Therefore,

$$\frac{\displaystyle\int_0^1 c(t)^T x(t)dt + \alpha}{\displaystyle\int_0^1 d(t)^T x(t)dt + \beta} \geqq \gamma.$$

□

Theorem 2.2 (Strong duality)**.** *Let* $\bar{x}$ *be an optimal solution of (P). Suppose that for any* $x \in \triangle$, $\int_0^1 d(t)^T x(t)dt + \beta > 0$. *Assume that there exists* $\widehat{x}(\cdot) \in intK$ *such that*

$$\int_0^1 a_i(t)^T \Big(\widehat{x}(t) - \bar{x}(t)\Big) = 0, \quad i = 1, \dots, m$$

and that $a_1, \dots, a_m$ *are linearly independent in* $L_n^2[0,1]$. *Then there exists* $y \in \mathbb{R}^m$ *such that* $\left(\frac{\int_0^1 c(t)^T \bar{x}(t)dt+\alpha}{\int_0^1 d(t)^T \bar{x}(t)dt+\beta}, y\right)$ *is an optimal solution of (D).*

Proof. Let $X = L_n^2[0,1]$, $Y = \mathbb{R}^m$ and $C = K$. Let $q(x) = \frac{\int_0^1 c(t)^T x(t)dt+\alpha}{\int_0^1 d(t)^T x(t)dt+\beta}$. Since $\bar{x}$ is an optimal solution of (P), $\bar{x}$ is an optimal solution of the following optimization problem:

$$\begin{array}{ll} \text{Minimize} & \int_0^1 c(t)^T x(t)dt + \alpha - q(\bar{x})\Big[\int_0^1 d(t)^T x(t)dt + \beta\Big] \\ \text{subject to} & x \in \triangle. \end{array}$$

Define $f : X \to \mathbb{R}$ by $f(x) = \int_0^1 c(t)^T x(t)dt + \alpha - q(\bar{x})\Big[\int_0^1 d(t)^T x(t)dt + \beta\Big]$ and $g : X \to Y$ by $g(x) = \begin{pmatrix} \int_0^1 a_1(t)^T x(t)dt - b_1 \\ \vdots \\ \int_0^1 a_m(t)^T x(t)dt - b_m \end{pmatrix}$. Then the Fréchet differential of f at $\bar{x}$ is $f'(\bar{x})h = \int_0^1 [c(t) - q(\bar{x})d(t)]^T h(t)dt$ for any $h \in X$, and the Fréchet differential of g at $\bar{x}$ is $g'(\bar{x})h = \begin{pmatrix} \int_0^1 a_1(t)^T h(t)dt \\ \vdots \\ \int_0^1 a_m(t)^T h(t)dt \end{pmatrix}$ for any $h \in X$. Also, g' is continuous at x_0. By assumption, there exists $\widehat{x} \in intC$ such that $g'(\bar{x})(\widehat{x} - \bar{x}) = 0$. Since $a_1, \ldots, a_m$ are linearly independent in $L_n^2[0,1]$, it follows from the lemma on the biorthogonal basis in [3], there exist $a_1^*, \ldots, a_m^* \in L_n^2[0,1]$ such that $\int_0^1 a_i^*(t)^T a_j(t)dt = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$ Let $\gamma = (\gamma_1, \ldots, \gamma_m)^T \in \mathbb{R}^m$ be any point. Then $g'(\bar{x})(\sum_{i=1}^m \gamma_i a_i^*) = \begin{pmatrix} \int_0^1 a_1(t)^T \sum_{i=1}^m \gamma_i a_i^*(t)dt \\ \vdots \\ \int_0^1 a_m(t)^T \sum_{i=1}^m \gamma_i a_i^*(t)dt \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_m \end{pmatrix} = \gamma$. Thus $g'(\bar{x})(\cdot)$ is surjective. By Theorem 1.3, there exists $y^* \in Y^*$ such that $\Big[f'(x_0) - y^* \circ g'(x_0)\Big](x - x_0) \geqq 0$ for any $x \in K$. Thus there exist $y_i \in \mathbb{R},\ i = 1, \ldots, m$ such that

$$\int_0^1 \Big[c(t) - q(\bar{x})d(t) - \sum_{i=1}^m y_i a_i(t)\Big]^T (x(t) - \bar{x}(t))dt \geqq 0 \ \text{ for any } x(\cdot) \in K.$$

Let $\bar{\gamma} = q(\bar{x})$. Then $\int_0^1 \Big[c(t) - \bar{\gamma}d(t) - \sum_{i=1}^m y_i a_i(t)\Big]^T x(t)dt \geqq 0$ for any $x(\cdot) \in K$. i.e., $c - \bar{\gamma}d - \sum_{i=1}^m y_i a_i \in K^*$. Moreover, we have

$$\int_0^1 \Big[c(t) - \bar{\gamma}d(t) - \sum_{i=1}^m y_i a_i(t)\Big]^T \bar{x}(t)dt = 0.$$

Since $\bar{x}$ is feasible for (P), $\int_0^1 a_i(t)^T\bar{x}(t)dt = b_i,\ i = 1,\dots,m$, and so,

$$\begin{aligned} -\beta\bar{\gamma} + \sum_{i=1}^{m} b_i y_i + \alpha &= -\beta\bar{\gamma} + \sum_{i=1}^{m} y_i \int_0^1 a_i(t)^T\bar{x}(t)dt + \alpha \\ &= -\beta\bar{\gamma} + \alpha + \int_0^1 \Big[c(t) - \bar{\gamma}d(t)\Big]^T \bar{x}(t)dt \\ &= \int_0^1 c(t)^T\bar{x}(t)dt + \alpha - \bar{\gamma}\Big[\int_0^1 d(t)^T\bar{x}(t)dt + \beta\Big] \\ &= 0. \end{aligned}$$

Thus, $(\bar{\gamma}, y)$ is feasible for (D). By Theorem 2.1 (Weak duality), for any feasible $(\gamma, \widetilde{y})$ for D,

$$\gamma \leqq \frac{\displaystyle\int_0^1 c(t)^T\bar{x}(t)dt + \alpha}{\displaystyle\int_0^1 d(t)^T\bar{x}(t)dt + \beta} = \bar{\gamma}.$$

So, $(\bar{\gamma}, y)$ is an optimal solution of (D). □

Theorem 2.3 (Converse duality)**.** *Assume that for any $x \in \triangle$, $\int_0^1 d(t)^T x(t)dt + \beta > 0$ and that $\triangle$ is bounded. Further assume that there exist $\widetilde{y} \in \mathbb{R}^m$ and $\widetilde{\gamma} \in \mathbb{R}$ such that $c - \sum_{i=1}^m \widetilde{y}_i a_i - \widetilde{\gamma} d \in intK^*$ and $\beta\widetilde{\gamma} - b^T\widetilde{y} < \alpha$. If $(\bar{\gamma}, \bar{y})$ is an optimal solution of (D), then there exists $\bar{x} \in \triangle$ such that $\bar{x}$ is an optimal solution of (P) and*

$$\bar{\gamma} = \frac{\displaystyle\int_0^1 c(t)^T\bar{x}(t)dt + \alpha}{\displaystyle\int_0^1 d(t)^T\bar{x}(t)dt + \beta}.$$

Proof. Let $(\bar{\gamma}, \bar{y})$ be an optimal solution of (D). By Lemma 4.1 in [4], there exist $v \in K$ and $\eta \in \mathbb{R}_+$ such that

$$-1 + \int_0^1 d(t)^T v(t)dt + \eta\beta = 0, \tag{2.1}$$

$$\int_0^1 a_i(t)^T v(t)dt - \eta b_i = 0, \quad i = 1,\dots,m, \tag{2.2}$$

$$\int_0^1 \Big[c(t) - \sum_{i=1}^{m} \bar{y}_i a_i(t) - \bar{\gamma} d(t)\Big]^T v(t)dt = 0, \tag{2.3}$$

$$\eta(\beta\bar{\gamma} - b^T\bar{y} - \alpha) = 0, \tag{2.4}$$

$$c - \sum_{i=1}^{m} \bar{y}_i a_i - \bar{\gamma} d \in K^* \text{ and } \beta\bar{\gamma} - b^T\bar{y} \leqq \alpha.$$

From (2.3) and (2.4),

$$\begin{aligned}\int_0^1 c(t)^T v(t)dt &= \int_0^1 \Big[\sum_{i=1}^m \bar{y}_i a_i(t) + \bar{\gamma} d(t)\Big]^T v(t)dt \\ \alpha\eta &= \eta\beta\bar{\gamma} - \eta b^T \bar{y}.\end{aligned}$$

From (2.1) and (2.2),

$$\begin{aligned}\int_0^1 c(t)^T v(t)dt + \alpha\eta &= \int_0^1 \Big[\sum_{i=1}^m \bar{y}_i a_i(t) + \bar{\gamma} d(t)\Big]^T v(t)dt + \eta\beta\bar{\gamma} - \eta b^T \bar{y} \\ &= \int_0^1 \sum_{i=1}^m \bar{y}_i a_i(t)^T v(t)dt - \eta b^T \bar{y} \\ &\quad + \bar{\gamma}\Big(\int_0^1 d(t)^T v(t)dt + \eta\beta\Big) \\ &= \bar{\gamma}.\end{aligned}$$

Thus we have

$$\int_0^1 c(t)^T v(t)dt + \alpha\eta = \bar{\gamma}, \tag{2.5}$$

$$\int_0^1 d(t)^T v(t)dt + \eta\beta = 1, \tag{2.6}$$

$$\int_0^1 a_i(t)^T v(t)dt - \eta b_i = 0,\ i = 1, \ldots, m. \tag{2.7}$$

Suppose that $\eta = 0$. From (2.6) and (2.7), $\|v\| := \int_0^1 \|v(t)\|dt > 0$ and $\int_0^1 a_i(t)^T v(t)dt = 0,\ i = 1 \ldots, m$. For any $x \in \triangle$ and any $\gamma > 0$, $x + \gamma v \in K$ and

$$\begin{aligned}\int_0^1 a_i(t)^T (x(t) + \bar{\gamma} v(t))dt &= \int_0^1 a_i(t)^T x(t)dt + \bar{\gamma} \int_0^1 a_i(t)^T v(t)dt \\ &= b_i\end{aligned}$$

and so $x + \gamma v \in \triangle$. This contradicts the boundedness of the set $\triangle$. Thus $\eta > 0$. Let $\bar{x} = \frac{1}{\eta} v$. Since $v \in K$, $\bar{x} \in K$. From (2.5), (2.6) and (2.7), $\eta = \frac{1}{\int_0^1 d(t)^T \bar{x}(t)dt + \beta}$, $\bar{\gamma} = \frac{\int_0^1 c(t)^T \bar{x}(t)dt + \alpha}{\int_0^1 d(t)^T \bar{x}(t)dt + \beta}$ and $\int_0^1 a_i(t)^T \bar{x}(t)dt = b_i,\ i = 1, \ldots, m$. By Theorem 2.1 (weak duality), $\bar{x}$ is an optimal solution of (P). $\square$

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