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STRONG CONVERGENCE THEOREMS FOR FINDING COMMON ATTRACTIVE POINTS OF NORMALLY 2-GENERALIZED HYBRID MAPPINGS AND APPLICATIONS

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Abstract. In this paper, using the ideas of mean convergence by Shimizu and Takahashi [23, 24], Atsushiba and Takahashi [3], and Kurokawa and Takahashi [20]. we prove two strong convergence theorems for finding common attractive and fixed points of two normally 2-generalized hybrid mappings in a Hilbert space. The mappings are not necessarily commutative. These two theorems are used to obtain well-known and new strong convergence theorems which are connected with normally 2-generalized hybrid mappings in a Hilbert space.

1. Introduction

In this paper, we denote a real Hilbert space by H, and its inner product and norm by $\langle \cdot, \ \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty subset of H, and let T be a mapping from C into H. The sets of fixed and *attractive points* [30] are denoted by

$$\begin{aligned} F(T) &= \{u \in C : Tu = u\} \text{ and} \\ A(T) &= \{u \in H : \|Ty - u\| \leq \|y - u\| \text{ for all } y \in C\}, \end{aligned}$$

respectively. The concept of attractive points was introduced by Takahashi and Takeuchi in their 2011's paper [30]. A mapping $T : C \to H$ is called

(i) *firmly nonexpansive* if $\|Tx - Ty\|^2 \leq \langle x - y, \ Tx - Ty \rangle$ for all $x, y \in C$;

(ii) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

It is well-known that a firmly nonexpansive mapping is nonexpansive. This mapping is deduced from the resolvent of a maximal monotone operator of H into itself. For nonexpansive mappings, approximation methods for finding fixed points have been studied. Wittmann [34] proved a strong convergence to a fixed point of T by using the Halpern's type iteration [4]:

$$x_{n+1} = \lambda_n x + (1 - \lambda_n) T x_n \quad \text{for all } n \in \mathbb{N}. \tag{1.1}$$

In (1.1), $x_1 = x \in C$ is given, $\{\lambda_n\}$ is a sequence of real numbers in the interval $[0, 1]$ that satisfies certain conditions, and $\mathbb{N}$ is the set of natural numbers.

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According to successive studies, the conditions imposed on mappings can be relaxed to include important classes of mappings. Kocourek et al. [12] defined a wide class of mappings. A mapping $T : C \to H$ is called

(iii) *generalized hybrid* [12] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + (1 - \alpha) \|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta) \|x - y\|^2$$

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers. The class of generalized hybrid mappings simultaneously includes nonexpansive mappings, *nonspreading mappings* [14], *hybrid mappings* [28], and λ-*hybrid mappings* [1] as special cases. A nonspreading mapping which is deduced from a firmly nonexpansive mapping is not necessarily continuous; see [10] or [32].

For nonspreading mappings, Kurokawa and Takahashi [20] used the following iteration:

$$x_{n+1} = \lambda_n w + (1 - \lambda_n) \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n, \tag{1.2}$$

for all $n \in \mathbb{N}$ and given $x_1, w \in C$, and established a strong convergence theorem for finding a fixed point of T. The idea of mean convergence as (1.2) based on Shimizu and Takahashi [23] [24], and Atsushiba and Takahashi [3]; see also Kohsaka [13], and Hojo and Takahashi [7]. For generalized hybrid mappings, Takahashi et al. [32] demonstrated a strong convergence theorem by using the iteration

$$x_{n+1} = \lambda_n w + (1 - \lambda_n) (\alpha_n x_n + (1 - \alpha_n) T x_n) \quad \text{for all } n \in \mathbb{N}. \tag{1.3}$$

In (1.3), $x_1, w \in C$ are given, and $\{\lambda_n\}$ and $\{\alpha_n\}$ are sequences in $[0, 1]$.

The class of generalized hybrid mappings has been further extended. A mapping $T : C \to C$ is called

(iv) *normally* 2-*generalized hybrid* [15] if there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that $\sum_{n=0}^{2} (\alpha_n + \beta_n) \geq 0$, $\alpha_2 + \alpha_1 + \alpha_0 > 0$, and

$$\alpha_2 \left\|T^2 x - Ty\right\|^2 + \alpha_1 \|Tx - Ty\|^2 + \alpha_0 \|x - Ty\|^2 + \beta_2 \left\|T^2 x - y\right\|^2 + \beta_1 \|Tx - y\|^2 + \beta_0 \|x - y\|^2 \leq 0$$

for all $x, y \in C$. This class of mappings contains generalized hybrid mappings, *normally generalized hybrid mappings* [31], and 2-*generalized hybrid mappings* [22] as special cases. Hojo et al. [8] gave examples that are 2-generalized hybrid but not generalized hybrid. It can be shown that if $\sum_{n=0}^{2} (\alpha_n + \beta_n) > 0$, then a normally 2-generalized hybrid mapping has at most one fixed point; see Theorem 4.3 in [17].

Let $T : C \to C$ be a normally 2-generalized hybrid mapping. Kondo and Takahashi [16] considered the following iteration:

$$x_{n+1} = \lambda_n w + (1 - \lambda_n) \left(a_n x_n + b_n T x_n + c_n T^2 x_n\right) \quad \text{for all } n \in \mathbb{N}. \tag{1.4}$$

In (1.4), $x_1, w \in C$ are given, and $a_n, b_n, c_n \in [0, 1]$ such that $a_n + b_n + c_n = 1$. They showed that the sequence $\{x_n\}$ converge strongly to an attractive point of T. Very recently, Kondo and Takahashi [18] applied the iteration (1.4) to common attractive point problems of two normally 2-generalized hybrid mappings. They considered the following iteration:

$$x_{n+1} = \lambda_n w_n + (1-\lambda_n)\left(a_n x_n + b_n S x_n + c_n S^2 x_n + d_n T x_n + e_n T^2 x_n\right) \tag{1.5}$$

for all $n \in \mathbb{N}$. In (1.5), $x_1 \in C$ is given, $a_n, b_n, c_n, d_n, e_n \in [0,1]$ such that $a_n + b_n + c_n + d_n + e_n = 1$, and the sequence $\{w_n\}$ in C is convergent. They proved a strong convergence theorem to a common attractive point of S and T. For common fixed or attractive point problems, see also Aoyama et al. [2], Iemoto and Takahashi [9], Hojo et al. [5], Takahashi [29], and Takahashi et al. [33].

In this paper, combining the ideas of the iterations (1.2) and (1.5), we consider two types of iterations as follows:

$$\begin{aligned} x_{n+1} &= \lambda_n w_n + (1-\lambda_n)\left(a_n x_n + b_n \frac{1}{n}\sum_{k=1}^{n} S^k x_n + c_n \frac{1}{n}\sum_{k=1}^{n} T^k x_n\right) \text{ and} \\ x_{n+1} &= \lambda_n w_n + (1-\lambda_n)\left(a_n x_n + b_n S x_n + c_n S^2 x_n + d_n \frac{1}{n}\sum_{k=1}^{n} T^k x_n\right), \end{aligned}$$

where S and T are normally 2-generalized hybrid mappings, which are not necessarily commutative. Using these iterations, we show that the sequence $\{x_n\}$ converges strongly to common attractive and fixed points of S and T (Theorem 3.1 and 3.2). These two theorems are used to obtain well-known and new strong convergence theorems which are connected with normally 2-generalized hybrid mappings in a Hilbert space.

2. Preliminaries

This section briefly presents definitions of basic concepts and preliminary results. In a real Hilbert space H, it is known that

$$2\langle x-y,\ y\rangle \leq \|x\|^2 - \|y\|^2 \leq 2\langle x-y,\ x\rangle \tag{2.1}$$

for all $x, y \in H$. The strong and weak convergence of a sequence $\{x_n\}$ in H to an element $x\,(\in H)$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Let T be a mapping from C into H, where C is a nonempty subset of H. Takahashi and Takeuchi [30] showed that the set of attractive points $A(T)$ is closed and convex in a Hilbert space. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if $\|Tx - u\| \leq \|x - u\|$ for all $x \in C$ and $u \in F(T)$. For a quasi-nonexpansive mapping T, it holds that $F(T) \subset A(T)$. We know from [15] that a normally 2-generalized hybrid mapping with $F(T) \neq \emptyset$ is quasi-nonexpansive. We also know from Itoh and Takahashi [11] that the set of fixed points $F(T)$ of a quasi-nonexpansive mapping is closed and convex.

Let D be a nonempty, closed, and convex subset of H. Let P_D be the *metric projection* from H onto D, that is, for any $x \in H$, $\|x - P_D x\| = \inf_{z \in D} \|x - z\|$. For the metric projection P_D from H onto D, it holds that $\langle x - P_D x,\ P_D x - z\rangle \geq 0$ for all $x \in H$ and $z \in D$; see [26]. It is easy to verify that the metric projection is firmly nonexpansive, and thus, it is nonexpansive.

We list lemmas that will be utilized in the proofs of the theorems in this paper. In Lemma 2.1, parts (a) and (b) were proved by Takahashi [27] and Maruyama et al. [22], respectively. For a proof of (c), see [18].

Lemma 2.1 ([27], [22]). *Let $x, y, z, w \in H$ and $a, b, c, d \in \mathbb{R}$. Then, the following hold:*

(a) *If $a + b = 1$, then $\|ax + by\|^2 = a\|x\|^2 + b\|y\|^2 - ab\|x - y\|^2$.*

(b) *If $a + b + c = 1$, then*

$$\begin{aligned} &\|ax + by + cz\|^2 \\ &= a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x - y\|^2 - bc\|y - z\|^2 - ca\|z - x\|^2. \end{aligned}$$

(c) *If $a + b + c + d = 1$, then*

$$\begin{aligned} \|ax + by + cz + dw\|^2 \;=\; & a\|x\|^2 + b\|y\|^2 + c\|z\|^2 + d\|w\|^2 \\ & - ab\|x - y\|^2 - ac\|x - z\|^2 - ad\|x - w\|^2 \\ & - bc\|y - z\|^2 - bd\|y - w\|^2 - cd\|z - w\|^2. \end{aligned}$$

The next lemma reveals a relationship between $A(T)$ and $F(T)$.

Lemma 2.2 ([30]). *Let C be a nonempty subset of H, and let T be a mapping from C into H. Then, $A(T) \cap C \subset F(T)$.*

According to Lemmas 2.3 and 2.4, a weak limit of a sequence in H is an attractive point of a nonlinear mapping. For Lemma 2.3, see also Kurokawa and Takahashi [20]. For Lemma 2.4, see also Kocourek et al. [12] and Maruyama et al. [22].

Lemma 2.3 ([16]). *Let C be a nonempty subset of H, and let $T : C \to C$ be a normally 2-generalized hybrid mapping from C into itself. Suppose that $A(T)$ is nonempty. Let $\{x_n\}$ be a bounded sequence in H, and define $z_n \equiv \frac{1}{n}\sum_{k=1}^{n} T^k x_n \, (\in H)$. If $z_{n_i} \rightharpoonup u$, then $u \in A(T)$, where $\{z_{n_i}\}$ is a subsequence of $\{z_n\}$.*

Lemma 2.4 ([15]). *Let C be a nonempty subset of H, let T be a normally 2-generalized hybrid mapping from C into itself, and let $\{x_n\}$ be a sequence in C. If $\{x_n\}$ satisfies $Tx_n - x_n \to 0$, $T^2 x_n - x_n \to 0$, and $x_n \rightharpoonup u$, then $u \in A(T)$.*

Lemmas 2.5 and 2.6 play key roles to derive strong convergence.

Lemma 2.5 ([2]; see also [35]). *Let $\{X_n\}$ be a sequence of nonnegative real numbers, let $\{Y_n\}$ be a sequence of real numbers such that $\limsup_{n\to\infty} Y_n \le 0$, and let $\{Z_n\}$ be a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} Z_n < \infty$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0, 1)$ such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. If $X_{n+1} \le (1 - \lambda_n) X_n + \lambda_n Y_n + Z_n$ for all $n \in \mathbb{N}$, then $X_n \to 0$ as $n \to \infty$.*

Lemma 2.6 ([21]). *Let $\{X_n\}$ be a sequence of real numbers. Assume that $\{X_n\}$ is not monotone decreasing for sufficiently large $n \in \mathbb{N}$, that is, there exists a subsequence $\{X_{n_i}\}$ of $\{X_n\}$ such that $X_{n_i} < X_{n_i+1}$ for all $i \in \mathbb{N}$. Let n_0 be a natural number such that $\{k \in \mathbb{N} : k \le n_0,\ X_k < X_{k+1}\}$ is nonempty. Define a sequence $\{\tau(n)\}_{n \ge n_0}$ of natural numbers as follows:*

$$\tau(n) = \max\{k \in \mathbb{N} : k \le n_0,\ X_k < X_{k+1}\} \quad \text{for all } n \ge n_0.$$

Then, the following hold:

(a) $\tau(n) \to \infty$ *as* $n \to \infty$;

(b) $X_n \leq X_{\tau(n)+1}$ *and* $X_{\tau(n)} < X_{\tau(n)+1}$ *for all* $n \geq n_0$.

3. Main Results

In this section, we present two alternative iterations under which sequences converge strongly to common attractive and fixed points. The proofs have been developed in [32], [16], [18], [6], and [25].

Theorem 3.1. *Let C be a nonempty and convex subset of H, let S and T be normally 2-generalized hybrid mappings from C into itself with $A(S) \cap A(T) \neq \emptyset$, and let P_A be the metric projection from H onto $A(S) \cap A(T)$. Let $a, b \in (0,1)$ such that $a \leq b$, and let $\{\lambda_n\}$, $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $(0,1)$ such that*

$$\lambda_n \to 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty,$$
$$a_n + b_n + c_n = 1, \quad 0 < a \leq a_n, b_n, c_n \leq b < 1 \quad \text{for all } n \in \mathbb{N}.$$

Let $\{w_n\}$ be a sequence in C such that $w_n \to w$. Define a sequence $\{x_n\}$ in C as follows:

$$x_1 \in C : \text{given},$$
$$x_{n+1} = \lambda_n w_n + (1 - \lambda_n)\left(a_n x_n + b_n \frac{1}{n}\sum_{k=1}^{n} S^k x_n + c_n \frac{1}{n}\sum_{k=1}^{n} T^k x_n\right)$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges strongly to a common attractive point $\overline{w} \in A(S) \cap A(T)$, where $\overline{w} = P_A w$. Additionally, if C is closed, then $\{x_n\}$ converges strongly to a common fixed point $\widehat{w} = P_F w \in F(S) \cap F(T)$, where P_F is the metric projection from H onto $F(S) \cap F(T)$.

Proof. Define

$$y_n = \frac{1}{n}\sum_{k=1}^{n} S^k x_n,$$
$$z_n = \frac{1}{n}\sum_{k=1}^{n} T^k x_n, \text{ and}$$
$$h_n = a_n x_n + b_n y_n + c_n z_n.$$

Then, $x_{n+1} = \lambda_n w_n + (1 - \lambda_n) h_n$. First, observe that

$$\|y_n - q\| \leq \|x_n - q\| \quad \text{and} \quad \|z_n - q\| \leq \|x_n - q\| \tag{3.1}$$

for all $q \in A(S) \cap A(T)$ and $n \in \mathbb{N}$. It can be easily ascertained as follows:

$$\|y_n - q\| = \left\|\frac{1}{n}\sum_{k=1}^{n} S^k x_n - q\right\| = \frac{1}{n}\left\|\sum_{k=1}^{n} S^k x_n - nq\right\|$$

$$= \frac{1}{n}\left\|\sum_{k=1}^{n}\left(S^k x_n - q\right)\right\| \le \frac{1}{n}\sum_{k=1}^{n}\left\|\left(S^k x_n - q\right)\right\|$$
$$\le \frac{1}{n}\sum_{k=1}^{n}\|x_n - q\| = \|x_n - q\|.$$

Similarly, the other part $\|z_n - q\| \le \|x_n - q\|$ can be verified. It follows from (3.1) that

$$\|h_n - q\| \le \|x_n - q\| \tag{3.2}$$

for all $q \in A(S) \cap A(T)$ and $n \in \mathbb{N}$. Indeed,

$$\begin{aligned} \|h_n - q\| &= \|a_n x_n + b_n y_n + c_n z_n - q\| \\ &\le a_n\|x_n - q\| + b_n\|y_n - q\| + c_n\|z_n - q\| \\ &\le a_n\|x_n - q\| + b_n\|x_n - q\| + c_n\|x_n - q\| \\ &= \|x_n - q\|. \end{aligned}$$

Next, we show that $\{x_n\}$ is bounded by using the mathematical induction. Choose $q \in A(S) \cap A(T)$ arbitrarily, and define

$$M = \max\left\{\sup_{k\in\mathbb{N}}\|w_k - q\|,\ \ \|x_1 - q\|\right\}.$$

Since $\{w_n\}$ is bounded, M is a real number. We prove that $\|x_n - q\| \le M$ for all $n \in \mathbb{N}$. (i) For the case of $n = 1$, it obviously holds. (ii) Assume that $\|x_k - q\| \le M$ for some $k \in \mathbb{N}$. It follows from (3.2) that

$$\begin{aligned} \|x_{k+1} - q\| &= \|\lambda_n w_n + (1-\lambda_n)h_n - q\| \\ &\le \lambda_k\|w_k - q\| + (1-\lambda_k)\|h_k - q\| \\ &\le \lambda_k\|w_k - q\| + (1-\lambda_k)\|x_k - q\| \\ &\le \lambda_k M + (1-\lambda_k)M = M. \end{aligned}$$

Hence, $\{x_n\}$ is bounded.

The following inequality is crucial for our purpose:

$$\begin{aligned} &a_n b_n\|x_n - y_n\|^2 + b_n c_n\|y_n - z_n\|^2 + c_n a_n\|z_n - x_n\|^2 \\ \le\ &\lambda_n\|w_n - q\|^2 + \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \end{aligned} \tag{3.3}$$

for all $q \in A(S) \cap A(T)$ and $n \in \mathbb{N}$. By using Lemma 2.1 and (3.1), we obtain

$$\begin{aligned} &\|x_{n+1} - q\|^2 \\ =\ &\|\lambda_n(w_n - q) + (1-\lambda_n)(h_n - q)\|^2 \\ \le\ &\lambda_n\|w_n - q\|^2 + (1-\lambda_n)\|h_n - q\|^2 \\ \le\ &\lambda_n\|w_n - q\|^2 + \|a_n(x_n - q) + b_n(y_n - q) + c_n(z_n - q)\|^2 \\ =\ &\lambda_n\|w_n - q\|^2 + a_n\|x_n - q\|^2 + b_n\|y_n - q\|^2 + c_n\|z_n - q\|^2 \\ &- a_n b_n\|x_n - y_n\|^2 - b_n c_n\|y_n - z_n\|^2 - c_n a_n\|z_n - x_n\|^2 \end{aligned}$$

$$\begin{aligned}
&\le \lambda_n \|w_n - q\|^2 + a_n \|x_n - q\|^2 + b_n \|x_n - q\|^2 + c_n \|x_n - q\|^2 \\
&\quad -a_n b_n \|x_n - y_n\|^2 - b_n c_n \|y_n - z_n\|^2 - c_n a_n \|z_n - x_n\|^2 \\
&= \lambda_n \|w_n - q\|^2 + \|x_n - q\|^2 \\
&\quad -a_n b_n \|x_n - y_n\|^2 - b_n c_n \|y_n - z_n\|^2 - c_n a_n \|z_n - x_n\|^2 .
\end{aligned}$$

Thus, (3.3) follows.

Next, we show that

$$\|x_{n+1} - x_n\| \le \lambda_n \|w_n - x_n\| + \|y_n - x_n\| + \|z_n - x_n\| \tag{3.4}$$

for all $n \in \mathbb{N}$. This inequality can be ascertained as follows:

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&= \|\lambda_n w_n + (1-\lambda_n) h_n - x_n\| \\
&\le \lambda_n \|w_n - x_n\| + (1-\lambda_n) \|h_n - x_n\| \\
&\le \lambda_n \|w_n - x_n\| + \|a_n x_n + b_n y_n + c_n z_n - (a_n + b_n + c_n) x_n\| \\
&\le \lambda_n \|w_n - x_n\| + b_n \|y_n - x_n\| + c_n \|z_n - x_n\| \\
&\le \lambda_n \|w_n - x_n\| + \|y_n - x_n\| + \|z_n - x_n\| .
\end{aligned}$$

Let $X_n = \|x_n - \overline{w}\|^2$, where $\overline{w} = P_A w$. Our purpose is to demonstrate that $X_n \to 0$ as $n \to \infty$. The rest of the proof is divided into two cases.

Case (A). Suppose that there exists a natural number n' such that $X_{n+1} \le X_n$ for all $n \ge n'$. In this case, the sequence $\{X_n\}$ is convergent. Since $\overline{w} \in A(S) \cap A(T)$, it holds from (3.3) that

$$\begin{aligned}
&a_n b_n \|x_n - y_n\|^2 + b_n c_n \|y_n - z_n\|^2 + c_n a_n \|z_n - x_n\|^2 \\
&\le \lambda_n \|w_n - \overline{w}\|^2 + \|x_n - \overline{w}\|^2 - \|x_{n+1} - \overline{w}\|^2 \\
&\equiv \lambda_n \|w_n - \overline{w}\|^2 + X_n - X_{n+1}
\end{aligned} \tag{3.5}$$

for all $n \in \mathbb{N}$. Since $\{w_n\}$ is bounded, $\lambda_n \to 0$, and $\{X_n\}$ is convergent, we have that

$$x_n - y_n \to 0 \quad \text{and} \quad x_n - z_n \to 0. \tag{3.6}$$

Since $\{w_n\}$ and $\{x_n\}$ are bounded, we have from (3.4) that

$$x_{n+1} - x_n \to 0. \tag{3.7}$$

We have from (2.1) and (3.2) that

$$\begin{aligned}
X_{n+1} &= \|x_{n+1} - \overline{w}\|^2 \\
&= \|\lambda_n (w_n - \overline{w}) + (1-\lambda_n)(h_n - \overline{w})\|^2 \\
&\le (1-\lambda_n)^2 \|h_n - \overline{w}\|^2 + 2\lambda_n \langle x_{n+1} - \overline{w},\ w_n - \overline{w}\rangle \\
&\le (1-\lambda_n) \|x_n - \overline{w}\|^2 + 2\lambda_n \langle x_{n+1} - \overline{w},\ w_n - \overline{w}\rangle \\
&\equiv (1-\lambda_n) X_n + 2\lambda_n \langle x_{n+1} - \overline{w},\ w_n - \overline{w}\rangle
\end{aligned}$$

for all $n \in \mathbb{N}$. From Lemma 2.5, it suffices to prove that

$$\limsup_{n\to\infty} \langle x_{n+1} - \overline{w},\ w_n - \overline{w} \rangle \leq 0.$$

Since the sequences $\{x_n\}$ is bounded, we can assume, without loss of generality, that there exists subsequence $\{x_{n_i}\}$ of $\{x_n\}$such that

$$\limsup_{n\to\infty} \langle x_{n+1} - \overline{w},\ w_n - \overline{w} \rangle = \lim_{i\to\infty} \langle x_{n_i} - \overline{w},\ w_{n_i-1} - \overline{w} \rangle$$

and $x_{n_i} \rightharpoonup u$ for some $u \in H$. Therefore, it follows from (3.6) that $y_{n_i} \rightharpoonup u$ and $z_{n_i} \rightharpoonup u$. From Lemma 2.3, we obtain $u \in A(S) \cap A(T)$. Since $w_n \to w$ and $\overline{w} \equiv P_A w$, we have that

$$\begin{aligned}\limsup_{n\to\infty} \langle x_{n+1} - \overline{w},\ w_n - \overline{w} \rangle &= \lim_{i\to\infty} \langle x_{n_i} - \overline{w},\ w_{n_i-1} - \overline{w} \rangle \\ &= \langle u - \overline{w},\ w - \overline{w} \rangle \leq 0.\end{aligned}$$

This completes the proof for Case (A).

Case (B). Suppose that there exists a subsequence $\{X_{n_i}\}$ of $\{X_n\}$ such that $X_{n_i} < X_{n_i+1}$ for all $i \in \mathbb{N}$. Let n_0 be a natural number such that $\{k \in \mathbb{N} : k \leq n_0, X_k < X_{k+1}\} \neq \emptyset$. Define

$$\tau(n) = \max\{k \in \mathbb{N} : k \leq n,\ X_k < X_{k+1}\} \quad \text{for all } n \geq n_0.$$

From Lemma 2.6, it holds that

$$\tau(n) \to \infty \quad \text{as } n \to \infty; \tag{3.8}$$

$$X_n \leq X_{\tau(n)+1} \quad \text{for all } n \geq n_0; \tag{3.9}$$

$$X_{\tau(n)} < X_{\tau(n)+1} \quad \text{for all } n \geq n_0. \tag{3.10}$$

From (3.9), it suffices to demonstrate that $X_{\tau(n)+1} \to 0$. From (3.2)–(3.4), the following hold:

$$\left\| h_{\tau(n)} - \overline{w} \right\| \leq \left\| x_{\tau(n)} - \overline{w} \right\|, \tag{3.11}$$

$$\begin{aligned} & a_{\tau(n)} b_{\tau(n)} \left\| x_{\tau(n)} - y_{\tau(n)} \right\|^2 + b_{\tau(n)} c_{\tau(n)} \left\| y_{\tau(n)} - z_{\tau(n)} \right\|^2 \\ & + c_{\tau(n)} a_{\tau(n)} \left\| z_{\tau(n)} - x_{\tau(n)} \right\|^2 \\ \leq\ & \lambda_{\tau(n)} \left\| w_{\tau(n)} - \overline{w} \right\|^2 + \left\| x_{\tau(n)} - \overline{w} \right\|^2 - \left\| x_{\tau(n)+1} - \overline{w} \right\|^2 \\ \equiv\ & \lambda_{\tau(n)} \left\| w_{\tau(n)} - \overline{w} \right\|^2 + X_{\tau(n)} - X_{\tau(n)+1}, \text{ and} \end{aligned} \tag{3.12}$$

$$\begin{aligned} & \left\| x_{\tau(n)+1} - x_{\tau(n)} \right\| \\ \leq\ & \lambda_{\tau(n)} \left\| w_{\tau(n)} - x_{\tau(n)} \right\| + \left\| y_{\tau(n)} - x_{\tau(n)} \right\| + \left\| z_{\tau(n)} - x_{\tau(n)} \right\| \end{aligned} \tag{3.13}$$

for all $n \geq n_0$. It holds from (3.10) and (3.12) that

$$\begin{aligned} & a_{\tau(n)} b_{\tau(n)} \left\| x_{\tau(n)} - y_{\tau(n)} \right\|^2 + b_{\tau(n)} c_{\tau(n)} \left\| y_{\tau(n)} - z_{\tau(n)} \right\|^2 \\ & \qquad + c_{\tau(n)} a_{\tau(n)} \left\| z_{\tau(n)} - x_{\tau(n)} \right\|^2 \leq \lambda_{\tau(n)} \left\| w_{\tau(n)} - \overline{w} \right\|^2. \end{aligned}$$

Since $\{w_{\tau(n)}\}$ is bounded and $\lambda_{\tau(n)} \to 0$, we obtain that

$$x_{\tau(n)} - y_{\tau(n)} \to 0, \quad x_{\tau(n)} - z_{\tau(n)} \to 0, \tag{3.14}$$

as $n \to \infty$. Thus, we have from (3.13) that

$$x_{\tau(n)+1} - x_{\tau(n)} \to 0.$$

Since $\{x_{\tau(n)}\}$ and $\{x_{\tau(n)+1}\}$ are bounded, it holds that

$$X_{\tau(n)+1} - X_{\tau(n)} \to 0.$$

Thus, it suffices to prove that $X_{\tau(n)} \to 0$. Using (2.1) and (3.11), we obtain

$$\begin{aligned} X_{\tau(n)+1} &= \left\|x_{\tau(n)+1} - \overline{w}\right\|^2 \\ &= \left\|\lambda_{\tau(n)}\left(w_{\tau(n)} - \overline{w}\right) + \left(1 - \lambda_{\tau(n)}\right)\left(h_{\tau(n)} - \overline{w}\right)\right\|^2 \\ &\leq \left(1 - \lambda_{\tau(n)}\right)^2 \left\|h_{\tau(n)} - \overline{w}\right\|^2 + 2\lambda_{\tau(n)} \left\langle x_{\tau(n)+1} - \overline{w},\ w_{\tau(n)} - \overline{w}\right\rangle \\ &\leq \left(1 - \lambda_{\tau(n)}\right) \left\|x_{\tau(n)} - \overline{w}\right\|^2 + 2\lambda_{\tau(n)} \left\langle x_{\tau(n)+1} - \overline{w},\ w_{\tau(n)} - \overline{w}\right\rangle \\ &= \left(1 - \lambda_{\tau(n)}\right) X_{\tau(n)} + 2\lambda_{\tau(n)} \left\langle x_{\tau(n)+1} - \overline{w},\ w_{\tau(n)} - \overline{w}\right\rangle, \end{aligned}$$

and hence,

$$\lambda_{\tau(n)} X_{\tau(n)} \leq X_{\tau(n)} - X_{\tau(n)+1} + 2\lambda_{\tau(n)} \left\langle x_{\tau(n)+1} - \overline{w},\ w_{\tau(n)} - \overline{w}\right\rangle.$$

From (3.10),

$$\lambda_{\tau(n)} X_{\tau(n)} \leq 2\lambda_{\tau(n)} \left\langle x_{\tau(n)+1} - \overline{w},\ w_{\tau(n)} - \overline{w}\right\rangle,$$

and hence,

$$X_{\tau(n)} \leq 2 \left\langle x_{\tau(n)+1} - \overline{w},\ w_{\tau(n)} - \overline{w}\right\rangle$$

We prove that

$$\limsup_{n\to\infty} \left\langle x_{\tau(n)} - \overline{w},\ w_{\tau(n)-1} - \overline{w}\right\rangle \leq 0.$$

Since $\{x_{\tau(n)}\}$ is bounded, we can assume, without loss of generality, that there is a subsequence $\{x_{\tau(n_i)}\}$ of $\{x_{\tau(n)}\}$ such that

$$\limsup_{n\to\infty} \left\langle x_{\tau(n)} - \overline{w},\ w_{\tau(n)-1} - \overline{w}\right\rangle = \lim_{i\to\infty} \left\langle x_{\tau(n_i)} - \overline{w},\ w_{\tau(n_i)-1} - \overline{w}\right\rangle$$

and $x_{\tau(n_i)} \rightharpoonup u$ for some $u \in H$. From (3.14), we obtain

$$y_{\tau(n_i)} \rightharpoonup u \quad\text{and}\quad z_{\tau(n_i)} \rightharpoonup u.$$

Using Lemma 2.3, we have that $u \in A(S) \cap A(T)$. Since $\overline{w} \equiv P_A w$, we obtain

$$\begin{aligned} \limsup_{n\to\infty} \left\langle x_{\tau(n)} - \overline{w},\ w_{\tau(n)-1} - \overline{w}\right\rangle &= \lim_{i\to\infty} \left\langle x_{\tau(n_i)} - \overline{w},\ w_{\tau(n_i)-1} - \overline{w}\right\rangle \\ &= \left\langle u - \overline{w},\ w - \overline{w}\right\rangle \leq 0. \end{aligned}$$

This completes the proof for Case (B), and we have shown that $x_n \to \overline{w} \equiv P_A w$.

Suppose, in addition to the other assumptions, that C is closed in H. We show that $x_n \to \widehat{w}\,(\equiv P_F w)$. Since $x_n \to \overline{w} \equiv P_A w$ and C is closed, it holds that $\overline{w} \in C \cap A(S) \cap A(T)$. We have from Lemma 2.2 that $\overline{w} \in F(S) \cap F(T)$, and hence, $F(S) \cap F(T) \neq \emptyset$. Since S and T are quasi-nonexpansive, $F(S) \cap F(T)$ is closed and convex. Consequently, there exists the metric projection P_F from H onto

$F(S) \cap F(T)$. We prove that $(\widehat{w} \equiv)\, P_F w = \overline{w}\, (\equiv P_A w)$. Since $\overline{w} \in F(S) \cap F(T)$, it suffices to demonstrate that $\|w - \overline{w}\| \leq \|w - v\|$ for all $v \in F(S) \cap F(T)$. Let $v \in F(S) \cap F(T)$. Since S and T are quasi-nonexpansive, it holds that $F(S) \cap F(T) \subset A(S) \cap A(T)$. Thus, we have that

$$\begin{aligned} \|w - \overline{w}\| &= \inf \{\|w - q\| : q \in A(S) \cap A(T)\} \\ &\leq \inf \{\|w - q\| : q \in F(S) \cap F(T)\} \\ &\leq \|w - v\| . \end{aligned}$$

This means that $\overline{w} = P_F w\, (\equiv \widehat{w})$. This completes the proof. □

Theorem 3.2. *Let C be a nonempty and convex subset of H, let S and T be normally 2-generalized hybrid mappings from C into itself with $A(S) \cap A(T) \neq \emptyset$, and let P_A be the metric projection from H onto $A(S) \cap A(T)$. Let $a, b \in (0,1)$ such that $a \leq b$, and let $\{\lambda_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers in the interval $(0,1)$ such that*

$$\lambda_n \to 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty,$$

$$a_n + b_n + c_n + d_n = 1, \quad 0 < a \leq a_n, b_n, c_n, d_n \leq b < 1 \quad \textit{for all } n \in \mathbb{N}.$$

Let $\{w_n\}$ be a sequence in C such that $w_n \to w$. Define a sequence $\{x_n\}$ in C as follows:

$$\begin{aligned} &x_1 \in C : \textit{given}, \\ &x_{n+1} = \lambda_n w_n + (1 - \lambda_n) \left(a_n x_n + b_n S x_n + c_n S^2 x_n + d_n \frac{1}{n} \sum_{k=1}^{n} T^k x_n \right) \end{aligned}$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges strongly to a common attractive point $\overline{w} \in A(S) \cap A(T)$, where $\overline{w} = P_A w$. Additionally, if C is closed, then $\{x_n\}$ converges strongly to a common fixed point $\widehat{w} = P_F w \in F(S) \cap F(T)$, where P_F is the metric projection from H onto $F(S) \cap F(T)$.

Proof. Let us define z_n and h_n as follows:

$$\begin{aligned} z_n &= \frac{1}{n} \sum_{k=1}^{n} T^k x_n, \text{ and} \\ h_n &= a_n x_n + b_n S x_n + c_n S^2 x_n + d_n z_n. \end{aligned}$$

Then, we have that $x_{n+1} = \lambda_n w_n + (1 - \lambda_n) h_n$. As in the proof of Theorem 3.1, we can verify that

$$\|z_n - q\| \leq \|x_n - q\| \tag{3.15}$$

for all $q \in A(T)$ and $n \in \mathbb{N}$. From (3.15), the following holds:

$$\|h_n - q\| \leq \|x_n - q\| \tag{3.16}$$

for all $q \in A(S) \cap A(T)$ and $n \in \mathbb{N}$. Indeed,

$$\|h_n - q\| = \left\|a_n x_n + b_n S x_n + c_n S^2 x_n + d_n z_n - q\right\|$$

$$\begin{aligned} &\leq a_n \|x_n - q\| + b_n \|Sx_n - q\| + c_n \left\|S^2 x_n - q\right\| + d_n \|z_n - q\| \\ &\leq a_n \|x_n - q\| + b_n \|x_n - q\| + c_n \|x_n - q\| + d_n \|x_n - q\| \\ &= \|x_n - q\| . \end{aligned}$$

Using this inequality, as in the proof of Theorem 3.1, we can prove that $\{x_n\}$ is bounded.

We show that

$$\begin{aligned} (3.17) \qquad & a_n b_n \|x_n - Sx_n\|^2 + a_n c_n \left\|x_n - S^2 x_n\right\|^2 + a_n d_n \|x_n - z_n\|^2 \\ & + b_n c_n \left\|Sx_n - S^2 x_n\right\|^2 + b_n d_n \|Sx_n - z_n\|^2 + c_n d_n \left\|S^2 x_n - z_n\right\|^2 \\ \leq\ & \lambda_n \|w_n - q\|^2 + \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \end{aligned}$$

for all $q \in A(S) \cap A(T)$ and $n \in \mathbb{N}$. By using Lemma 2.1-(c) and (3.15), we obtain

$$\begin{aligned} & \|x_{n+1} - q\|^2 \\ =\ & \|\lambda_n (w_n - q) + (1 - \lambda_n)(h_n - q)\|^2 \\ \leq\ & \lambda_n \|w_n - q\|^2 + (1 - \lambda_n) \|h_n - q\|^2 \\ =\ & \lambda_n \|w_n - q\|^2 + (1 - \lambda_n) \left\|a_n x_n + b_n S x_n + c_n S^2 x_n + d_n z_n - q\right\|^2 \\ \leq\ & \lambda_n \|w_n - q\|^2 \\ & + \left\|a_n (x_n - q) + b_n (Sx_n - q) + c_n \left(S^2 x_n - q\right) + d_n (z_n - q)\right\|^2 \\ =\ & \lambda_n \|w_n - q\|^2 \\ & + a_n \|x_n - q\|^2 + b_n \|Sx_n - q\|^2 + c_n \left\|S^2 x_n - q\right\|^2 + d_n \|z_n - q\|^2 \\ & - a_n b_n \|x_n - Sx_n\|^2 - a_n c_n \left\|x_n - S^2 x_n\right\|^2 - a_n d_n \|x_n - z_n\|^2 \\ & - b_n c_n \left\|Sx_n - S^2 x_n\right\|^2 - b_n d_n \|Sx_n - z_n\|^2 - c_n d_n \left\|S^2 x_n - z_n\right\|^2 \\ \leq\ & \lambda_n \|w_n - q\|^2 \\ & + a_n \|x_n - q\|^2 + b_n \|x_n - q\|^2 + c_n \|x_n - q\|^2 + d_n \|x_n - q\|^2 \\ & - a_n b_n \|x_n - Sx_n\|^2 - a_n c_n \left\|x_n - S^2 x_n\right\|^2 - a_n d_n \|x_n - z_n\|^2 \\ & - b_n c_n \left\|Sx_n - S^2 x_n\right\|^2 - b_n d_n \|Sx_n - z_n\|^2 - c_n d_n \left\|S^2 x_n - z_n\right\|^2 \\ =\ & \lambda_n \|w_n - q\|^2 + \|x_n - q\|^2 \\ & - a_n b_n \|x_n - Sx_n\|^2 - a_n c_n \left\|x_n - S^2 x_n\right\|^2 - a_n d_n \|x_n - z_n\|^2 \\ & - b_n c_n \left\|Sx_n - S^2 x_n\right\|^2 - b_n d_n \|Sx_n - z_n\|^2 - c_n d_n \left\|S^2 x_n - z_n\right\|^2 . \end{aligned}$$

Therefore, we obtain (3.17).

Our next aim is to prove that

$$(3.18) \qquad \|x_{n+1} - x_n\| \leq \lambda_n \|w_n - x_n\| + \|Sx_n - x_n\| + \left\|S^2 x_n - x_n\right\| + \|z_n - x_n\|$$

for all $n \in \mathbb{N}$. Indeed, we have that

$$\|x_{n+1} - x_n\|$$

$$
\begin{aligned}
&= \|\lambda_n w_n + (1-\lambda_n) h_n - x_n\| \\
&\le \lambda_n \|w_n - x_n\| + (1-\lambda_n)\|h_n - x_n\| \\
&\le \lambda_n \|w_n - x_n\| \\
&\quad + \left\|a_n x_n + b_n S x_n + c_n S^2 x_n + d_n z_n - (a_n + b_n + c_n + d_n) x_n\right\| \\
&\le \lambda_n \|w_n - x_n\| + b_n \|S x_n - x_n\| + c_n \left\|S^2 x_n - x_n\right\| + d_n \|z_n - x_n\| \\
&\le \lambda_n \|w_n - x_n\| + \|S x_n - x_n\| + \left\|S^2 x_n - x_n\right\| + \|z_n - x_n\|.
\end{aligned}
$$

Define $X_n = \|x_n - \overline{w}\|^2$, where $\overline{w} = P_A w$. Our goal is to prove that $X_n \to 0$ as $n \to \infty$. We divide the rest of the proof into two cases.

Case (A). Suppose that there exists a natural number n' such that $X_{n+1} \le X_n$ for all $n \ge n'$. In this case, the sequence $\{X_n\}$ is convergent. Since $\overline{w} \in A(S) \cap A(T)$, it holds from (3.17) that

$$
\begin{aligned}
& a_n b_n \|x_n - S x_n\|^2 + a_n c_n \left\|x_n - S^2 x_n\right\|^2 + a_n d_n \|x_n - z_n\|^2 \\
& + b_n c_n \left\|S x_n - S^2 x_n\right\|^2 + b_n d_n \|S x_n - z_n\|^2 + c_n d_n \left\|S^2 x_n - z_n\right\|^2 \\
\le\ & \lambda_n \|w_n - \overline{w}\|^2 + \|x_n - \overline{w}\|^2 - \|x_{n+1} - \overline{w}\|^2 \\
\equiv\ & \lambda_n \|w_n - \overline{w}\|^2 + X_n - X_{n+1}
\end{aligned}
$$

for all $n \in \mathbb{N}$. Since $\{w_n\}$ is bounded, $\lambda_n \to 0$, and $\{X_n\}$ is convergent, we have that

$$
x_n - S x_n \to 0, \quad x_n - S^2 x_n \to 0, \quad x_n - z_n \to 0, \tag{3.19}
$$

Since $\{w_n\}$ and $\{x_n\}$ are bounded, it follows from (3.18) and (3.19) that

$$
x_{n+1} - x_n \to 0.
$$

We obtain from (2.1) and (3.16) that

$$
\begin{aligned}
X_{n+1} &= \|x_{n+1} - \overline{w}\|^2 \\
&= \|\lambda_n (w_n - \overline{w}) + (1-\lambda_n)(h_n - \overline{w})\|^2 \\
&\le (1-\lambda_n)^2 \|h_n - \overline{w}\|^2 + 2\lambda_n \langle x_{n+1} - \overline{w},\ w_n - \overline{w}\rangle \\
&\le (1-\lambda_n) \|x_n - \overline{w}\|^2 + 2\lambda_n \langle x_{n+1} - \overline{w},\ w_n - \overline{w}\rangle \\
&\equiv (1-\lambda_n) X_n + 2\lambda_n \langle x_{n+1} - \overline{w},\ w_n - \overline{w}\rangle
\end{aligned}
$$

for all $n \in \mathbb{N}$. From Lemma 2.5, it suffices to prove that

$$
\limsup_{n\to\infty} \langle x_{n+1} - \overline{w},\ w_n - \overline{w}\rangle \le 0.
$$

Since the sequences $\{x_n\}$ is bounded and $\{w_n\}$ is convergent, we can assume, without loss of generality, that there exists subsequence $\{x_{n_i}\}$ of $\{x_n\}$such that

$$
\limsup_{n\to\infty} \langle x_{n+1} - \overline{w},\ w_n - \overline{w}\rangle = \lim_{i\to\infty} \langle x_{n_i} - \overline{w},\ w_{n_i - 1} - \overline{w}\rangle
$$

and $x_{n_i} \rightharpoonup u$ for some $u \in H$. From Lemma 2.4 and (3.19), we have that $u \in A(S)$. Furthermore, it follows from (3.19) that $z_{n_i} \rightharpoonup u$. From Lemma 2.3, we obtain

$u \in A(T)$. Thus, $u \in A(S) \cap A(T)$. Since $w_n \to w$ and $\overline{w} = P_A w$, we have that

$$\begin{aligned}\limsup_{n\to\infty} \langle x_{n+1} - \overline{w},\ w_n - \overline{w}\rangle &= \lim_{i\to\infty} \langle x_{n_i} - \overline{w},\ w_{n_i-1} - \overline{w}\rangle \\ &= \langle u - \overline{w},\ w - \overline{w}\rangle \le 0.\end{aligned}$$

This completes the proof for Case (A).

Case (B). Suppose that there exists a subsequence $\{X_{n_i}\}$ of $\{X_n\}$ such that $X_{n_i} < X_{n_i+1}$ for all $i \in \mathbb{N}$. Let n_0 be a natural number such that $\{k \in \mathbb{N} : k \le n_0,\ X_k < X_{k+1}\}$ is nonempty. Define

$$\tau(n) = \max\{k \in \mathbb{N} : k \le n,\ X_k < X_{k+1}\} \quad \text{for all } n \ge n_0.$$

From Lemma 2.6, it holds that

$$\tau(n) \to \infty \text{ as } n \to \infty; \tag{3.20}$$

$$X_n \le X_{\tau(n)+1} \text{ for all } n \ge n_0; \tag{3.21}$$

$$X_{\tau(n)} < X_{\tau(n)+1} \text{ for all } n \ge n_0. \tag{3.22}$$

From (3.21), it suffices to demonstrate that $X_{\tau(n)+1} \to 0$. Since $\overline{w}(\equiv P_A w) \in A(S) \cap A(T)$, we have from (3.16)–(3.18) that

$$\left\|h_{\tau(n)} - \overline{w}\right\| \le \left\|x_{\tau(n)} - \overline{w}\right\|, \tag{3.23}$$

$$\begin{aligned}&a_{\tau(n)}b_{\tau(n)}\left\|x_{\tau(n)} - Sx_{\tau(n)}\right\|^2 + a_{\tau(n)}c_{\tau(n)}\left\|x_{\tau(n)} - S^2x_{\tau(n)}\right\|^2 \\ &+ a_{\tau(n)}d_{\tau(n)}\left\|x_{\tau(n)} - z_{\tau(n)}\right\|^2 \\ &+ b_{\tau(n)}c_{\tau(n)}\left\|Sx_{\tau(n)} - S^2x_{\tau(n)}\right\|^2 + b_{\tau(n)}d_{\tau(n)}\left\|Sx_{\tau(n)} - z_{\tau(n)}\right\|^2 \\ &+ c_{\tau(n)}d_{\tau(n)}\left\|S^2x_{\tau(n)} - z_{\tau(n)}\right\|^2 \\ &\le \lambda_{\tau(n)}\left\|w_{\tau(n)} - \overline{w}\right\|^2 + \left\|x_{\tau(n)} - \overline{w}\right\|^2 - \left\|x_{\tau(n)+1} - \overline{w}\right\|^2 \\ &\equiv \lambda_{\tau(n)}\left\|w_{\tau(n)} - \overline{w}\right\|^2 + X_{\tau(n)} - X_{\tau(n)+1}, \text{ and}\end{aligned} \tag{3.24}$$

$$\begin{aligned}&\left\|x_{\tau(n)+1} - x_{\tau(n)}\right\| \\ &\le \lambda_{\tau(n)}\left\|w_{\tau(n)} - x_{\tau(n)}\right\| \\ &+ \left\|Sx_{\tau(n)} - x_{\tau(n)}\right\| + \left\|S^2x_{\tau(n)} - x_{\tau(n)}\right\| + \left\|z_{\tau(n)} - x_{\tau(n)}\right\|\end{aligned} \tag{3.25}$$

for all $n \ge n_0$. It follows from (3.22) and (3.24) that

$$\begin{aligned}&a_{\tau(n)}b_{\tau(n)}\left\|x_{\tau(n)} - Sx_{\tau(n)}\right\|^2 + a_{\tau(n)}c_{\tau(n)}\left\|x_{\tau(n)} - S^2x_{\tau(n)}\right\|^2 \\ &+ a_{\tau(n)}d_{\tau(n)}\left\|x_{\tau(n)} - z_{\tau(n)}\right\|^2 \\ &+ b_{\tau(n)}c_{\tau(n)}\left\|Sx_{\tau(n)} - S^2x_{\tau(n)}\right\|^2 + b_{\tau(n)}d_{\tau(n)}\left\|Sx_{\tau(n)} - z_{\tau(n)}\right\|^2 \\ &+ c_{\tau(n)}d_{\tau(n)}\left\|S^2x_{\tau(n)} - z_{\tau(n)}\right\|^2 \\ &\le \lambda_{\tau(n)}\left\|w_{\tau(n)} - \overline{w}\right\|^2.\end{aligned}$$

Since $\{w_{\tau(n)}\}$ is bounded and $\lambda_{\tau(n)} \to 0$, we obtain that

$$x_{\tau(n)} - Sx_{\tau(n)} \to 0, \quad x_{\tau(n)} - S^2 x_{\tau(n)} \to 0, \quad x_{\tau(n)} - z_{\tau(n)} \to 0 \tag{3.26}$$

as $n \to \infty$. Thus, we have from (3.25) that

$$x_{\tau(n)+1} - x_{\tau(n)} \to 0.$$

Consequently, it holds that

$$X_{\tau(n)+1} - X_{\tau(n)} \to 0.$$

Thus, our goal is to prove that $X_{\tau(n)} \to 0$. From (2.1) and (3.23), we obtain

$$\begin{aligned} X_{\tau(n)+1} &= \left\| x_{\tau(n)+1} - \overline{w} \right\|^2 \\ &= \left\| \lambda_{\tau(n)} \left(w_{\tau(n)} - \overline{w} \right) + \left(1 - \lambda_{\tau(n)} \right) \left(h_{\tau(n)} - \overline{w} \right) \right\|^2 \\ &\leq \left(1 - \lambda_{\tau(n)} \right)^2 \left\| h_{\tau(n)} - \overline{w} \right\|^2 + 2\lambda_{\tau(n)} \left\langle x_{\tau(n)+1} - \overline{w},\ w_{\tau(n)} - \overline{w} \right\rangle \\ &\leq \left(1 - \lambda_{\tau(n)} \right) \left\| x_{\tau(n)} - \overline{w} \right\|^2 + 2\lambda_{\tau(n)} \left\langle x_{\tau(n)+1} - \overline{w},\ w_{\tau(n)} - \overline{w} \right\rangle \\ &\equiv \left(1 - \lambda_{\tau(n)} \right) X_{\tau(n)} + 2\lambda_{\tau(n)} \left\langle x_{\tau(n)+1} - \overline{w},\ w_{\tau(n)} - \overline{w} \right\rangle. \end{aligned}$$

This yields that

$$\lambda_{\tau(n)} X_{\tau(n)} \leq X_{\tau(n)} - X_{\tau(n)+1} + 2\lambda_{\tau(n)} \left\langle x_{\tau(n)+1} - \overline{w},\ w_{\tau(n)} - \overline{w} \right\rangle.$$

We have from (3.22) that

$$\lambda_{\tau(n)} X_{\tau(n)} \leq 2\lambda_{\tau(n)} \left\langle x_{\tau(n)+1} - \overline{w},\ w_{\tau(n)} - \overline{w} \right\rangle,$$

Dividing it by $\lambda_{\tau(n)}\,(> 0)$, we obtain

$$X_{\tau(n)} \leq 2 \left\langle x_{\tau(n)+1} - \overline{w},\ w_{\tau(n)} - \overline{w} \right\rangle$$

We show that

$$\limsup_{n\to\infty} \left\langle x_{\tau(n)} - \overline{w},\ w_{\tau(n)-1} - \overline{w} \right\rangle \leq 0.$$

Since $\{x_{\tau(n)}\}$ is bounded, we can assume, without loss of generality, that there is a subsequence $\{x_{\tau(n_i)}\}$ of $\{x_{\tau(n)}\}$ such that

$$\limsup_{n\to\infty} \left\langle x_{\tau(n)} - \overline{w},\ w_{\tau(n)-1} - \overline{w} \right\rangle = \lim_{i\to\infty} \left\langle x_{\tau(n_i)} - \overline{w},\ w_{\tau(n_i)-1} - \overline{w} \right\rangle$$

and $x_{\tau(n_i)} \rightharpoonup u$ for some $u \in H$. From (3.26) and Lemma 2.4, we have that $u \in A(S)$. Furthermore, from (3.26), it holds that

$$z_{\tau(n_i)} \rightharpoonup u.$$

As a consequence from Lemma 2.3, we have that $u \in A(T)$. Thus, $u \in A(S) \cap A(T)$. Since $\overline{w} = P_A w$, we obtain

$$\begin{aligned} \limsup_{n\to\infty} \left\langle x_{\tau(n)} - \overline{w},\ w_{\tau(n)-1} - \overline{w} \right\rangle &= \lim_{i\to\infty} \left\langle x_{\tau(n_i)} - \overline{w},\ w_{\tau(n_i)-1} - \overline{w} \right\rangle \\ &= \langle u - \overline{w},\ w - \overline{w} \rangle \leq 0. \end{aligned}$$

This completes the proof for Case (B), and we have shown that $x_n \to \overline{w} \equiv P_A w$ as claimed.

Additionally, suppose that C is closed in H. As in the proof of Theorem 3.1, we can show that $x_n \to \widehat{w} \,(= P_F w)$, where P_F is the metric projection from H onto $F\,(S) \cap F\,(T)$. This completes the proof. □

4. Applications

In this section, using Theorems 3.1 and 3.2, we obtain well-known and new strong convergence theorems which are connected with normally 2-generalized hybrid mappings in a Hilbert space. We can first prove the following two results from Theorems 3.1 and 3.2.

Theorem 4.1. *Let C be a nonempty and convex subset of H, let S be a normally 2-generalized hybrid mapping from C into itself with $A\,(S) \neq \emptyset$, and let $P_{A(S)}$ be the metric projection from H onto $A\,(S)$. Let $a, b \in (0,1)$ such that $a \leq b$, and let $\{\lambda_n\}$, $\{a_n\}$ and $\{d_n\}$ be sequences of real numbers in the interval $(0,1)$ such that*

$$\lambda_n \to 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty,$$
$$a_n + d_n = 1, \quad 0 < a \leq a_n, d_n \leq b < 1 \quad \text{for all } n \in \mathbb{N}.$$

Let $\{w_n\}$ be a sequence in C such that $w_n \to w$. Define a sequence $\{x_n\}$ in C as follows:

$$x_1 \in C : \text{given},$$
$$x_{n+1} = \lambda_n w_n + (1-\lambda_n)\left(a_n x_n + d_n \frac{1}{n}\sum_{k=1}^{n} S^k x_n\right)$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges strongly to an attractive point $\overline{w} \in A\,(S)$, where $\overline{w} = P_{A(S)} w$. Additionally, if C is closed, then $\{x_n\}$ converges strongly to a fixed point $\widehat{w} = P_{F(S)} w \in F\,(S)$, where $P_{F(S)}$ is the metric projection from H onto $F\,(S)$.

Proof. Putting $b_n = c_n = \frac{d_n}{2}$ and $T = S$ in Theorem 3.1, we have the desired result from Theorem 3.1. □

Theorem 4.2. *Let C be a nonempty and convex subset of H, let S be a normally 2-generalized hybrid mapping from C into itself with $A\,(S) \neq \emptyset$, and let $P_{A(S)}$ be the metric projection from H onto $A\,(S)$. Let $a, b \in (0,1)$ such that $a \leq b$, and let $\{\lambda_n\}$, $\{f_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $(0,1)$ such that*

$$\lambda_n \to 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty,$$
$$f_n + b_n + c_n = 1, \quad 0 < a \leq f_n, b_n, c_n \leq b < 1 \quad \text{for all } n \in \mathbb{N}.$$

Let $\{w_n\}$ be a sequence in C such that $w_n \to w$. Define a sequence $\{x_n\}$ in C as follows:

$$x_1 \in C : \text{given},$$

$$x_{n+1} = \lambda_n w_n + (1 - \lambda_n) \left(f_n x_n + b_n S x_n + c_n S^2 x_n \right)$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges strongly to an attractive point $\overline{w} \in A(S)$, where $\overline{w} = P_{A(S)} w$. Additionally, if C is closed, then $\{x_n\}$ converges strongly to a fixed point $\widehat{w} = P_{F(S)} w \in F(S)$, where $P_{F(S)}$ is the metric projection from H onto $F(S)$.

Proof. Putting $a_n = d_n = \frac{f_n}{2}$ and $T = I$ in Theorem 3.2, we have the desired result from Theorem 3.2. □

We have that a generalized hybrid mapping is a normally 2-generalized hybrid mapping. As direct results of Theorems 3.1 and 3.2, we have the following two theorems.

Theorem 4.3. *Let C be a nonempty and convex subset of H, let S and T be generalized hybrid mappings from C into itself such that $A(S) \cap A(T) \neq \emptyset$, and let P_A be the metric projection from H onto $A(S) \cap A(T)$. Let $a, b \in (0,1)$ such that $a \leq b$, and let $\{\lambda_n\}$, $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $(0,1)$ such that*

$$\lambda_n \to 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty,$$

$$a_n + b_n + c_n = 1, \quad 0 < a \leq a_n, b_n, c_n \leq b < 1 \quad \textit{for all } n \in \mathbb{N}.$$

Let $\{w_n\}$ be a sequence in C such that $w_n \to w$. Define a sequence $\{x_n\}$ in C as follows:

$$x_1 \in C : \textit{given},$$

$$x_{n+1} = \lambda_n w_n + (1 - \lambda_n) \left(a_n x_n + b_n \frac{1}{n} \sum_{k=1}^{n} S^k x_n + c_n \frac{1}{n} \sum_{k=1}^{n} T^k x_n \right)$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges strongly to a common attractive point $\overline{w} \in A(S) \cap A(T)$, where $\overline{w} = P_A w$. Additionally, if C is closed, then $\{x_n\}$ converges strongly to a common fixed point $\widehat{w} = P_F w \in F(S) \cap F(T)$, where P_F is the metric projection from H onto $F(S) \cap F(T)$.

Theorem 4.4. *Let C be a nonempty and convex subset of H, let S and T be generalized hybrid and normally 2-generalized hybrid mappings from C into itself, respectively, such that $A(S) \cap A(T) \neq \emptyset$, and let P_A be the metric projection from H onto $A(S) \cap A(T)$. Let $a, b \in (0,1)$ such that $a \leq b$, and let $\{\lambda_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers in the interval $(0,1)$ such that*

$$\lambda_n \to 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty,$$

$$a_n + b_n + c_n + d_n = 1, \quad 0 < a \leq a_n, b_n, c_n, d_n \leq b < 1 \quad \textit{for all } n \in \mathbb{N}.$$

Let $\{w_n\}$ be a sequence in C such that $w_n \to w$. Define a sequence $\{x_n\}$ in C as follows:

$$x_1 \in C : \textit{given},$$

$$x_{n+1} = \lambda_n w_n + (1 - \lambda_n) \left(a_n x_n + b_n S x_n + c_n S^2 x_n + d_n \frac{1}{n} \sum_{k=1}^{n} T^k x_n \right)$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges strongly to a common attractive point $\overline{w} \in A(S) \cap A(T)$, where $\overline{w} = P_A w$. Additionally, if C is closed, then $\{x_n\}$ converges strongly to a common fixed point $\widehat{w} = P_F w \in F(S) \cap F(T)$, where P_F is the metric projection from H onto $F(S) \cap F(T)$.

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