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INERTIAL VISCOSITY-TYPE ALGORITHMS FOR A CLASS OF SPLIT FEASIBILITY PROBLEMS AND FIXED POINT PROBLEMS IN HILBERT SPACES

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Abstract. In this paper, we introduce a new iterative algorithm based on the inertial method and vicosity-type algorithm for finding a common solution of a class of split feasibility problem and fixed point problem in Hilbert spaces. By assuming the existence of solutions, we show the strong convergence theorems of the constructed sequences and some applications of the considered problem are also discussed.

1. Introduction

Split feasibility problem (SFP) was first introduced by Censor and Elfving [7], which is the problem of finding a point

$$x^* \in C \quad \text{ such that } \quad Lx^* \in Q, \tag{1.1}$$

where C and Q are nonempty closed convex subsets of $\mathbb{R}^n$, and L is an $n \times n$ matrix. SFP problems have many applications in various fields of science and technology, such as in signal processing, medical image reconstruction, and intensity-modulated radiation therapy; for more information, see [3, 4, 6, 7] and the references therein. The popular algorithm for solving the problem (1.1) is the following CQ algorithm, suggested by Byrne [3]: for arbitrary $x_1 \in \mathbb{R}^n$,

$$x_{n+1} = P_C\big(x_n - \gamma L^\top (I - P_Q)Lx_n\big), \qquad \forall n \in \mathbb{N}, \tag{1.2}$$

where $\gamma \in \big(0, 2/\|L\|^2\big)$, L is a real $m \times n$ matrix and $L^\top$ is the transpose of the matrix L. Subsequently, in 2010, Xu [26] considered SFP in infinite-dimensional Hilbert spaces: let H_1 and H_2 be real Hilbert spaces, C and Q be nonempty closed convex subsets of H_1 and H_2, respectively, and $L : H_1 \to H_2$ be a bounded linear operator. They proposed the following algorithm: for a given $x_1 \in H_1$,

$$x_{n+1} = P_C\big(x_n - \gamma L^* (I - P_Q)Lx_n\big), \qquad \forall n \in \mathbb{N}, \tag{1.3}$$

where $\gamma \in \big(0, 2/\|L\|^2\big)$ and L^* is the adjoint operator of L. In [26], the conditions to guarantee the weak convergence of the sequence $\{x_n\}$ to a solution of SFP was considered. In addition, by considering the CQ algorithm (1.2), López et al. [11]

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suggested to use the stepsizes γ_n without the norm of operator L,

$$\gamma_n = \frac{\delta_n \|(I - P_Q)Lx_n\|^2}{2\|L^*(I - P_Q)Lx_n\|^2}, \tag{1.4}$$

where $0 < \delta_n < 4$ and $L^*(I - P_Q)Lx_n \neq 0$, and proved weakly convergence theorem. They point out that, the higher dimensions of L may be hard to compute the operator norm and it may affect the computing in the iteration process, for example, the CPU time, and the algorithm with stepsizes (1.4) gives faster results.

On the other hand, for a Hilbert space H, variational inclusion problem (VIP) has the following formal form: find $x^* \in H$ such that

$$0 \in Bx^*, \tag{1.5}$$

where $B : H \to 2^H$ is a set-valued operator. The problem (1.5) was introduced by Martinet [15], and the popular method for solving the problem (1.5) is the proximal point algorithm: for a given $x_1 \in H$,

$$x_{n+1} = J^B_{\lambda_n} x_n, \qquad \forall n \in \mathbb{N},$$

where $\{\lambda_n\} \subset (0, \infty)$ and $J^B_{\lambda_n} = (I + \lambda_n B)^{-1}$ is the resolvent of the maximal monotone operator B corresponding to λ_n; see [9, 14, 25, 27] for more details. Subsequently, by using the concept of SFP in Hilbert spaces, Byrne et al. [5] proposed the following split null point problem (SNPP): let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be set-valued mappings, then SNPP is the problem of finding a point $x^* \in H_1$ such that

$$0 \in B_1(x^*) \quad \text{and} \quad 0 \in B_2(Lx^*). \tag{1.6}$$

They considered the following iterative algorithm: for $\lambda > 0$ and an arbitrary $x_1 \in H_1$,

$$x_{n+1} = J^{B_1}_{\lambda}\big(x_n - \gamma L^*(I - J^{B_2}_{\lambda})Lx_n\big), \qquad \forall n \in \mathbb{N}, \tag{1.7}$$

where $\gamma \in \big(0, 2/\|L\|^2\big)$, and $J^{B_1}_{\lambda}$ and $J^{B_2}_{\lambda}$ are the resolvent of maximal monotone operators B_1 and B_2, respectively. They showed that, under some suitable control conditions, the sequence $\{x_n\}$ converges weakly to a point in the solution set of problem (1.6). Furthermore, in 2015, Takahashi et al. [23] considered the problem of finding a point

$$x^* \in B^{-1}0 \cap L^{-1}F(T), \tag{1.8}$$

where $B : H_1 \to 2^{H_1}$ is a maximal monotone operator and $T : H_2 \to H_2$ is a nonexpansive mapping. They considered the following iterative algorithm: for any $x_1 \in H_1$,

$$x_{n+1} = J^B_{\lambda_n}\big(I - \gamma_n L^*(I - T)L\big)x_n, \qquad \forall n \in \mathbb{N}, \tag{1.9}$$

where $\{\lambda_n\}$ and $\{\gamma_n\}$ satisfy some suitable control conditions, and $J^B_{\lambda_n}$ is the resolvent of a maximal monotone operator B associated to λ_n. They provided the weak convergence theorem of algorithm (1.9) to a solution set of the problem (1.8). Moreover, in [23], Takahashi et al. also considered the problem of finding a point

$$x^* \in F(S) \cap B^{-1}0 \cap L^{-1}F(T) =: \Omega, \tag{1.10}$$

where $S : H_1 \to H_1$ is a nonexpansive mapping. They suggested the following iterative algorithm: for any $x_1 \in H_1$,

$$\begin{aligned} y_n &= J^B_{\lambda_n}\big(x_n - \lambda_n L^*(I-T)Lx_n\big), \\ x_{n+1} &= \alpha_n x_n + (1-\alpha_n)Sy_n, \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{1.11}$$

where $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy some suitable control conditions and provided the weak convergence theorem of algorithm (1.11) to a solution point of the problem (1.10).

For the study of the inertial technique, was first presented by Polyak in 1964, to speed up the rate of convergence; see [18]. The inertial method is a two-step iterative method, in which each iteration involves the previous two iterates. Recently, many authors used this technique because of the faster convergence rate of the algorithm; see [1, 2, 8, 19, 24] for more information.

In 2001, Alvarez and Attouch [1] proposed the inertial proximal point method for solving the problem (1.5): for arbitrary $x_0, x_1 \in H$,

$$\begin{aligned} y_n &= x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} &= J^B_{\lambda_n} y_n, \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{1.12}$$

where $\{\lambda_n\}$ and $\{\mu_n\}$ satisfy some suitable control conditions with $\sum_{n=1}^{\infty} \mu_n \|x_n - x_{n-1}\|^2 < \infty$, and proved weakly convergence theorem.

In 2017, Dang et al. [8] proposed the following innertial relaxed CQ algorithms for solving SFP in Hilbert spaces: for arbitrary $x_0, x_1 \in H_1$,

$$\begin{aligned} y_n &= x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} &= P_{C_n}\big(y_n - \gamma_n L^\top (I - P_{Q_n})Ly_n\big), \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{1.13}$$

and

$$\begin{aligned} y_n &= x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} &= (1-\beta_n)y_n + \beta_n P_{C_n}\big(y_n - \gamma_n L^\top (I - P_{Q_n})Ly_n\big), \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{1.14}$$

where $\mu_n \in [0, \bar{\mu}_n]$, $\bar{\mu}_n = \min\big\{\mu, \big(\max\{n^2\|x_n - x_{n-1}\|, n^2\|x_n - x_{n-1}\|^2\}\big)^{-1}\big\}$, $\mu \in [0,1)$ and $\beta_n \in (0,1)$. They proved that both sequences $\{x_n\}$ converge weakly to a point in the solution set of SFP.

Recently, by combining the inertial method, the algorithm (1.7) and Mann iteration [13], Anh et al. [2] proposed the following algorithm for solving the problem (1.6): for arbitrary $x_0, x_1 \in H_1$,

$$\begin{aligned} z_n &= x_n + \mu_n(x_n - x_{n-1}), \\ y_n &= J^{B_1}_{\lambda}\big(z_n - \gamma_n L^*(I - J^{B_2}_{\lambda})Lz_n\big), \\ x_{n+1} &= (1-\theta_n-\alpha_n)x_n + \theta_n y_n, \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{1.15}$$

where $\{\mu_n\} \subset [0, \mu)$ for some $\mu > 0$, $\{\theta_n\} \subset (a, b) \subset (0, 1-\alpha_n)$ and $\{\alpha_n\}$ satisfies $\lim_{n\to\infty} \frac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$ and some suitable conditions. They provided the strong converges theorem of algotirhm (1.15) to a solution set of the problem (1.6).

In this paper, motivated and inspired by the above literature, we are going to consider a class of SFP problems and fixed point problems, the problem (1.10). We aim to suggest a new algorithm, based on the inertial method and viscosity-type algorithm [16], for finding a solution of the problem (1.10). In our main Theorem, we provide some suitable conditions to guarantee that the constructed sequence $\{x_n\}$ converges strongly to a point in Ω.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ for the set of positive integers, and $\mathbb{R}$ for the set of real numbers. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$, respectively. When $\{x_n\}$ is a sequence in H, we denote the strong convergence and weak convergence of $\{x_n\}$ to x in H by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Let $T : H \to H$ be a mapping. We say that T is a Lipschitz mapping if there exists $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \qquad \forall x, y \in H.$$

The number L, associated with T, is called a Lipschitz constant. If $L \in [0, 1)$, we say that T is a contraction mapping, and T is a nonexpansive mapping if $L = 1$.

We will say that T is firmly nonexpansive if

$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2, \qquad \forall x, y \in H.$$

The set of fixed points of a self-mapping T will be denoted by $F(T)$, that is $F(T) = \{x \in H : Tx = x\}$. It is well known that if T is nonexpansive, then $F(T)$ is closed and convex.

Let $A : H \to H$ be a single-valued mapping. For a positive real number β, we will say that A is β-inverse strongly monotone (β-ism) if

$$\langle Ax - Ay, x - y\rangle \geq \beta\|Ax - Ay\|^2, \qquad \forall x, y \in H.$$

Notice that, A mapping $T : H \to H$ is nonexpansive if and only if $I - T$ is $\frac{1}{2}$-ism.

Let $B : H \to 2^H$ be a set-valued mapping. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. Recall that B is said to be monotone if

$$\langle x - y, u - v\rangle \geq 0, \qquad \forall x, y \in D(B), u \in Bx, v \in By.$$

A monotone mapping B is said to be maximal if its graph is not properly contained in the graph of any other monotone operator. To a maximal monotone operator $B : H \to 2^H$ and $\lambda > 0$, its resolvent J_λ^B is defined by

$$J_\lambda^B := (I + \lambda B)^{-1} : H \to D(B).$$

It is well known that if B is a maximal monotone operator and λ is a positive number, then the resolvent J_λ^B is a single-valued and firmly nonexpansive, and $F(J_\lambda^B) = B^{-1}0 \equiv \{x \in H : 0 \in Bx\}$, $\forall \lambda > 0$; see [21, 23].

The following fundamental results and inequalities are needed in our proof.

Let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called a metric projection of H onto C; see [22]. The following property of P_C is well known and useful:

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall x \in H, y \in C.$$

For each $x, y, z \in H$, then the following equalities are valid for inner product spaces,

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2, \tag{2.1}$$

and

$$\begin{aligned} \|\alpha x + \beta y + \gamma z\|^2 &= \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 \\ &\quad -\alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2, \end{aligned} \tag{2.2}$$

for any $\alpha, \beta, \gamma \in [0,1]$ such that $\alpha + \beta + \gamma = 1$; see [17, 21].

We also use the following lemmas for proving the main results.

Lemma 2.1 ([20]). *Let C be a closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping. Then, $U := I - T$ is demiclosed, that is, $x_n \rightharpoonup x_0$ and $Ux_n \to y_0$ imply $Ux_0 = y_0$.*

Lemma 2.2 ([10, 25]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \delta_n, \qquad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$, $\{\sigma_n\}$ and $\{\delta_n\}$ are sequences of real numbers satisfying

(i) $\{\alpha_n\} \subset [0,1]$, $\sum_{i=1}^\infty \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$;
(iii) $\delta_n \geq 0$, $\sum_{i=1}^\infty \delta_n < \infty$.

Then, $a_n \to 0$ as $n \to \infty$.

3. Main results

We start by introducing the following assumptions and our main algorithm that will be used to provide the convergence theorems.

(A1) $B : H_1 \to 2^{H_1}$ is a maximal monotone operator;
(A2) $L : H_1 \to H_2$ is a bounded linear operator;
(A3) $T : H_2 \to H_2$ is a nonexpansive mapping;
(A4) $S : H_1 \to H_1$ is a nonexpansive mapping;
(A5) $f : H_1 \to H_1$ is a contraction mapping with coefficient $\kappa \in (0, 1)$.

Algorithm 3.1. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\theta_n\}$ be sequences in $(0, 1)$ with $\alpha_n+\beta_n+\theta_n = 1$ and the initial $x_0, x_1 \in H_1$ be arbitrary, define

$$\begin{aligned} z_n &= x_n + \mu_n(x_n - x_{n-1}), \\ y_n &= J^B_\lambda\big(z_n - \gamma_n L^*(I - T)Lz_n\big), \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \theta_n S y_n, \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{3.1}$$

where $\{\mu_n\} \subset [0, \mu)$ with $\mu \in [0, 1)$ and $\{\gamma_n\}$ is depend on $\delta_n \in [a, b] \subset (0, 1)$ by

$$\gamma_n = \begin{cases} \frac{\delta_n \left\|(I-T)Lz_n\right\|^2}{\left\|L^*(I-T)Lz_n\right\|^2}, & \text{if } L^*(I - T)Lz_n \neq 0; \\ \gamma, & \text{otherwise,} \end{cases}$$

where γ is any nonnegative value.

Remark 3.2. The sequence $\{\gamma_n\}$ is bounded. Indeed, for each $n \in \mathbb{N}$,

$$\left\|L^*(I - T)Lz_n\right\| \leq \|L^*\|\left\|(I - T)Lz_n\right\|,$$

which implies

$$\frac{\left\|(I - T)Lz_n\right\|^2}{\left\|L^*(I - T)Lz_n\right\|^2} \geq \frac{1}{\|L^*\|^2}.$$

Let $w_n = \frac{\|(I-T)Lz_n\|^2}{\|L^*(I-T)Lz_n\|^2}$ and $L^*(I - T)Lz_n \neq 0$, for each $n \in \mathbb{N}$. It follows from the definition of $\{\gamma_n\}$, we can see that $\sup \gamma_n < \inf w_n < \infty$. This means $\{\gamma_n\}$ is bounded.

By considering the above assumptions and Algorithm 3.1, we will show the following strong convergence theorem.

Theorem 3.3. *Let H_1 and H_2 be two real Hilbert spaces. Let $\{x_n\}$ be generated by Algorithm 3.1. Suppose that the assumptions (A1)-(A5) hold, $\Omega \neq \emptyset$ and the following control conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^\infty \alpha_n = \infty$;
(iii) $0 < a \leq \beta_n$ *and* $0 < a \leq \theta_n$;
(iv) $\lim_{n\to\infty} \frac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

Then, $\{x_n\}$ converges strongly to $\bar{x} \in \Omega$, where $\bar{x} = P_\Omega f(\bar{x})$.

Proof. Firstly, we will prove that $\{x_n\}$ is bounded. Let $p \in \Omega$, we have that $p \in F(S)$, $p = J_\lambda^B p$ and $Lp = TLp$. We note that $I - T$ is $\frac{1}{2}$-ism. It follows that, for each $n \in \mathbb{N}$,

$$\big\langle (I-T)Lz_n - (I-T)Lp, Lz_n - Lp \big\rangle \geq \frac{1}{2}\big\|(I-T)Lz_n - (I-T)Lp\big\|^2.$$

Since $Lp = TLp$, the above inequality is reduced to

$$\big\langle (I-T)Lz_n, Lz_n - Lp \big\rangle \geq \frac{1}{2}\big\|(I-T)Lz_n\big\|^2, \tag{3.2}$$

for each $n \in \mathbb{N}$. By using (3.2), we see that

$$\begin{aligned}
\|y_n - p\|^2 &= \big\|J_\lambda^B\big(z_n - \gamma_n L^*(I-T)Lz_n\big) - p\big\|^2 \\
&\leq \big\|(z_n - p) - \gamma_n L^*(I-T)Lz_n\big\|^2 \\
&\leq \|z_n - p\|^2 - 2\gamma_n\big\langle z_n - p, L^*(I-T)Lz_n\big\rangle + \gamma_n^2\big\|L^*(I-T)Lz_n\big\|^2 \\
&= \|z_n - p\|^2 - 2\gamma_n\big\langle Lz_n - Lp, (I-T)Lz_n\big\rangle + \gamma_n^2\big\|L^*(I-T)Lz_n\big\|^2 \\
&\leq \|z_n - p\|^2 - \gamma_n\big\|(I-T)Lz_n\big\|^2 + \gamma_n^2\big\|L^*(I-T)Lz_n\big\|^2 \\
&= \|z_n - p\|^2 - \gamma_n\Big(\big\|(I-T)Lz_n\big\|^2 - \gamma_n\big\|L^*(I-T)Lz_n\big\|^2\Big), \qquad (3.3)
\end{aligned}$$

for each $n \in \mathbb{N}$. By the definition of γ_n, we have

$$\gamma_n\Big(\big\|(I-T)Lz_n\big\|^2 - \gamma_n\big\|L^*(I-T)Lz_n\big\|^2\Big) \geq 0,$$

for each $n \in \mathbb{N}$. Thus, from (3.3) we get

$$\|y_n - p\| \leq \|z_n - p\|,$$

for each $n \in \mathbb{N}$. Furthermore, for each $n \in \mathbb{N}$,

$$\begin{aligned}
\|z_n - p\| &= \big\|x_n + \mu_n(x_n - x_{n-1}) - p\big\| \\
&\leq \|x_n - p\| + \mu_n\|x_n - x_{n-1}\| \\
&= \|x_n - p\| + \alpha_n\frac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| \\
&\leq \|x_n - p\| + \alpha_n M_1, \qquad (3.4)
\end{aligned}$$

for some $M_1 > 0$. Now, by the definition of x_{n+1} and (3.4), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \big\|\alpha_n f(x_n) + \beta_n x_n + \theta_n S y_n - p\big\| \\
&\leq \alpha_n\big\|f(x_n) - p\big\| + \beta_n\|x_n - p\| + \theta_n\|Sy_n - p\| \\
&\leq \alpha_n\big\|f(x_n) - f(p)\big\| + \alpha_n\big\|f(p) - p\big\| + \beta_n\|x_n - p\| + \theta_n\|y_n - p\| \\
&\leq \alpha_n\kappa\|x_n - p\| + \alpha_n\big\|f(p) - p\big\| + \beta_n\|x_n - p\| + \theta_n\|x_n - p\| + \theta_n\alpha_n M_1 \\
&\leq \big(\alpha_n\kappa + \beta_n + \theta_n\big)\|x_n - p\| + \alpha_n\Big(\big\|f(p) - p\big\| + M_1\Big)
\end{aligned}$$

$$
\begin{aligned}
&\le \big(1-\alpha_n(1-\kappa)\big)\|x_n-p\| + \alpha_n\bigg(\big\|f(p)-p\big\| + M_1\bigg)\\
&= \big(1-\alpha_n(1-\kappa)\big)\|x_n-p\| + \alpha_n(1-\kappa)\bigg(\frac{\big\|f(p)-p\big\| + M_1}{1-\kappa}\bigg)\\
&\le \max\left\{\|x_n-p\|, \frac{\big\|f(p)-p\big\| + M_1}{1-\kappa}\right\}\\
&\ \vdots\\
&\le \max\left\{\|x_1-p\|, \frac{\big\|f(p)-p\big\| + M_1}{1-\kappa}\right\}, \qquad (3.5)
\end{aligned}
$$

for each $n \in \mathbb{N}$. Therefore, $\big\{\|x_n - p\|\big\}$ is a bounded sequence. This implies that $\{x_n\}$ is bounded. Consequently, $\{z_n\}$, $\{y_n\}$ and $\{f(x_n)\}$ are also bounded.

Next, we notice that $P_\Omega f(\cdot)$ is a contraction mapping. Let $\bar{x}$ be a unique fixed point of $P_\Omega f(\cdot)$, that is $\bar{x} = P_\Omega f(\bar{x})$. Consider,

$$
\begin{aligned}
\|z_n-\bar{x}\|^2 &= \big\|x_n + \mu_n(x_n - x_{n-1}) - \bar{x}\big\|^2\\
&\le \big\|x_n-\bar{x}\big\|^2 + \mu_n^2\|x_n - x_{n-1}\|^2 + 2\mu_n\langle x_n-\bar{x}, x_n - x_{n-1}\rangle\\
&\le \big\|x_n-\bar{x}\big\|^2 + \mu_n^2\|x_n - x_{n-1}\|^2 + 2\mu_n\|x_n-\bar{x}\|\|x_n - x_{n-1}\|, \qquad (3.6)
\end{aligned}
$$

for each $n \in \mathbb{N}$. By the definition of x_{n+1} and (3.6), we obtain

$$
\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &= \big\langle \alpha_n f(x_n) + \beta_n x_n + \theta_n S y_n - \bar{x}, x_{n+1}-\bar{x}\big\rangle\\
&= \alpha_n\big\langle f(x_n) - f(\bar{x}), x_{n+1}-\bar{x}\big\rangle + \alpha_n\big\langle f(\bar{x}) - \bar{x}, x_{n+1}-\bar{x}\big\rangle\\
&\quad +\beta_n\big\langle x_n-\bar{x}, x_{n+1}-\bar{x}\big\rangle + \theta_n\big\langle Sy_n-\bar{x}, x_{n+1}-\bar{x}\big\rangle\\
&\le \frac{\alpha_n}{2}\bigg(\big\|f(x_n)-f(\bar{x})\big\|^2 + \big\|x_{n+1}-\bar{x}\big\|^2\bigg)\\
&\quad +\alpha_n\big\langle f(\bar{x}) - \bar{x}, x_{n+1}-\bar{x}\big\rangle\\
&\quad +\frac{\beta_n}{2}\bigg(\big\|x_n-\bar{x}\big\|^2 + \big\|x_{n+1}-\bar{x}\big\|^2\bigg)\\
&\quad +\frac{\theta_n}{2}\bigg(\big\|Sy_n-\bar{x}\big\|^2 + \big\|x_{n+1}-\bar{x}\big\|^2\bigg)\\
&\le \bigg(\frac{\alpha_n\kappa^2}{2} + \frac{\beta_n}{2}\bigg)\|x_n-\bar{x}\|^2\\
&\quad +\frac{\theta_n}{2}\big\|y_n-\bar{x}\big\|^2 + \frac{\alpha_n+\beta_n+\theta_n}{2}\|x_{n+1}-\bar{x}\|^2\\
&\quad +\alpha_n\big\langle f(\bar{x}) - \bar{x}, x_{n+1}-\bar{x}\big\rangle\\
&\le \bigg(\frac{\alpha_n\kappa^2+\beta_n+\theta_n}{2}\bigg)\|x_n-\bar{x}\|^2 + \frac{1}{2}\|x_{n+1}-\bar{x}\|^2 \qquad (3.7)\\
&\quad +\frac{\theta_n\mu_n^2}{2}\|x_n-x_{n-1}\|^2 + \theta_n\mu_n\|x_n-\bar{x}\|\|x_n-x_{n-1}\|
\end{aligned}
$$

$$+\alpha_n\big\langle f(\bar{x})-\bar{x}, x_{n+1}-\bar{x}\big\rangle,$$

for each $n\in\mathbb{N}$. Then,

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &\le \big(1-\alpha_n(1-\kappa^2)\big)\|x_n-\bar{x}\|^2+\mu_n^2\|x_n-x_{n-1}\|^2\\
&\quad+2\mu_n\|x_n-\bar{x}\|\|x_n-x_{n-1}\|+2\alpha_n\big\langle f(\bar{x})-\bar{x},x_{n+1}-\bar{x}\big\rangle\\
&\le \big(1-\alpha_n(1-\kappa^2)\big)\|x_n-\bar{x}\|^2\\
&\quad+\mu_n\|x_n-x_{n-1}\|\big(\mu_n\|x_n-x_{n-1}\|+2\|x_n-\bar{x}\|\big)\\
&\quad+2\alpha_n\big\langle f(\bar{x})-\bar{x},x_{n+1}-\bar{x}\big\rangle\\
&\le \big(1-\alpha_n(1-\kappa^2)\big)\|x_n-\bar{x}\|^2\\
&\quad+3M_2\mu_n\|x_n-x_{n-1}\|+2\alpha_n\big\langle f(\bar{x})-\bar{x},x_{n+1}-\bar{x}\big\rangle\\
&\le \big(1-\alpha_n(1-\kappa^2)\big)\|x_n-\bar{x}\|^2\\
&\quad+\alpha_n(1-\kappa^2)\left(\frac{3M_2}{1-\kappa^2}\frac{\mu_n}{\alpha_n}\|x_n-x_{n-1}\|+\frac{2}{1-\kappa^2}\big\langle f(\bar{x})-\bar{x},x_{n+1}-\bar{x}\big\rangle\right), \qquad (3.8)
\end{aligned}$$

where $M_2=\sup_n\big\{\mu\|x_n-x_{n-1}\|,\|x_n-\bar{x}\|\big\}>0$, for each $n\in\mathbb{N}$.

We will consider the following two cases.

Case 1: Suppose that there exists $n_0\in\mathbb{N}$ such that $\big\{\|x_n-\bar{x}\|\big\}$ is monotonically non-increasing. By the boundedness of $\big\{\|x_n-\bar{x}\|\big\}$, it is a convergent sequence. Now, consider

$$\begin{aligned}
\|z_n-\bar{x}\|^2 &= \big(\|x_n-\bar{x}\|+\alpha_nM_1\big)^2\\
&= \|x_n-\bar{x}\|^2+2\alpha_nM_1\|x_n-\bar{x}\|+\alpha_n^2M_1^2\\
&= \|x_n-\bar{x}\|^2+\alpha_n\big(2M_1\|x_n-\bar{x}\|+\alpha_nM_1^2\big)\\
&= \|x_n-\bar{x}\|^2+\alpha_nM_3, \qquad (3.9)
\end{aligned}$$

where $M_3=\sup_n\big\{2M_1\|x_n-\bar{x}\|+\alpha_nM_1^2\big\}>0$, for each $n\in\mathbb{N}$. By using (2.2), (3.3) and (3.9), we have

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &= \big\|\alpha_nf(x_n)+\beta_nx_n+\theta_nSy_n-\bar{x}\big\|^2\\
&= \big\|\alpha_n\big(f(x_n)-\bar{x}\big)+\beta_n(x_n-\bar{x})+\theta_n(Sy_n-\bar{x})\big\|^2\\
&\le \alpha_n\big\|f(x_n)-\bar{x}\big\|^2+\beta_n\|x_n-\bar{x}\|^2+\theta_n\|Sy_n-\bar{x}\|^2\\
&\quad-\alpha_n\beta_n\big\|f(x_n)-x_n\big\|^2-\alpha_n\theta_n\big\|f(x_n)-Sy_n\big\|^2-\beta_n\theta_n\|x_n-Sy_n\|^2\\
&\le \alpha_n\big\|f(x_n)-\bar{x}\big\|^2+\beta_n\|x_n-\bar{x}\|^2+\theta_n\|Sy_n-\bar{x}\|^2\\
&\le \alpha_n\big\|f(x_n)-\bar{x}\big\|^2+\beta_n\|x_n-\bar{x}\|^2+\theta_n\|y_n-\bar{x}\|^2 \qquad (3.10)\\
&\le \alpha_n\big\|f(x_n)-\bar{x}\big\|^2+\beta_n\|x_n-\bar{x}\|^2+\theta_n\|z_n-\bar{x}\|^2\\
&\quad-\theta_n\gamma_n\left(\big\|(I-T)Lz_n\big\|^2-\gamma_n\big\|L^*(I-T)Lz_n\big\|^2\right)
\end{aligned}$$

$$\begin{aligned} &\leq \alpha_n\big\|f(x_n)-\bar{x}\big\|^2+\beta_n\|x_n-\bar{x}\|^2+\theta_n\|x_n-\bar{x}\|^2+\alpha_n\theta_nM_3 \\ &\quad -\theta_n\gamma_n\Bigg(\big\|(I-T)Lz_n\big\|^2-\gamma_n\big\|L^*(I-T)Lz_n\big\|^2\Bigg), \end{aligned}$$

for each $n\in\mathbb{N}$. It follows that

$$\begin{aligned} \theta_n\gamma_n\Bigg(\big\|(I-T)Lz_n\big\|^2-\gamma_n\big\|L^*(I-T)Lz_n\big\|^2\Bigg) &\leq \|x_n-\bar{x}\|^2-\|x_{n+1}-\bar{x}\|^2 \\ &\quad +\alpha_n\|f(x_n)-\bar{x}\|^2+\alpha_n\theta_nM_3 \\ &\leq \|x_n-\bar{x}\|^2-\|x_{n+1}-\bar{x}\|^2 \\ &\quad +\alpha_n\big(\|f(x_n)-\bar{x}\|^2+M_3\big), \end{aligned}$$

for each $n\in\mathbb{N}$. Consequently, by condition (i) and (iii), we have

$$\gamma_n\Bigg(\big\|(I-T)Lz_n\big\|^2-\gamma_n\big\|L^*(I-T)Lz_n\big\|^2\Bigg)\to 0,$$

as $n\to\infty$. Moreover, by the definition of γ_n, we see that

$$\gamma_n\Bigg(\big\|(I-T)Lz_n\big\|^2-\gamma_n\big\|L^*(I-T)Lz_n\big\|^2\Bigg)=\delta_n(1-\delta_n)\frac{\big\|(I-T)Lz_n\big\|^4}{\big\|L^*(I-T)Lz_n\big\|^2},$$

for each $n\in\mathbb{N}$. This implies

$$\delta_n(1-\delta_n)\frac{\big\|(I-T)Lz_n\big\|^4}{\big\|L^*(I-T)Lz_n\big\|^2}\to 0,$$

as $n\to\infty$. Since $\delta_n\in[a,b]\subset(0,1)$, we get

$$\lim_{n\to\infty}\frac{\big\|(I-T)Lz_n\big\|^2}{\big\|L^*(I-T)Lz_n\big\|}=0. \tag{3.11}$$

In addition, we observe that the fact $\big\|L^*(I-T)Lz_n\big\|\leq\|L^*\|\big\|(I-T)Lz_n\big\|$, implies

$$\big\|(I-T)Lz_n\big\|\leq\|L^*\|\frac{\big\|(I-T)Lz_n\big\|^2}{\big\|L^*(I-T)Lz_n\big\|},$$

for each $n\in\mathbb{N}$. Thus, by (3.11), we obtain

$$\lim_{n\to\infty}\big\|(I-T)Lz_n\big\|=0. \tag{3.12}$$

This forces

$$\lim_{n\to\infty}\big\|L^*(I-T)Lz_n\big\|=0. \tag{3.13}$$

At this point, we note that J_λ^B is firmly nonexpansive. Then, we have

$$\begin{aligned} \|y_n-\bar{x}\|^2 &= \big\|J_\lambda^B\big(z_n-\gamma_nL^*(I-T)Lz_n\big)-\bar{x}\big\|^2 \\ &\leq \big\langle J_\lambda^B\big(z_n-\gamma_nL^*(I-T)Lz_n\big)-\bar{x},z_n-\gamma_nL^*(I-T)Lz_n-\bar{x}\big\rangle \\ &= \big\langle y_n-\bar{x},z_n-\bar{x}\big\rangle-\gamma_n\big\langle y_n-\bar{x},L^*(I-T)Lz_n\big\rangle \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\|y_n - \bar{x}\|^2 + \frac{1}{2}\|z_n - \bar{x}\|^2 - \frac{1}{2}\|y_n - z_n\|^2 \\
&\quad -\gamma_n\langle y_n - \bar{x}, L^*(I-T)Lz_n\rangle,
\end{aligned}
$$

for each $n \in \mathbb{N}$. This gives,

$$
\begin{aligned}
\|y_n - \bar{x}\|^2 &\leq \|z_n - \bar{x}\|^2 - \|y_n - z_n\|^2 - 2\gamma_n\langle y_n - \bar{x}, L^*(I-T)Lz_n\rangle \\
&\leq \|z_n - \bar{x}\|^2 - \|y_n - z_n\|^2 + 2\gamma_n\|y_n - \bar{x}\|\|L^*(I-T)Lz_n\| \\
&\leq \|x_n - \bar{x}\|^2 + \alpha_n M_3 - \|y_n - z_n\|^2 + 2\gamma_n\|y_n - \bar{x}\|\|L^*(I-T)Lz_n\|, \qquad (3.14)
\end{aligned}
$$

for each $n \in \mathbb{N}$. From (3.10), by using (3.14) we have

$$
\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &\leq \alpha_n\big\|f(x_n) - \bar{x}\big\|^2 + \beta_n\|x_n - \bar{x}\|^2 + \theta_n\|y_n - \bar{x}\|^2 \\
&\leq \alpha_n\big\|f(x_n) - \bar{x}\big\|^2 + \beta_n\|x_n - \bar{x}\|^2 + \theta_n\|x_n - \bar{x}\|^2 + \alpha_n\theta_n M_3 \\
&\quad -\theta_n\|y_n - z_n\|^2 + 2\theta_n\gamma_n\|y_n - \bar{x}\|\|L^*(I-T)Lz_n\|,
\end{aligned}
$$

for each $n \in \mathbb{N}$. Thus

$$
\begin{aligned}
\theta_n\|y_n - z_n\|^2 &\leq \|x_n - \bar{x}\|^2 - \|x_{n+1} - \bar{x}\|^2 + \alpha_n\Big(\big\|f(x_n) - \bar{x}\big\|^2 + M_3\Big) \\
&\quad +2\theta_n\gamma_n\|y_n - \bar{x}\|\|L^*(I-T)Lz_n\|,
\end{aligned}
$$

for each $n \in \mathbb{N}$. By the convergent of the sequence $\{\|x_n - \bar{x}\|\}$, (3.13) and condition (i) and (iii), we obtain that

$$
\lim_{n\to\infty} \|y_n - z_n\| = 0. \qquad (3.15)
$$

From above relation, we know that

$$
\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &\leq \alpha_n\big\|f(x_n) - \bar{x}\big\|^2 + \beta_n\|x_n - \bar{x}\|^2 + \theta_n\|Sy_n - \bar{x}\|^2 \\
&\quad -\alpha_n\beta_n\|f(x_n) - \bar{x}\|^2 - \alpha_n\theta_n\|f(x_n) - Sy_n\|^2 - \beta_n\theta_n\|x_n - Sy_n\|^2,
\end{aligned}
$$

for each $n \in \mathbb{N}$. It follows that

$$
\begin{aligned}
\beta_n\theta_n\|x_n - Sy_n\|^2 &\leq \alpha_n\big\|f(x_n) - \bar{x}\big\|^2 + \beta_n\|x_n - \bar{x}\|^2 + \theta_n\|Sy_n - \bar{x}\|^2 \\
&\quad -\|x_{n+1} - \bar{x}\|^2, \qquad (3.16)
\end{aligned}
$$

for each $n \in \mathbb{N}$. Moreover, we have

$$
\|Sy_n - \bar{x}\|^2 \leq \|y_n - \bar{x}\|^2 \leq \|z_n - \bar{x}\|^2 \leq \|x_n - \bar{x}\|^2 + \alpha_n M_3,
$$

for each $n \in \mathbb{N}$. Thus, from (3.16) we obtain

$$
\begin{aligned}
\beta_n\theta_n\|x_n - Sy_n\|^2 &\leq \alpha_n\big\|f(x_n) - \bar{x}\big\|^2 + \beta_n\|x_n - \bar{x}\|^2 + \theta_n\|x_n - \bar{x}\|^2 \\
&\quad +\alpha_n\theta_n M_3 - \|x_{n+1} - \bar{x}\|^2 \\
&\leq \alpha_n\Big(\big\|f(x_n) - \bar{x}\big\|^2 + M_3\Big) + \|x_n - \bar{x}\|^2 - \|x_{n+1} - \bar{x}\|^2, \qquad (3.17)
\end{aligned}
$$

for each $n \in \mathbb{N}$. By the convergent of the sequence $\{\|x_n - \bar{x}\|\}$ and conditions (i) and (iii), we get

$$\lim_{n\to\infty} \|x_n - Sy_n\| = 0. \tag{3.18}$$

Using the definition of x_{n+1}, we have

$$\begin{aligned} \|x_{n+1} - Sy_n\| &= \big\|\alpha_n f(x_n) + \beta_n x_n + \theta_n Sy_n - Sy_n\big\| \\ &\le \alpha_n\big\|f(x_n) - Sy_n\big\| + \beta_n\|x_n - Sy_n\|, \end{aligned} \tag{3.19}$$

for each $n \in \mathbb{N}$. By using (3.18) and condition (i), we get

$$\lim_{n\to\infty} \|x_{n+1} - Sy_n\| = 0. \tag{3.20}$$

Next, we will show that $\lim_{n\to\infty} \|y_n - Sy_n\| = 0$. Consider the following inequality

$$\|y_n - Sy_n\| \le \|y_n - z_n\| + \|z_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - Sy_n\|, \tag{3.21}$$

for each $n \in \mathbb{N}$. In the second term of (3.21), we consider

$$\begin{aligned} \|z_n - x_n\| &= \big\|x_n + \mu_n(x_n - x_{n-1}) - x_n\big\| \\ &\le \alpha_n \frac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\|, \end{aligned}$$

for each $n \in \mathbb{N}$. Thus, by conditions (i) and (iv) we get

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.22}$$

And, in the third term of (3.21), we consider

$$\begin{aligned} \|x_{n+1} - x_n\| &= \big\|\alpha_n f(x_n) + \beta_n x_n + \theta_n Sy_n - x_n\big\| \\ &\le \alpha_n\big\|f(x_n) - x_n\big\| + \theta_n\|Sy_n - x_n\|, \end{aligned}$$

for each $n \in \mathbb{N}$. By using (3.18) and condition (i), we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.23}$$

Therefore, by (3.15), (3.20), (3.22) and (3.23) we get

$$\lim_{n\to\infty} \|y_n - Sy_n\| = 0. \tag{3.24}$$

Next, since $\{x_n\}$ is bounded on H_1, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to $x^* \in H_1$. We will prove that $x^* \in \Omega$. From (3.24) and $y_{n_i} \rightharpoonup x^*$, we obtain from Lemma 2.1 that $x^* \in F(S)$. Next, we will show that $x^* \in B^{-1}0$. Consider, for each $i \in \mathbb{N}$,

$$\begin{aligned} \big\|x^* - J_\lambda^B x^*\big\|^2 &\le \big\langle x^* - J_\lambda^B x^*, x^* - x_{n_i}\big\rangle + \big\langle x^* - J_\lambda^B x^*, x_{n_i} - J_\lambda^B x_{n_i}\big\rangle \\ &\quad + \big\langle x^* - J_\lambda^B x^*, J_\lambda^B x_{n_i} - J_\lambda^B x^*\big\rangle. \end{aligned} \tag{3.25}$$

Observe that

$$\begin{aligned} \big\|y_n - J_\lambda^B x_n\big\| &\leq \big\|J_\lambda^B\big(z_n - \gamma_n L^*(I-T)Lz_n\big) - J_\lambda^B x_n\big\| \\ &\leq \|z_n - x_n\| + \gamma_n\|L^*\|\big\|(I-T)Lz_n\big\|, \end{aligned} \tag{3.26}$$

for each $n \in \mathbb{N}$. By (3.12) and (3.22) we have

$$\lim_{n\to\infty} \big\|y_n - J_\lambda^B x_n\big\| = 0. \tag{3.27}$$

Consider the following inequality

$$\big\|x_n - J_\lambda^B x_n\big\| \leq \|x_n - z_n\| + \|z_n - y_n\| + \|y_n - J_\lambda^B x_n\|,$$

for each $n \in \mathbb{N}$. Thus, by (3.15), (3.22) and (3.27) we get

$$\lim_{n\to\infty} \big\|x_n - J_\lambda^B x_n\big\| = 0. \tag{3.28}$$

Since $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$, so consequence from (3.28) we have

$$\lim_{i\to\infty} \big\|x_{n_i} - J_\lambda^B x_{n_i}\big\| = 0. \tag{3.29}$$

From (3.25), by using (3.29) and together with $x_{n_i} \rightharpoonup x^*$, we obtain

$$\lim_{i\to\infty} \big\|x^* - J_\lambda^B x^*\big\| = 0.$$

Therefore, $x^* = J_\lambda^B x^*$ and hence $x^* \in B^{-1}0$.

Next, we will show that $Lx^* \in F(T)$. Similarly, we consider, for each $i \in \mathbb{N}$,

$$\begin{aligned} \big\|Lx^* - TLx^*\big\|^2 &\leq \big\langle Lx^* - TLx^*, Lx^* - Lx_{n_i}\big\rangle \\ &\quad + \big\langle Lx^* - TLx^*, Lx_{n_i} - TLx_{n_i}\big\rangle \\ &\quad + \big\langle Lx^* - TLx^*, TLx_{n_i} - TLx^*\big\rangle. \end{aligned} \tag{3.30}$$

To estimate the second term in (3.30), we first consider the following inequality,

$$\begin{aligned} \big\|(I-T)Lx_n\big\| &\leq \big\|(I-T)Lx_n - (I-T)Lz_n\big\| + \big\|(I-T)Lz_n\big\| \\ &\leq \|Lx_n - Lz_n\| + \big\|TLx_n - TLz_n\big\| + \big\|(I-T)Lz_n\big\| \\ &\leq 2\|L\|\|x_n - z_n\| + \big\|(I-T)Lz_n\big\|, \end{aligned}$$

for each $n \in \mathbb{N}$. Then, by (3.12) and (3.22), we have

$$\lim_{n\to\infty} \big\|(I-T)Lx_n\big\| = 0.$$

Thus, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, we also have

$$\lim_{i\to\infty} \big\|(I-T)Lx_{n_i}\big\| = 0.$$

Moreover, by the linearity and continuity of L, $Lx_{n_i} \rightharpoonup Lx^*$, as $i \to \infty$. Hence, from (3.30) we obtain that

$$\lim_{i\to\infty} \big\|Lx^* - TLx^*\big\| = 0.$$

Therefore, $Lx^* = TLx^*$, that is $Lx^* \in F(T)$. Consequently, we have $x^* \in \Omega$.

Finally, we will prove that $\{x_n\}$ converges strongly to $\bar{x} = P_\Omega f(\bar{x})$. Now, we know that $\{x_n\}$ is bounded and from (3.23) we have $\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$. With loss of generality, we may assume that a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ converges weakly to $x^* \in H_1$. Thus, we obtain

$$\begin{aligned}
\limsup_{n\to\infty} \frac{2}{1-\kappa^2}\big\langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\big\rangle &= \lim_{i\to\infty} \frac{2}{1-\kappa^2}\big\langle f(\bar{x}) - \bar{x}, x_{n_i+1} - \bar{x}\big\rangle \\
&= \frac{2}{1-\kappa^2}\big\langle f(\bar{x}) - \bar{x}, x^* - \bar{x}\big\rangle \leq 0. \qquad (3.31)
\end{aligned}$$

By using (3.31) and together with condition (iv) we get

$$\limsup_{n\to\infty}\left(\frac{3M}{1-\kappa^2}\frac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| + \frac{2}{1-\kappa^2}\big\langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\big\rangle\right) \leq 0.$$

From (3.8), by using Lemma 2.2, we can conclude that $\|x_n - \bar{x}\| \to 0$, as $n \to \infty$. Thus $x_n \to \bar{x}$, as $n \to \infty$.

Case 2: In the case that $\big\{\|x_n - \bar{x}\|\big\}$ is not monotonically decreasing sequence. Set $\Gamma_n = \|x_n - \bar{x}\|$, $\forall n \in \mathbb{N}$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\big\{k \in \mathbb{N} : k \leq n, \quad \Gamma_k \leq \Gamma_{k+1}\big\}.$$

Then, $\{\tau(n)\}$ is a nondecreasing sequence, such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \forall n \geq n_0.$$

Consequently, we have $\|x_{\tau(n)} - \bar{x}\|^2 - \|x_{\tau(n)+1} - \bar{x}\|^2 \leq 0$, for each $n \geq n_0$. From (3.11) we obtain the following relation,

$$\begin{aligned}
&\delta_{\tau(n)}\gamma_{\tau(n)}\left(\big\|(I-T)Lz_{\tau(n)}\big\|^2 - \gamma_{\tau(n)}\big\|L^*(I-T)Lz_{\tau(n)}\big\|^2\right) \\
&\leq \|x_{\tau(n)} - \bar{x}\|^2 - \|x_{\tau(n)+1} - \bar{x}\|^2 \\
&\quad + \alpha_{\tau(n)}\big(\|f(x_{\tau(n)}) - \bar{x}\|^2 + M_3\big) \\
&\leq \alpha_{\tau(n)}\big(\|f(x_{\tau(n)}) - \bar{x}\|^2 + M_3\big), \qquad (3.32)
\end{aligned}$$

for each $n \geq n_0$. By the similar argument as in **Case 1**, we can conclude that

$$\lim_{n\to\infty}\big\|(I-T)Lz_{\tau(n)}\big\| = 0,$$

$$\lim_{n\to\infty}\big\|L^*(I-T)Lz_{\tau(n)}\big\| = 0,$$

$$\lim_{n\to\infty}\|x_{\tau(n)+1} - x_{\tau(n)}\| = 0$$

and

$$\limsup_{n\to\infty}\left(\frac{3M}{1-\kappa^2}\frac{\mu_{\tau(n)}}{\alpha_{\tau(n)}}\|x_{\tau(n)} - x_{\tau(n)-1}\| + \frac{2}{1-\kappa^2}\big\langle f(\bar{x}) - \bar{x}, x_{\tau(n)+1} - \bar{x}\big\rangle\right) \leq 0.$$

Since $\{x_{\tau(n)}\}$ is bounded, we can find a subsequence of $\{x_{\tau(n)}\}$, still denoted by $\{x_{\tau(n)}\}$, which converges weakly to $x^* \in F(S) \cap B^{-1}0 \cap L^{-1}F(T)$. It follows from (3.8) that

$$\begin{aligned} \|x_{\tau(n)+1} - \bar{x}\|^2 &\leq \big(1 - \alpha_{\tau(n)}(1-\kappa^2)\big)\|x_{\tau(n)} - \bar{x}\|^2 \\ &\quad + \alpha_{\tau(n)}(1-\kappa^2)T_{\tau(n)}, \end{aligned} \tag{3.33}$$

where $T_{\tau(n)} = \frac{3M}{1-\kappa^2}\frac{\mu_{\tau(n)}}{\alpha_{\tau(n)}}\|x_{\tau(n)} - x_{\tau(n)-1}\| + \frac{2}{1-\kappa^2}\big\langle f(\bar{x}) - \bar{x}, x_{\tau(n)+1} - \bar{x}\big\rangle$, for each $n \geq n_0$. Then,

$$\begin{aligned} \alpha_{\tau(n)}(1-\kappa^2)\|x_{\tau(n)} - \bar{x}\|^2 &\leq \|x_{\tau(n)} - \bar{x}\|^2 - \|x_{\tau(n)+1} - \bar{x}\|^2 \\ &\quad + \alpha_{\tau(n)}(1-\kappa^2)T_{\tau(n)} \\ &\leq \alpha_{\tau(n)}(1-\kappa^2)T_{\tau(n)}, \end{aligned} \tag{3.34}$$

for each $n \geq n_0$. We note that $\alpha_{\tau(n)}(1-\kappa^2) > 0$. Thus, from (3.34) we get

$$\limsup_{n\to\infty} \|x_{\tau(n)} - \bar{x}\|^2 \leq 0.$$

This implies that

$$\lim_{n\to\infty} \|x_{\tau(n)} - \bar{x}\|^2 = 0,$$

and hence

$$\lim_{n\to\infty} \|x_{\tau(n)} - \bar{x}\| = 0. \tag{3.35}$$

Using $\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0$ and (3.35), we obtain that

$$\|x_{\tau(n)+1} - \bar{x}\| \leq \|x_{\tau(n)+1} - x_{\tau(n)}\| + \|x_{\tau(n)} - \bar{x}\| \to 0, \tag{3.36}$$

as $n \to \infty$. Furthermore, for each, $n \geq n_0$, if $\tau(n) < n$, we can see that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, because $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As a consequence, we obtain for each $n \geq n_0$

$$0 \leq \Gamma_n \leq \max\left\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\right\} = \Gamma_{\tau(n)+1}.$$

By using (3.36), we obtain $\lim_{n\to\infty} \Gamma_n = 0$. That is, we can conclude that $\{x_n\}$ converges strongly to $\bar{x}$. This completes the proof.

Remark 3.4. (a) The condition (iv) is easily implemented in numerical computation because we can find the valued of $\|x_n - x_{n-1}\|$ before choosing μ_n. Indeed, we can choose the parameter μ_n such that $0 \leq \mu_n \leq \bar{\mu}_n$, where

$$\bar{\mu}_n = \begin{cases} \min\left\{\mu, \dfrac{\omega_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}; \\ \mu, & \text{otherwise,} \end{cases}$$

where ω_n is a positive sequence such that $\omega_n = o(\alpha_n)$. The readers may see the reference [19] for more detail.

(b) The following choice is the special case of (a); we choose $\alpha_n = \frac{1}{n+1}$, $\omega_n = \frac{1}{(n+1)^2}$ and $\mu = \frac{n-1}{n+\kappa-1} \in [0,1)$. Then, we have

$$\bar{\mu}_n = \begin{cases} \min\left\{\dfrac{n-1}{n+\kappa-1}, \frac{1}{(n+1)^2\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}; \\ \dfrac{n-1}{n+\kappa-1}, & \text{otherwise.} \end{cases}$$

(c) If $S = I$ (the identity operator), then problem (1.10) reduces to problem (1.8). And, if $L = I$, we see that problem (1.10) reduces to a type of common fixed points of nonexpansive mappings (see [12] for more detail).

4. Applications

In this section, we will show some applications of the problem (1.10) via Theorem 3.3.

4.1. **Split feasibility problem.**

Recall that the normal cone to C at $u \in C$ is defined as

$$N_C(u) = \{z \in H : \langle z, y-u\rangle \leq 0, \quad \forall y \in C\},$$

where C is a nonempty closed convex subset of H. It is well known that N_C is a maximal monotone operator. So, in the case $B := N_C : H_1 \to 2^{H_1}$, we get $J_\lambda^B =: P_C$ (P_C is the metric projections onto C). Also, we get $F(J_\lambda^B) = F(P_C) = C$. By the setting $T =: P_Q$ (P_Q is the metric projections onto Q), we can verify that the problem (1.8) is reduced to the split feasibility problem (1.1). Subsequently, the problem (1.10) is reduced to a problem of finding a point

$$x^* \in F(S) \cap C \cap L^{-1}Q =: \Omega_{C,Q}^S$$

By following Algorithm 3.1, we introduce the following algorithm.

Algorithm 4.1. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\theta_n\}$ be sequences in $(0,1)$ with $\alpha_n+\beta_n+\theta_n = 1$ and the initial $x_0, x_1 \in H_1$ be arbitrary, define

$$\begin{aligned} z_n &= x_n + \mu_n(x_n - x_{n-1}), \\ y_n &= P_C\big(z_n - \gamma_n L^*(I-P_Q)Lz_n\big), \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \theta_n S y_n, \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{4.1}$$

where $\{\mu_n\} \subset [0,\mu]$ with $\mu \in [0,1)$ and $\{\gamma_n\}$ is depend on $\delta_n \in [a,b] \subset (0,1)$ by

$$\gamma_n = \begin{cases} \frac{\delta_n \left\|(I-P_Q)Lz_n\right\|^2}{\left\|L^*(I-P_Q)Lz_n\right\|^2}, & \text{if } L^*(I-P_Q)Lz_n \neq 0; \\ \gamma, & \text{otherwise,} \end{cases}$$

where γ is any nonnegative value.

Subsequently, by applying Theorem 3.3, we obtain the following theorem.

Theorem 4.2. *Let H_1 and H_2 be two real Hilbert spaces and let C and Q be a nonempty closed convex subset of H_1 and H_2, respectively. Let $\{x_n\}$ be generated by Algorithm 4.1. Suppose that the assumptions (A2), (A4) and (A5) hold, $\Omega^S_{C,Q} \neq \emptyset$ and the following control conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < a \leq \beta_n$ *and* $0 < a \leq \theta_n$;
(iv) $\lim_{n\to\infty} \frac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

Then, $\{x_n\}$ converges strongly to $\bar{x} \in \Omega^S_{C,Q}$, where $\bar{x} = P_{\Omega^S_{C,Q}} f(\bar{x})$.

Proof. It follows immediately from Theorem 3.3 and the above setting.

4.2. **Split variational inclusion problem.**

We will consider a maximal monotone operator $\tilde{B} : H_2 \to 2^{H_2}$. By setting $T := J^{\tilde{B}}_\lambda = (I + \lambda\tilde{B})^{-1}$, we obtain $F(T) := \tilde{B}^{-1}0$. In this case, we can verify that the problem (1.8) is reduced to the split null point problem (1.6). Subsequently, the problem (1.10) is reduced to a problem of finding a point

$$x^* \in F(S) \cap B^{-1}0 \cap L^{-1}(\tilde{B}^{-1}0) =: \Omega^S_{B,\tilde{B}}$$

Then, we obtain the following results.

Algorithm 4.3. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\theta_n\}$ be sequences in $(0,1)$ with $\alpha_n+\beta_n+\theta_n = 1$ and the initial $x_0, x_1 \in H_1$ be arbitrary, define

$$\begin{aligned} z_n &= x_n + \mu_n(x_n - x_{n-1}), \\ y_n &= J^B_\lambda\big(z_n - \gamma_n L^*(I - J^{\tilde{B}}_\lambda)Lz_n\big), \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \theta_n S y_n, \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{4.2}$$

where $\{\mu_n\} \subset [0,\mu]$ with $\mu \in [0,1)$ and $\{\gamma_n\}$ is depend on $\delta_n \in [a,b] \subset (0,1)$ by

$$\gamma_n = \begin{cases} \frac{\delta_n \left\|(I-J^{\tilde{B}}_\lambda)Lz_n\right\|^2}{\left\|L^*(I-J^{\tilde{B}}_\lambda)Lz_n\right\|^2}, & \text{if } L^*(I - J^{\tilde{B}}_\lambda)Lz_n \neq 0; \\ \gamma, & \text{otherwise}, \end{cases}$$

where γ is any nonnegative value.

Theorem 4.4. *Let H_1 and H_2 be two real Hilbert spaces and let $\tilde{B} : H_2 \to 2^{H_2}$ be a maximal monotone operator. Let $\{x_n\}$ be generated by Algorithm 4.3. Suppose that the assumptions (A1)-(A2) and (A4)-(A5) hold, $\Omega^S_{B,\tilde{B}} \neq \emptyset$ and the following control conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < a \leq \beta_n$ *and* $0 < a \leq \theta_n$*;*
(iv) $\lim_{n\to\infty} \frac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| = 0.$

Then, $\{x_n\}$ *converges strongly to* $\bar{x} \in \Omega^S_{B,\tilde{B}}$*, where* $\bar{x} = P_{\Omega^S_{B,\tilde{B}}} f(\bar{x})$*.*

Proof. By the above setting, we get this result follows from Theorem 3.3.

5. Concluding remarks

In this work, we present an algorithm for finding a solution of a class of split feasibility problems and fixed point problems in Hilbert spaces. We suggest the inertial viscosity-type algorithm and provide some suitable control conditions to the process. The strong convergence theorem of the proposed algorithm, Theorem 3.3, is presented. We showed some applications of the considered problem and the presented main result to the split feasibility problem and split variational inclusion problem in Hilbert spaces.

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