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CONVERGENCE UNDER SOME CONDITIONS OF A GENERAL ITERATIVE ALGORITHM FOR CONTINUOUS PSEUDOCONTRACTIVE MAPPINGS

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Abstract. In this paper, we consider a general iterative algorithm for a continuous pseudocontractive mapping in a Hilbert space. Utilizing weaker control conditions than previous ones, we establish the strong convergence of the sequence generated by the proposed iterative method to a fixed point of the mapping, which is the unique solution of a certain variational inequality.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and let $S : C \to C$ be a self-mapping on C. We denote by $F(S)$ the set of fixed points of S.

The class of pseudocontractive mappings is one of the most important classes of mappings among nonlinear mappings. We recall ([2, 3]) that a mapping $T : C \to H$ is said to be *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C,$$

and T is said to be *k-strictly pseudocontractive* ([3]) if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C,$$

where I is the identity mapping. The class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, T is *nonexpansive* (*i.e.*, $\|Tx - Ty\| \leq \|x - y\|$, $\forall x,\ y \in C$) if and only if T is 0-strictly pseudocontractive. Clearly, the class of k-strictly pseudocontractive mappings falls into the one between classes of nonexpansive mappings and pseudocontractive mappings. Recently, many authors have been devoting the studies on the problems of finding fixed points for pseudocontractive mappings, see, for example, [1, 5, 6, 7, 8, 10, 18] and the references therein.

In 2019, by combining Yamada's method [16] and Marino and Xu's method [9], Jung [8] considered the following general iterative algorithm for a continuous pseudocontractive mapping T:

$$x_{n+1} = P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F)T_{r_n} x_n], \quad \forall n \geq 0, \tag{1.1}$$

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where $x_0 \in C$ is an arbitrary initial guess; $\{\alpha_n\} \subset (0,1)$, $\{r_n\} \subset (0,\infty)$; P_C is the metric projection of H ont C; $F : C \to H$ is a ρ-Lipschitzian and η-strongly monotone operator with constants $\rho > 0$ and $\eta > 0$ (i.e., $\|Fx - Fy\| \leq \rho\|x - y\|$ and $\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2$, $x, y \in H$, respectively); $V : C \to H$ is an l-Lipschitzian mapping with a constant $l \geq 0$; $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 \leq \gamma l < \tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$; and $T_{r_n} : H \to C$ is a mapping defined by

$$T_{r_n}x = \left\{z \in C : \langle y - z, Tz\rangle - \frac{1}{r_n}\langle y - z, (1 + r_n)z - x\rangle \leq 0, \quad \forall y \in C\right\}$$

for $r_n \in (0,\infty)$. In particular, by using following control conditions on $\{\alpha_n\}$ and $\{r_n\}$

(C1) $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty} \alpha_n = 0$,
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(C3) $|\alpha_{n+1} - \alpha_n| \leq o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty} \sigma_n < \infty$ (the perturbed control condition), and
(C4) $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$ and $r_n > b > 0$ for $n \geq 0$,

he proved that the sequence $\{x_n\}$ generated by (1.1) converges strongly to a fixed point q of T, which is the unique solution of a certain variational inequality related to the operator F. His results improved the corresponding results of Ceng *et al.* [4], Jung [6, 7] and Tian [13, 14] from the class of nonexpansive mappings or the class of strictly pseudocontractive mappings to the class of continuous pseudocontractive mappings.

The following problem arises:

Question. Can we relax the conditions (C3) and (C4) in [8] on control parameters $\{\alpha_n\}$ and $\{r_n\}$ to the more weaker control condition?

In this paper, in order to give an affirmative answer to the above question, we consider the following general iterative algorithm for a continuous pseudocontractive mapping T in a Hilbert space:

$$x_{n+1} = \alpha_n\gamma Vx_n + (I - \alpha_n\mu F)T_{r_n}x_n, \quad \forall n \geq 0, \tag{1.2}$$

where $x_0 \in C$ is an arbitrary initial guess; $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$. By using weaker control conditions than previous ones, we establish the strong convergence of the sequence generated by the proposed algorithm (1.2) to a fixed point of T, which is a solution of a certain variational inequality related to F, where the constraint set is $Fix(T)$. The results in this paper improve and develop the corresponding results given in [4, 5, 6, 7, 8, 9, 13, 14] and references therein.

2. Preliminaries and Lemmas

Throughout this paper, when $\{x_n\}$ is a sequence in E, $x_n \to x$ (resp., $x_n \rightharpoonup x$) will denote strong (resp., weak) convergence of the sequence $\{x_n\}$ to x.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C ([12]). It is well known that P_C is nonexpansive and that for $x \in H$,

$$z = P_C x \Longleftrightarrow \langle x - z, y - z \rangle \le 0, \quad \forall y \in C. \tag{2.1}$$

The following is proven easily by the property of inner product.

Lemma 2.1. *In a real Hilbert space H, the following inequality holds:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x,\ y \in H.$$

Let LIM be a Banach limit. According to time and circumstances, we use $LIM_n(a_n)$ instead of $LIM(a)$ for every $a = \{a_n\} \in \ell^\infty$. The following properties are well-known ([12]):

(i) for all $n \ge 1, a_n \le c_n$ implies $LIM_n(a_n) \le LIM_n(c_n)$,
(ii) $LIM_n(a_{n+N}) = LIM_n(a_n)$ for any fixed positive integer N,
(iii) $\liminf_{n\to\infty} a_n \le LIM_n(a_n) \le \limsup_{n\to\infty} a_n$ for all $\{a_n\} \in l^\infty$.

The following lemma was given in [11].

Lemma 2.2. ([11]) *Let $a \in \mathbb{R}$ be a real number and a sequence $\{a_n\} \in l^\infty$ satisfy the condition $LIM_n(a_n) \le a$ for all Banach limit LIM. If $\limsup_{n\to\infty}(a_{n+1} - a_n) \le 0$, then $\limsup_{n\to\infty} a_n \le a$.*

We also need the following lemmas for the proof of our main results.

Lemma 2.3. ([15]) *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \le (1 - \beta_n)s_n + \beta_n\delta_n + \gamma_n, \quad \forall n \ge 0,$$

where $\{\beta_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\beta_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=0}^{\infty} \beta_n|\delta_n| < \infty$,
(iii) $\gamma_n \ge 0$ $(n \ge 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4. ([17]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a continuous pseudocontractive mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle Tz, y - z \rangle - \frac{1}{r}\langle y - z, (1 + r)z - x \rangle \le 0, \quad \forall y \in C.$$

For $r > 0$ and $x \in H$, define $T_r : H \to C$ by

$$T_r x = \left\{ z \in C : \langle Tz, y - z \rangle - \frac{1}{r}\langle y - z, (1 + r)z - x \rangle \le 0, \quad \forall y \in C \right\}.$$

Then the following hold:

(i) *T_r is single-valued;*
(ii) *T_r is firmly nonexpansive, that is,*

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y \rangle, \quad \forall x,\ y \in H;$$

(iii) *$Fix(T_r) = Fix(T)$;*

(iv) *$Fix(T)$ is a closed convex subset of C.*

The following lemmas can be easily proven, and therefore, we omit the proofs (see [16]).

Lemma 2.5. *Let H be a real Hilbert space. Let $V : H \to H$ be an l-Lipschitzian mapping with a constant $l \geq 0$ and let $F : H \to H$ be a ρ-Lipschitzian and η-strongly monotone operator with constants $\rho > 0$ and $\eta > 0$. Then for $0 \leq \gamma l < \mu\eta$,*

$$\langle(\mu F - \gamma V)x - (\mu F - \gamma V)y, x - y\rangle \geq (\mu\eta - \gamma l)\|x - y\|^2, \quad \forall x,\ y \in H.$$

That is, $\mu F - \gamma V$ is strongly monotone with a constant $\mu\eta - \gamma l$.

Lemma 2.6. *Let H be a real Hilbert space H. Let $F : H \to H$ be a ρ-Lipschitzian and η-strongly monotone operator with constants $\rho > 0$ and $\eta > 0$. Let $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 < t < \varsigma \leq 1$. Then $S := \varsigma I - t\mu F : H \to H$ is a contractive mapping with a constant $\varsigma - t\tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$.*

Finally, we recall that the sequence $\{x_n\}$ in H is said to be *weakly asymptotically regular* if

$$w - \lim_{n\to\infty}(x_{n+1} - x_n) = 0, \quad \text{that is,} \quad x_{n+1} - x_n \rightharpoonup 0$$

and *asymptotically regular* if

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$$

respectively.

3. Main results

Throughout the rest of this paper, we always assume the following:

- H is a real Hilbert space;
- $T : H \to H$ is a continuous pseudocontractive mapping with $Fix(T) \neq \emptyset$;
- $T_r : H \to H$ is a mapping defined by

$$T_r x = \left\{ z \in H : \langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in H \right\}$$

  for $r \in (0, \infty)$;
- $T_{r_n} : H \to H$ is a mapping defined by

$$T_{r_n} x = \left\{ z \in H : \langle y - z, Tz\rangle - \frac{1}{r_n}\langle y - z, (1 + r_n)z - x\rangle \leq 0, \quad \forall y \in H \right\}$$

  for $r_n \in (0, \infty)$ and $\lim_{n\to\infty} r_n = r$;
- $V : H \to H$ is an l-Lipschitzian mapping with constant $l \in [0, \infty)$;
- $F : H \to H$ is a ρ-Lipschitzian and η-strongly monotone mapping with constants $\rho > 0$ and $\eta > 0$;
- Constants μ, l, τ, and γ satisfy $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$;
- $Fix(T) \neq \emptyset$;

By Lemma 2.4, T_r and T_{r_n} are nonexpansive and $Fix(T) = Fix(T_r) = Fix(T_{r_n})$.

In this section, we consider the following iterative algorithm which generates a sequence in an explicit way:

$$x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_{r_n} x_n, \quad \forall n \geq 0, \tag{3.1}$$

where $\{\alpha_n\} \subset (0, 1)$, $r_n \in (0, \infty)$ and $x_0 \in H$ is an arbitrary initial guess, and establish strong convergence of this sequence to a fixed point q of T, which is the unique solution of the variational inequality:

$$\langle (\mu F - \gamma V)q, q - p \rangle \leq 0, \quad \forall p \in Fix(T). \tag{3.2}$$

(Equivalently, by (2.1), we have $P_{Fix(T)}(I - \mu F + \gamma V)q) = q$).

First, we consider the following iterative algorithm that generates a net $\{x_t\}_{t\in(0,1)}$ in an implicit way:

$$x_t = t\gamma V x_t + (I - t\mu F) T_r x_t. \tag{3.3}$$

Indeed, for $t \in (0, 1)$, consider a mapping $Q_t : H \to H$ defined by

$$Q_t x = t\gamma V x + (I - t\mu F) T_r x, \quad \forall x \in H.$$

It is easy to see that Q_t is a contractive mapping with constant $1 - t(\tau - \gamma l)$. Indeed, by Lemma 2.6, we have

$$\begin{aligned}
\|Q_t x - Q_t y\| &\leq t\gamma \|Vx - Vy\| + \|(I - t\mu F) T_r x - (I - t\mu F) T_r y\| \\
&\leq t\gamma l \|x - y\| + (1 - t\tau) \|x - y\| \\
&= (1 - t(\tau - \gamma l)) \|x - y\|.
\end{aligned}$$

Hence Q_t has a unique fixed point, denoted x_t, which uniquely solves the fixed point equation (3.3).

By utilizing the same method as in Theorem 3.1 of Jung [8] along with $r_t = r$ for $t \in (0, 1)$, we obtain the following proposition for strong convergence of the net $\{x_t\}$ as $t \to 0$, which guarantees the existence of solutions of the variational inequality (3.2). We omit its proof .

Proposition 3.1. ([8 , Theorem 3.1]) *The net $\{x_t\}$ defined by* (3.3) *converges strongly to a fixed point q of T as $t \to 0$, which solves the variational inequality* (3.2).

First, we give the following result in order to establish strong convergence of the sequence generated by the explicit algorithm (3.1).

Theorem 3.2. *Let $\{x_n\}$ be the sequence generated iteratively by the algorithm* (3.1) *and let LIM be a Banach limit. If $\{\alpha_n\}$ satisfies the following condition:*

(C1) $\{\alpha_n\} \subset (0, 1)$ *and* $\lim_{n\to\infty} \alpha_n = 0$,

then

$$LIM_n(\langle \mu F q - \gamma V q, q - x_n \rangle) \leq 0,$$

where $q = \lim_{t\to 0^+} x_t$ *with x_t being defined by* (3.3).

Proof. First, note that from the condition (C1), without loss of generality, we assume that $\alpha_n\tau < 1$ for all $n \geq 0$.

Let $\{x_t\}$ be the net generated by (3.3). By Proposition 3.1, there exists $\lim_{t\to 0} x_t \in Fix(T)$. Denote it by q. Moreover q is the unique solution of the variational inequality (3.2). By (3.3), we have

$$\begin{aligned}\|x_t - x_{n+1}\| &= \|t\gamma Vx_t + (I - t\mu F)T_r x_t - x_{n+1}\| \\ &= \|(I - t\mu F)T_r x_t - (I - t\mu F)x_{n+1} + t(\gamma Vx_t - \mu Fx_{n+1})\|.\end{aligned}$$

Applying Lemma 2.1 and Lemma 2.6, we have

$$\|x_t - x_{n+1}\|^2 \leq (1 - t\tau)^2\|T_r x_t - x_{n+1}\|^2 + 2t\langle\gamma Vx_t - \mu Fx_{n+1}, x_t - x_{n+1}\rangle. \tag{3.4}$$

First of all, we show that $\{x_t\}$ is bounded, and so $\{Vx_t\}$, $\{Tx_t\}$, $\{T_r x_t\}$, $\{Fx_t\}$ and $\{FT_r x_t\}$ are bounded. To this end, let $p \in Fix(T)$. Then, observing $Fix(T) = Fix(T_r)$ by Lemma 2.4, from (3.3), we derive that

$$\begin{aligned}\|x_t - p\| &\leq \|t\gamma Vx_t + (I - t\mu F)T_r x_t - p\| \\ &= \|t(\gamma Vx_t - \mu Fp) + (I - t\mu F)T_r x_t - (I - t\mu F)p\| \\ &\leq (1 - t\tau)\|x_t - p\| + t\|\gamma Vx_t - \mu Fp\|,\end{aligned}$$

and hence

$$\begin{aligned}\|x_t - p\| &\leq \frac{1}{\tau}\|\gamma Vx_t - \mu Fp\| \\ &\leq \frac{1}{\tau}[\|\gamma Vx_t - \gamma Vp\| + \|\gamma Vp - \mu Fp\|] \\ &\leq \frac{1}{\tau}[\gamma l\|x_t - p\| + \|\gamma Vp - \mu Fp\|].\end{aligned}$$

This implies that

$$\|x_t - p\| \leq \frac{1}{\tau - \gamma l}\|\gamma Vp - \mu Fp\|.$$

Hence $\{x_t\}$, $\{Vx_t\}$, $\{Tx_t\}$, $\{T_r x_t\}$, $\{Fx_t\}$ and $\{FT_r x_t\}$ are bounded.

Now we show that $\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|\mu Fp - \gamma Vp\|}{\tau - \gamma l}\}$ for all $n \geq 0$ and all $p \in Fix(T)$. Indeed, let $p \in Fix(T)$. Noticing $p = T_{r_n}p$, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n(\gamma Vx_n - \mu Fp) + (I - \alpha_n\mu F)T_{r_n}x_n - (I - \alpha_n\mu F)T_{r_n}p\| \\ &\leq (1 - \alpha_n\tau)\|x_n - p\| + \alpha_n\|\gamma Vx_n - \mu Fp\| \\ &\leq (1 - \alpha_n\tau)\|x_n - p\| + \alpha_n(\|\gamma Vx_n - \gamma Vp\| + \|\gamma Vp - \mu Fp\|) \\ &\leq [1 - (\tau - \gamma l)\alpha_n]\|x_n - p\| + (\tau - \gamma l)\alpha_n\frac{\|\gamma Vp - \mu Fp\|}{\tau - \gamma l} \\ &\leq \max\left\{\|x_n - p\|, \frac{\|\gamma Vp - \mu Fp\|}{\tau - \gamma l}\right\}.\end{aligned}$$

Using an induction, we have $\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|\gamma Vp - \mu Fp\|}{\tau - \gamma l}\}$. Hence $\{x_n\}$ is bounded, and so are $\{Vx_n\}$, $\{Tx_n\}$ $\{T_{r_n}x_n\}$, $\{FT_{r_n}x_n\}$, and $\{Fx_n\}$. As a consequence of condition (C1), we get

$$\|x_{n+1} - T_{r_n}x_n\| = \alpha_n\|\gamma Vx_n - \mu FT_{r_n}x_n\| \to 0 \quad (n \to \infty). \tag{3.5}$$

First of all, we show that $\|T_r x_n - T_{r_n} x_n\| \leq \frac{1}{r}|r - r_n|K_1$, where $K_1 = \sup\{\|T_r x_n - x_n\| : n \geq 1\}$. Indeed, let $z_n := T_{r_n} x_n$ and $z_r = T_r x_n$. Then, from definitions of T_r and T_{r_n}, we deduce

$$\langle y - z_r, Tz_r\rangle - \frac{1}{r}\langle y - z_r, (1+r)z_r - x_n\rangle \leq 0, \quad \forall y \in H, \tag{3.6}$$

and

$$\langle y - z_n, Tz_n\rangle - \frac{1}{r_n}\langle y - z_n, (1+r_n)z_n - x_n\rangle \leq 0, \quad \forall y \in H. \tag{3.7}$$

Putting $y = z_n$ in (3.6) and $y = z_r$ in (3.7), we obtain

$$\langle z_n - z_r, Tz_r\rangle - \frac{1}{r}\langle z_n - z_r, (1+r)z_r - x_n\rangle \leq 0, \tag{3.8}$$

and

$$\langle z_r - z_n, Tz_n\rangle - \frac{1}{r_n}\langle z_r - z_n, (1+r_n)z_n - x_n\rangle \leq 0. \tag{3.9}$$

Adding up (3.8) and (3.9), we have

$$\begin{aligned}&\langle z_r - z_n, Tz_n - Tz_r\rangle \\ &\qquad - \left\langle z_r - z_n, \frac{(1+r_n)z_n - x_n}{r_n} - \frac{(1+r)z_r - x_n}{r}\right\rangle \leq 0,\end{aligned}$$

which implies that

$$\begin{aligned}&\langle w_r - w_n, (w_r - Tw_r) - (w_n - Tw_n)\rangle \\ &\qquad\qquad - \left\langle w_r - w_n, \frac{w_n - x_n}{r_n} - \frac{w_r - x_n}{r}\right\rangle \leq 0.\end{aligned}$$

Now, using the fact that T is pseudocontractive, we deduce

$$\left\langle z_r - z_n, \frac{z_n - x_n}{r_n} - \frac{z_r - x_n}{r}\right\rangle \geq 0,$$

and hence

$$\left\langle z_r - z_n, z_n - z_r + z_r - x_n - \frac{r_n}{r}(z_r - x_n)\right\rangle \geq 0. \tag{3.10}$$

By (3.10), we have

$$\begin{aligned}\|z_r - z_n\|^2 &\leq \left\langle z_r - z_n, \left(1 - \frac{r_n}{r}\right)(z_r - x_n)\right\rangle \\ &\leq \|z_n - z_r\|\frac{1}{r}|r - r_n|\|z_r - x_n\|,\end{aligned}$$

which implies

$$\|T_r x_n - T_{r_n} x_n\| \leq \frac{1}{r}|r - r_n|K_1, \tag{3.11}$$

where $K_1 = \sup\{\|T_r x_n - x_n\| : n \geq 1\}$. Thus, by (3.11), we obtain

$$\begin{aligned}\|T_r x_t - x_{n+1}\| &\leq \|T_r x_t - T_r x_n\| + \|T_r x_n - T_{r_n} x_n\| + \|T_{r_n} x_n - x_{n+1}\| \\ &\leq \|x_t - x_n\| + \frac{1}{r}|r - r_n|\|x_n - T_r x_n\| + \|T_{r_n} x_n - x_{n+1}\| \\ &\leq \|x_t - x_n\| + \frac{1}{r}|r - r_n|K_1 + \|T_{r_n} x_n - x_{n+1}\| \\ &= \|x_t - x_n\| + e_n,\end{aligned}$$

where $e_n = \frac{K_1}{r}|r - r_n| + \|x_{n+1} - T_{r_n} x_n\| \to 0$ as $n \to \infty$ (by $\lim_{n\to\infty} r_n = r$ and (3.5)). Also observing that F is η-strongly monotone, we have

$$\langle \mu F x_t - \mu F x_n, x_t - x_n \rangle \geq \mu\eta \|x_t - x_n\|^2 \geq \tau \|x_t - x_n\|^2. \tag{3.12}$$

So, by combining (3.10) and (3.12), we obtain

$$\begin{aligned}&\|x_t - x_{n+1}\|^2 \\ &\leq (1 - t\tau)^2(\|x_t - x_n\| + e_n)^2 \\ &\quad + 2t\langle \gamma V x_t - \mu F x_t, x_t - x_{n+1}\rangle + 2t\langle \mu F x_t - \mu F x_{n+1}, x_t - x_{n+1}\rangle \\ &\leq (t^2\tau - 2t)\tau\|x_t - x_n\|^2 + \|x_t - x_n\|^2 \\ &\quad + (1 - t\tau)^2 e_n(2\|x_t - x_n\| + e_n) \\ &\quad + 2t\langle \gamma V x_t - \mu F x_t, x_t - x_{n+1}\rangle + 2t\langle \mu F x_t - \mu F x_{n+1}, x_t - x_{n+1}\rangle \\ &\leq (t^2\tau - 2t)\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle + \|x_t - x_n\|^2 \\ &\quad + e_n(K_2 + e_n) + 2t\langle \gamma V x_t - \mu F x_t, x_t - x_{n+1}\rangle \\ &\quad + 2t\langle \mu F x_t - \mu F x_{n+1}, x_t - x_{n+1}\rangle \\ &= t^2\tau\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle + \|x_t - x_n\|^2 \\ &\quad + e_n(K_2 + e_n) + 2t\langle \gamma V x_t - \mu F x_t, x_t - x_{n+1}\rangle \\ &\quad + 2t(\langle \mu F x_t - \mu F x_{n+1}, x_t - x_{n+1}\rangle - \langle \mu F x_t - \mu F x_n, x_t - x_n\rangle),\end{aligned} \tag{3.13}$$

where $K_2 = \sup\{2\|x_t - x_n\| : t, n \geq 0\}$. Applying the Banach limit LIM to (3.13) together with $\lim_{n\to\infty} e_n = 0$, we have

$$\begin{aligned}&LIM_n(\|x_t - x_{n+1}\|^2) \\ &\quad \leq t^2\tau LIM_n(\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle) + LIM_n(\|x_t - x_n\|^2) \\ &\qquad + 2t LIM_n(\langle \gamma V x_t - \mu F x_t, x_t - x_{n+1}\rangle) \\ &\qquad + 2t[LIM_n(\langle \mu F x_t - \mu F x_{n+1}, x_t - x_{n+1}\rangle) \\ &\qquad\qquad - LIM_n(\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle)].\end{aligned} \tag{3.14}$$

Using the property $LIM_n(a_n) = LIM_n(a_{n+1})$ of Banach limit in (3.14), we obtain

$$
\begin{aligned}
&LIM_n(\langle \mu F x_t - \gamma V x_t, x_t - x_n\rangle) = LIM_n(\langle \mu F x_t - \gamma V x_t, x_t - x_{n+1}\rangle) \\
&\le \frac{t\tau}{2} LIM_n(\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle) \\
&\quad + \frac{1}{2t}[LIM_n(\|x_t - x_n\|^2) - LIM_n(\|x_t - x_n\|^2)] \\
&\quad + [LIM_n(\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle) - LIM_n(\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle)] \\
&= \frac{t\tau}{2} LIM_n(\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle).
\end{aligned} \tag{3.15}
$$

Since

$$
\begin{aligned}
t\langle \mu F x_t - \mu F x_n, & x_t - x_n\rangle \\
&\le t\mu\rho\|x_t - x_n\|^2 \\
&\le t\mu\rho(\|x_t - p\| + \|p - x_n\|)^2 \\
&\le t\mu\rho\left(\frac{\|\gamma V p - \mu F p\|}{\tau - \gamma l} + \|x_0 - p\|\right)^2 \to 0 \quad (\text{as } t \to 0),
\end{aligned} \tag{3.16}
$$

we conclude from (3.15) and (3.16) that

$$
\begin{aligned}
LIM_n(\langle \mu F\widetilde{x} - \gamma V\widetilde{x}, \widetilde{x} - x_n\rangle) &\le \limsup_{t\to 0} LIM_n(\langle \mu F x_t - \gamma V x_t, x_t - x_n\rangle) \\
&\le \limsup_{t\to 0} \frac{t\tau}{2} LIM_n(\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle) \le 0.
\end{aligned}
$$

This completes the proof. □

Now, using Theorem 3.2, we establish strong convergence of the sequence generated by the explicit iterative algorithm (3.1) to a fixed point q of T, which is the unique solution of the variational inequality (3.2).

Theorem 3.3. *Let $\{x_n\}$ be the sequence generated iteratively by the algorithm* (3.1), *where $\{\alpha_n\}$ satisfies the following conditions:*

(C1) $\{\alpha_n\} \subset (0,1)$ *and* $\lim_{n\to\infty} \alpha_n = 0$.
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

If $\{x_n\}$ is weakly asymptotically regular, then $\{x_n\}$ converges strongly to $q \in Fix(T)$, where is the unique solution of the variational inequality (3.2).

Proof. First, note that from the condition (C1), without loss of generality, we assume that $\alpha_n\tau < 1$ and $\frac{2\alpha_n(\tau-\gamma l)}{1-\alpha_n\gamma l} < 1$ for all $n \ge 0$.

Let x_t be defined by (3.3), that is, $x_t = t\gamma V x_t + (I - t\mu F)T_r x_t$ for $0 < t < 1$, and let $\lim_{t\to 0} x_t := q \in Fix(T) = Fix(T_r)$ (by Lemma 2.4). Then q is the unique solution of the variational inequality (3.2).

We divides the proof into three steps:

Step 1. We see that $\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|\gamma V p - \mu F p\|}{\tau - \gamma l}\right\}$ for all $n \ge 0$ and all $p \in Fix(T)$ as in the proof of Theorem 3.2. Hence $\{x_n\}$ is bounded and so are $\{T_{r_n}x_n\}$, $\{FT_{r_n}x_n\}$ and $\{Vx_n\}$.

Step 2. We show that $\limsup_{n\to\infty}\langle \mu Fq - \gamma Vq, q - x_n\rangle \le 0$. To this end, put

$$a_n := \langle \mu Fq - \gamma Vq, q - x_n)\rangle, \quad \forall n \ge 0.$$

Then Theorem 3.2 implies that $LIM_n(a_n) \le 0$ for any Banach limit LIM. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \lim_{j\to\infty}(a_{n_j+1} - a_{n_j})$$

and $x_{n_j} \rightharpoonup v \in H$. This implies that $x_{n_j+1} \rightharpoonup v$ since $\{x_n\}$ is weakly asymptotically regular. Therefore, we have

$$w - \lim_{j\to\infty}(q - x_{n_j+1}) = w - \lim_{j\to\infty}(q - x_{n_j}) = (q - v),$$

and so

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \lim_{j\to\infty}\langle \mu Fq - \gamma Vq, (q - x_{n_j+1}) - (q - x_{n_j})\rangle = 0.$$

Then Lemma 2.2 implies that $\limsup_{n\to\infty} a_n \le 0$, that is,

$$\limsup_{n\to\infty}\langle \mu Fq - \gamma Vq, q - x_n\rangle \le 0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_n - q\| = 0$. By using (3.1), we have

$$x_{n+1} - q = \alpha_n(\gamma Vx_n - \mu Fq) + (I - \alpha_n\mu F)T_{r_n}x_n - (I - \alpha_n\mu F)q.$$

Applying Lemma 2.1 and Lemma 2.6, we obtain

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|(I - \alpha_n\mu F)T_{r_n}x_n - (I - \alpha_n\mu F)q + \alpha_n(\gamma Vx_n - \mu Fq)\|^2 \\
&\le \|(I - \alpha_n\mu F)T_{r_n}x_n - (I - \alpha_n\mu F)T_{r_n}q\|^2 \\
&\quad + 2\alpha_n\langle \gamma Vx_n - \mu Fq, x_{n+1} - q\rangle \\
&\le (1 - \alpha_n\tau)^2\|x_n - q\|^2 + 2\alpha_n\langle \gamma Vx_n - \gamma Vq, x_{n+1} - q\rangle \\
&\quad + 2\alpha_n\langle \gamma Vq - \mu Fq, x_{n+1} - q\rangle \\
&\le (1 - \alpha_n\tau)^2\|x_n - q\|^2 + \alpha_n\gamma l(\|x_n - q\|^2 + \|x_{n+1} - q\|^2) \\
&\quad + 2\alpha_n\langle \gamma Vq - \mu Fq, x_{n+1} - q\rangle.
\end{aligned} \tag{3.17}$$

It then follows from (3.17) that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le \frac{(1 - \alpha_n\tau)^2 + \alpha_n\gamma l}{1 - \alpha_n\gamma l}\|x_n - q\|^2 + \frac{2\alpha_n}{1 - \alpha_n\gamma l}\langle \gamma Vq - \mu Fq, x_{n+1} - q\rangle \\
&\le \left(1 - \frac{2\alpha_n(\tau - \gamma l)}{1 - \alpha_n\gamma l}\right)\|x_n - q\|^2 \\
&\quad + \frac{2\alpha_n(\tau - \gamma l)}{1 - \alpha_n\gamma l}\left(\frac{1}{\tau - \gamma l}\langle \gamma Vq - \mu Fq, x_{n+1} - q\rangle + \frac{\alpha_n\tau^2}{2(\tau - \gamma l)}K_3\right),
\end{aligned} \tag{3.18}$$

where $K_3 = \sup\{\|x_n - q\|^2 : n \geq 0\}$. Put

$$\beta_n = \frac{2\alpha_n(\tau - \gamma l)}{1 - \alpha_n \gamma l} \quad \text{and} \quad \delta_n = \frac{1}{\tau - \gamma l}\langle \mu F q - \gamma V q, q - x_{n+1}\rangle + \frac{\alpha_n \tau^2}{2(\tau - \gamma l)} K_3.$$

From (C1), (C2) and Step 2, it follows that $\beta_n \to 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$ and

$$\limsup_{n\to\infty} \delta_n \leq 0.$$

Since (3.18) reduces to

$$\|x_{n+1} - q\|^2 \leq (1 - \beta_n)\|x_n - q\|^2 + \beta_n \delta_n,$$

from Lemma 2.3 with $\gamma_n = 0$, we conclude that $\lim_{n\to\infty} \|x_n - q\| = 0$. This completes the proof. □

Corollary 3.4. *Let $\{x_n\}$ be the sequence generated iteratively by the algorithm* (3.1), *where $\{\alpha_n\}$ satisfies the following conditions:*

(C1) $\{\alpha_n\} \subset (0, 1)$ *and* $\lim_{n\to\infty} \alpha_n = 0$.
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

If $\{x_n\}$ is asymptotically regular, then $\{x_n\}$ converges strongly to $\widetilde{x} \in F(T)$, where is the unique solution of the variational inequality (3.2).

Remark 3.5. If $\{\alpha_n\}$ and $\{r_n\}$ in Corollary 3.4 satisfy conditions (C1), (C2),

(C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; or
(C4) $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$ or, equivalently, $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}} = 0$; or,
(C5) $|\alpha_{n+1} - \alpha_n| \leq o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty} \sigma_n < \infty$ (the perturbed control condition); and
(C6) $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$ and $0 < b \leq r_n$ for $n \geq 0$,

then, by using method of [8], we can prove that the sequence $\{x_n\}$ generated by (3.1) is asymptotically regular.

Remark 3.6. Theorem 3.3 extends the corresponding results of Ceng *et al.* [4], Cho et al. [5], Jung [6, 7, 8], Marino and Xu [9] and Tian [13, 14] in the following aspects:

(a) The class of strictly pseudocontractive mappings in [5, 6, 7] was extended to the class of continuous pseudocontractive mappings.
(b) The class of nonexpansive mappings in [4, 9, 13, 14] was extended to the class of continuous pseudocontractive mappings.
(c) The conditions $|\alpha_{n+1} - \alpha_n| \leq o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty} \sigma_n < \infty$ (the perturbed control condition) and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$ in [8] was relaxed to weak asymptotic regularity on $\{x_n\}$ along with $\lim_{n\to\infty} r_n = r$.
(d) The condition $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ in [4, 5, 9, 13, 14] was also weakened to the weak asymptotic regularity on $\{x_n\}$.
(e) A strongly positive bounded linear operator A in [5, 9] was extended to the case of a ρ-Lipschitzian and η-strongly monotone operator F. (In fact, from the definitions, it follows that a strongly positive bounded linear operator A

(i.e., there exists a constant $\overline{\gamma} > 0$ with the property: $\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \ \ x \in H$) is a $\|A\|$-Lipschitzian and $\overline{\gamma}$-strongly monotone operator).

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