Linear and Nonlinear Analysis
Yokohama Publishers
ISSN 2188-8167
Volume 6, Number 3, 2020, 347–369

USING INVERTIBLE FUNCTIONS TO CONSTRUCT NCP FUNCTIONS

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Abstract. In this paper, we provide a novel idea to construct NCP functions, which is new to the literature. More specifically, we present under what conditions the category of invertible functions can be employed to construct new NCP functions. Demonstrations of generated NCP functions with their figures are shown as well. There are many possible research directions theorectically and numerically based on the newly discovered NCP functions.

1. Introduction

The nonlinear complementarity problem (NCP) is to find a vector $x \in \mathbb{R}^n$ satisfying the following:

$$x \geq 0, \quad F(x) \geq 0, \quad x^T F(x) = 0, \tag{1.1}$$

or equivalently,

$$x_i \geq 0, \quad F_i(x) \geq 0, \quad \text{and} \quad x_i F_i(x) = 0, \ i = 1, \cdots, n, \tag{1.2}$$

where F is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$.

During the past few decades, the nonlinear complementarity problem has attracted much attention due to its various applications in operations research, economics, and engineering, see [6, 7] and references therein. There have many methods proposed for solving the NCP. Among which, one of the most popular and powerful approaches is to reformulate the NCP as a system of nonlinear equations [15] or as an unconstrained minimization problem [8, 10]. For these two approaches, they indeed rely on the so-called NCP functions. Officially, a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is said to be an NCP function if it satisfies

$$\phi(a, b) = 0 \quad \Longleftrightarrow \quad a \geq 0, \ b \geq 0 \text{ and } ab = 0. \tag{1.3}$$

In light of the above the NCP function, one can define the vector-valued function $\Phi_F : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi_F(x) = \begin{pmatrix} \phi(x_1, F_1(x)) \\ \vdots \\ \phi(x_n, F_n(x)) \end{pmatrix}.$$

2010 *Mathematics Subject Classification.* 90C25, 90C33, 26A27.

Key words and phrases. NCP function, Nonlinear complementarity problem, Fischer-Burmeister.

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Then, it is clear to see that original NCP (1.1) is equivalent to the problem of solving $\Phi_{F}(x) = 0$. Equivalently, one can also transform the problem to finding the global minima solutions of its merit function, that is,

$$\min_{x\in\mathbb{R}^n} \Psi_{F}(x) := \frac{1}{2}\|\Phi_{F}(x)\|^2.$$

In the literature, there have many NCP functions which were discovered gradually. One of them is the well-known Fischer-Burmeister function $\phi_{\rm FB}(a,b) = \sqrt{a^2+b^2}-a-b$. An advantage of this function is that it is convex and is twice differentiable at every point except at the origin. A natural extension of $\phi_{\rm FB}$ is the generalized Fischer-Burmeister function given by

$$\phi_p(a,b) = \|(a,b)\|_p - a - b,\ p > 1$$

which is also an NCP function, where $\|(a,b)\|_p = (|a|^p + |b|^p)^{1/p}$. The function ϕ_p can be viewed as a natural "continuous extension" of $\phi_{\rm FB}$, and its geometric view is depicted in [16]. Another different type of popular NCP function is the natural residual function $\phi_{\rm NR} : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\phi_{\rm NR}(a,b) = a - (a-b)_+ = \min\{a,b\}.$$

Albeit the function $\phi_{\rm NR}$ is not differentiable on the line $a = b$, Chen, Ko, and Wu [4] figured out a way, which is called "discrete generalization", to smoothize this function. To achieve it, they generalize it by adding a p-power to each term, that is, by defining

$$\phi^p_{\rm NR}(a,b) = a^p - (a-b)^p_+,$$

where $p > 1$ is an odd integer. Note that p being positive odd integer is very crucial, more details can be found in [4].

In fact, there exist some systematic ways to construct NCP functions. The first general way was given by Magasarian in [15]. Some subsequent ways were proposed by Luo and Tseng [13]; and by Kanzow, Yamashita, and Fukushima [11]. Recently, more rigorous discussions and further study on how to construct NCP functions were investigated by Galantai in [9], and by Alcantara and Chen, et al. in [1, 2]. In particular, in contrast to "discrete generalization" in [4], Alcantara and Chen proposed a novel "continuous extension" of $\phi_{\rm NR}$ in [1]. More specifically, a family of NCP functions is constructed, which is defined by

$$\tilde{\phi}^p_{\rm NR} = \mathrm{sgn}(a)|a|^p - [(a-b)_+]^p,$$

where $p \in (0,\infty)$. The main idea behind is that if $\phi = \phi_1 - \phi_2$ is an NCP function and a given function f is one to one, then $\tilde{\phi} = f(\phi_1) - f(\phi_2)$ is also an NCP function. Indeed, it can be verified that

$$\tilde{\phi}^p_{\rm NR} = f(a) - f((a-b)_+),$$

where $f(t) = \mathrm{sgn}(t)|t|^p$ is an one to one function. By this construction, the value p (positive odd integer) is extended to any positive real number. Moreover, $\tilde{\phi}^p_{\rm NR}$ reduces to $\phi^p_{\rm NR}$ when p is an odd integer and $p > 1$.

In this paper, we provide another novel idea to construct NCP functions, which is new to the literature. More specifically, we present under what conditions the category of invertible functions can be employed to construct new NCP functions. The motivation comes from discovering three specific NCP functions and observing their structures. Demonstrations of generated NCP functions with their figures are shown as well. There definitely have many possible research directions theorectically and numerically based on the newly discovered NCP functions in this paper.

2. Three specific new NCP functions

As mentioned in Section 1, the motivation of our study comes from three specific new NCP functions. First, we show out the three new NCP functions as below:

$$\phi_{\ln-\max}(a,b) = \ln(e^{|a|} + e^{|b|} - 1) - \max(a,b); \tag{2.1}$$

$$\phi_{\ln-\mathrm{sum}}(a,b) = \ln(e^{|a|} + e^{|b|} - 1) - (a+b); \tag{2.2}$$

$$\phi_{\mathrm{abs-exp}}(a,b) = |a| + |b| - e^a b - e^b a. \tag{2.3}$$

The first two functions $\phi_{\ln-\max}$ and $\phi_{\ln-\mathrm{sum}}$ are composed of the functions, $\exp(\cdot)$ and $\ln(\cdot)$, with an additional term to keep the function value zero on the nonnegative sides of the axes. These three new NCP functions are discovered from observing the construction ways in [1, 2]. Note that these three NCP functions are not differentiable since they involve the absolute value terms. Therefore, we need to compute their subdifferentials [5] due to subsequent analysis. To this end, we will use sign function

$$\mathrm{sgn}(t) := \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0. \end{cases}$$

and the definition of convex hull of all limits points of Jacobian sequence.

Proposition 2.1. *Let $\phi_{\ln-\max} : \mathbb{R}^2 \to \mathbb{R}$ be defined in (2.1), that is,*

$$\phi_{\ln-\max}(a,b) = \ln(e^{|a|} + e^{|b|} - 1) - \max(a,b).$$

Then, the following hold.

(a) *The function $\phi_{\ln-\max}$ is an NCP function.*

(b) *The subdifferential of* $\phi_{\ln-\max}$ *is described by*

$$\partial\phi_{\ln-\max}(a,b) =$$
$$\begin{cases}
\left\{\left(\frac{e^a}{e^a+e^b-1}-1,\ \frac{e^b}{e^a+e^b-1}\right)\right\} & \text{if } (a,b)\in \mathrm{I}_1=\{(a,b)|a,b>0 \text{ and } a>b\}.\\
\left\{\left(\frac{e^a}{e^a+e^b-1},\ \frac{e^b}{e^a+e^b-1}-1\right)\right\} & \text{if} (a,b)\in \mathrm{I}_2=\{(a,b)|a,b>0 \text{ and } b>a\}.\\
\left\{\left(\frac{-e^{-a}}{e^{-a}+e^b-1},\ \frac{e^b}{e^{-a}+e^b-1}-1\right)\right\} & \text{if } (a,b)\in \mathrm{I}_3=\{(a,b)|a<0,\ b>0\}.\\
\left\{\left(\frac{-e^{-a}}{e^{-a}+e^{-b}-1},\ \frac{-e^{-b}}{e^{-a}+e^{-b}-1}-1\right)\right\} & \text{if } (a,b)\in \mathrm{I}_4=\{(a,b)|a,b<0 \text{ and } b>a\}.\\
\left\{\left(\frac{-e^{-a}}{e^{-a}+e^{-b}-1}-1,\ \frac{-e^{-b}}{e^{-a}+e^{-b}-1}\right)\right\} & \text{if } (a,b)\in \mathrm{I}_5=\{(a,b)|a,b<0, \text{ and } a>b\}\\
\left\{\left(\frac{e^a}{e^a+e^{-b}-1}-1,\ \frac{-e^{-b}}{e^a+e^{-b}-1}\right)\right\} & \text{if } (a,b)\in \mathrm{I}_6=\{(a,b)|a>0,\ b<0\}\\
\left\{(0,\rho)\,|\,\frac{-1}{e^a}\le\rho\le\frac{1}{e^a}\right\} & \text{if } (a,b)\in \mathrm{L}_1=\{(a,b)|a>0,\ b=0\}\\
\{(\rho,0)\,|\,-1\le\rho\le 1\} & \text{if } (a,b)\in \mathrm{L}_3=\{(a,b)|a=0,\ b>0\}\\
co\left\{\begin{pmatrix}\frac{e^a}{2e^a-1}-1\\ \frac{e^a}{2e^a-1}\end{pmatrix},\begin{pmatrix}\frac{e^a}{2e^a-1}\\ \frac{e^a}{2e^a-1}-1\end{pmatrix}\right\} & \text{if } (a,b)\in \mathrm{L}_2=\{(a,b)|a,b>0, \text{ and } a=b\}.\\
\{(-1,\rho-1)\,|-e^a\le\rho\le e^a\} & \text{if } (a,b)\in \mathrm{L}_4=\{(a,b)|a<0,\ b=0\}.\\
\left\{(\rho-1,-1)\,|-e^b\le\rho\le e^b\right\} & \text{if } (a,b)\in \mathrm{L}_6=\{(a,b)|a=0,\ b<0\}.\\
co\left\{\begin{pmatrix}\frac{-e^{-a}}{2e^{-a}-1}\\ \frac{-e^{-a}}{2e^{-a}-1}-1\end{pmatrix},\begin{pmatrix}\frac{-e^{-a}}{2e^{-a}-1}-1\\ \frac{-e^{-a}}{2e^{-a}-1}\end{pmatrix}\right\} & \text{if } (a,b)\in \mathrm{L}_5=\{(a,b)|a,b<0, \text{ and } a=b\}.\\
co\left\{\begin{matrix}(0,1),(1,0),\\ (-1,-2),(-2,-1)\end{matrix}\right\} & \text{if } (a,b)=(0,0).
\end{cases}$$

where $co(S)$ *denotes the convex hull of the set* S.

Proof. (a) "⇒" Suppose $\phi_{\ln-\max}(a,b)=0$, we need to show $a\ge 0,\ b\ge 0,\ ab=0$. To proceed, we discuss two cases.

(i) If $a\ge b$, then $e^{|a|}+e^{|b|}-1=e^a$. The left-hand side of this equality is greater than or equal to 1 since the absolute value is always nonnegative. Hence, e^a must be greater than or equal to 1. This leads to a being greater than or equal to 0. Thus, $a\ge 0$ and $e^{|b|}-1=0$, which says $a\ge 0$ and $b=0$.

(ii) If $b\ge a$, by the symmetric form of the function $\phi_{\ln-\max}$, we see that $a=0,\ b\ge 0$. Therefore, $a\ge 0,\ b=0$ or $a=0,\ b\ge 0$. This is equivalent to $a\ge 0,\ b\ge 0,\ ab=0$.

"⇐" Conversely, if $a\ge 0,\ b\ge 0,\ ab=0$, then $a\ge 0,\ b=0$ or $a=0,\ b\ge 0$. For $a\ge 0,\ b=0$, then $\phi_{\ln-\max}(a,b)=\ln(e^{|a|}+e^{|b|}-1)-\max(a,b)=|a|-a=0$. For $a=0,\ b\ge 0$, then $\phi_{\ln-\max}(a,b)=\ln(e^{|a|}+e^{|b|}-1)-\max(a,b)=|b|-b=0$. Thus, the proof is done.

(b) Note that $\phi_{\ln-\max}$ is differentiable at $(a,b)\in \mathrm{I}_1\sim\mathrm{I}_6$, whereas it is not differentiable in other cases. Hence, we need to calculate each case separately.

Case (1): $(a,b) \in \mathrm{I}_1 = \{(a,b)\,|\, a,b>0 \text{ and } a>b\}$.

$$\begin{aligned}\nabla\phi_{\ln-\max}(a,b) &= \left(\frac{e^{|a|}a}{(e^{|a|}+e^{|b|}-1)|a|}-1,\ \frac{e^{|b|}b}{(e^{|a|}+e^{|b|}-1)|b|}\right)\\ &= \left(\frac{e^{a}}{e^{a}+e^{b}-1}-1,\ \frac{e^{b}}{e^{a}+e^{b}-1}\right)\end{aligned}$$

Case (2): $(a,b) \in \mathrm{I}_2 = \{(a,b)\,|\, a,b>0 \text{ and } b>a\}$.

$$\begin{aligned}\nabla\phi_{\ln-\max}(a,b) &= \left(\frac{e^{|a|}a}{(e^{|a|}+e^{|b|}-1)|a|},\ \frac{e^{|b|}b}{(e^{|a|}+e^{|b|}-1)|b|}-1\right)\\ &= \left(\frac{e^{a}}{e^{a}+e^{b}-1},\ \frac{e^{b}}{e^{a}+e^{b}-1}-1\right)\end{aligned}$$

Case (3): $(a,b) \in \mathrm{I}_3 = \{(a,b)\,|\, a<0,\ b>0\}$.

$$\begin{aligned}\nabla\phi_{\ln-\max}(a,b) &= \left(\frac{e^{|a|}a}{(e^{|a|}+e^{|b|}-1)|a|},\ \frac{e^{|b|}b}{(e^{|a|}+e^{|b|}-1)|b|}-1\right)\\ &= \left(\frac{-e^{-a}}{e^{-a}+e^{b}-1},\ \frac{e^{b}}{e^{-a}+e^{b}-1}-1\right)\end{aligned}$$

Case (4): $(a,b) \in \mathrm{I}_4 = \{(a,b)\,|\, a,b<0, \text{ and } b>a\}$.

$$\begin{aligned}\nabla\phi_{\ln-\max}(a,b) &= \left(\frac{e^{|a|}a}{(e^{|a|}+e^{|b|}-1)|a|},\ \frac{e^{|b|}b}{(e^{|a|}+e^{|b|}-1)|b|}-1\right)\\ &= \left(\frac{-e^{-a}}{e^{-a}+e^{-b}-1},\ \frac{-e^{-b}}{e^{-a}+e^{-b}-1}-1\right)\end{aligned}$$

Case (5): $(a,b) \in \mathrm{I}_5 = \{(a,b)\,|\, a,b<0, \text{ and } a>b\}$.

$$\begin{aligned}\nabla\phi_{\ln-\max}(a,b) &= \left(\frac{e^{|a|}a}{(e^{|a|}+e^{|b|}-1)|a|}-1,\ \frac{e^{|b|}b}{(e^{|a|}+e^{|b|}-1)|b|}\right)\\ &= \left(\frac{-e^{-a}}{e^{-a}+e^{-b}-1}-1,\ \frac{-e^{-b}}{e^{-a}+e^{-b}-1}\right)\end{aligned}$$

Case (6): $(a,b) \in \mathrm{I}_6 = \{(a,b)\,|\, a>0,\ b<0\}$.

$$\begin{aligned}\nabla\phi_{\ln-\max}(a,b) &= \left(\frac{e^{|a|}a}{(e^{|a|}+e^{|b|}-1)|a|}-1,\ \frac{e^{|b|}b}{(e^{|a|}+e^{|b|}-1)|b|}\right)\\ &= \left(\frac{e^{a}}{e^{a}+e^{-b}-1}-1,\ \frac{-e^{-b}}{e^{a}+e^{-b}-1}\right)\end{aligned}$$

Case (7): $(a,b) \in \mathrm{L}_1 = \{(a,b)\,|\, a>0,\ b=0\}$.
Since the point $(a,0)$ is adjacent to the region I_1 and I_6, let $\{(a_k,b_k)\}$ be the sequence

such that $\lim_{k\to\infty}(a_k, b_k) = (a, 0)$.
If $\{(a_k, b_k)\} \subseteq \mathrm{I}_1 = \{(a, b)|a, b > 0 \text{ and } a > b\}$, then

$$\begin{aligned}\lim_{k\to\infty} \nabla\phi_{\ln-\max}(a_k, b_k) &= \lim_{k\to\infty} \left(\frac{e^{a_k}}{e^{a_k} + e^{b_k} - 1} - 1,\ \frac{e^{b_k}}{e^{a_k} + e^{b_k} - 1} \right) \\ &= \left(\frac{e^{a}}{e^{a} + e^{0} - 1} - 1,\ \frac{e^{0}}{e^{a} + e^{0} - 1} \right) \\ &= \left(0, \frac{1}{e^a}\right).\end{aligned}$$

If $\{(a_k, b_k)\} \subseteq \mathrm{I}_6 = \{(a, b)|a > 0,\ b < 0\}$, then

$$\begin{aligned}\lim_{k\to\infty} \nabla\phi_{\ln-\max}(a_k, b_k) &= \lim_{k\to\infty} (\frac{e^{a_k}}{e^{a_k} + e^{-b_k} - 1} - 1,\ \frac{-e^{-b_k}}{e^{a_k} + e^{-b_k} - 1}) \\ &= (\frac{e^{a}}{e^{a} + e^{0} - 1} - 1,\ \frac{-e^{0}}{e^{a} + e^{0} - 1}) \\ &= (0,\ \frac{-1}{e^a})\end{aligned}$$

Thus, by definition of subdifferential, we have

$$\begin{aligned}\partial\phi_{\ln-\max}(a, b) &= \mathrm{co}\left\{ \begin{pmatrix} 0 \\ \frac{1}{e^a} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{-1}{e^a} \end{pmatrix} \right\} \\ &= \left\{ (0, \rho) \mid \frac{-1}{e^a} \le \rho \le \frac{1}{e^a} \right\},\end{aligned}$$

Case (8) $\sim$ (12): For the region $\mathrm{L_i}$, $i = 2, \cdots, 6$ and the origin point $(0, 0)$, the way to calculate the subdifferential is similar to L_1, so we omit them here. □

The next proposition is about the second NCP function $\phi_{\ln-\mathrm{sum}}$. In fact, computing the subdifferential of $\phi_{\ln-\max}$ is more complicated than $\phi_{\ln-\mathrm{sum}}$ since the term $\max(a, b)$ together with the absolute value function partition the plane to more pieces where it is differentiable. In addition, we can obtain the results regarding $\phi_{\ln-\mathrm{sum}}$ by using some parts in Proposition 2.1. Before presenting the proof, we point out two special observations regarding $\phi_{\ln-\mathrm{sum}}$. Note that $\max(a, b) = a + b$ when $a \ge 0$, $b \ge 0$, $ab = 0$, hence $\phi_{\ln-\max}(a, b) = \phi_{\ln-\mathrm{sum}}(a, b) = 0$ when $a \ge 0$, $b \ge 0$, $ab = 0$. In view of these, we make a guess that $\phi_{\ln-\mathrm{sum}}$ is also an NCP function. In order to verify that $\phi_{\ln-\mathrm{sum}}$ is indeed an NCP function, we need to check the value of the the function $\phi_{\ln-\mathrm{sum}}$ on other regions.

Proposition 2.2. *Let $\phi_{\ln-\mathrm{sum}} : \mathbb{R}^2 \to \mathbb{R}$ be defined in (2.2), that is,*

$$\phi_{\ln-\mathrm{sum}}(a, b) = \ln(e^{|a|} + e^{|b|} - 1) - (a + b).$$

Then, the following hold.

(a) *The function $\phi_{\ln-\mathrm{sum}}$ is an NCP function.*

(b) *The subdifferential of* $\phi_{\ln-\text{sum}}$ *is described by*

$$\partial\phi_{\ln-\text{sum}}(a,b)=\begin{cases}\left\{\left(\frac{e^{|a|}a}{(e^{|a|}+e^{|b|}-1)|a|}-1,\ \frac{e^{|b|}b}{(e^{|a|}+e^{|b|}-1)|b|}-1\right)\right\} & \text{if } a\neq 0 \text{ and } b\neq 0.\\ \left\{(\rho-1,0)\mid\ \leq\frac{-1}{e^{|b|}}\leq\rho\leq\frac{1}{e^{|b|}}\right\} & \text{if } a=0,b>0.\\ \left\{(\rho-1,-2)\mid\frac{-1}{e^{|b|}}\leq\rho\leq\frac{1}{e^{|b|}}\right\} & \text{if } a=0,b<0.\\ \left\{(0,\rho-1)\mid\frac{-1}{e^{|a|}}\leq\rho\leq\frac{1}{e^{|a|}}\right\} & \text{if } a>0,b=0.\\ \left\{(-2,\rho-1)\mid\frac{-1}{e^{|a|}}\leq\rho\leq\frac{1}{e^{|a|}}\right\} & \text{if } a<0,b=0.\\ \{(\xi,\eta)\mid-2\leq\xi,\eta\leq 0\} & \text{if } a=b=0.\end{cases}$$

Proof. To prove (a), we need to show $\phi_{\ln-\text{sum}}$ satisfies condition (1.3). From Proposition 2.1, it is obvious that $\phi_{\ln-\text{sum}}(a,b)$ is zero on the nonnegative sides of the a,b-axes, and positive on the negative sides of the a,b-axes. Thus, we only have to check the four quadrants of the ab-plane. Suppose $\phi_{\ln-\text{sum}}(a,b)=0$. To proceed, we consider four cases as below:

Case (i): If $a>0$ and $b>0$, then $e^a+e^b-1=e^ae^b$. Then, we have

$$0=e^a(e^b-1)+(e^b-1)=(e^a-1)(e^b-1).$$

Since $a>0$ and $b>0$, we should have $(e^a-1)(e^b-1)>0$ which leads to a contradiction. Thus, $\phi_{\ln-\text{sum}}(a,b)\neq 0$ in case (i).

Case (ii): If $a<0$ and $b>0$, then $e^{-a}+e^b-1=e^ae^b$ and hence $\frac{1}{e^a}+e^b-1=e^ae^b$, which gives $(e^a)^2e^b-1-e^ae^b+e^a=0$. It follows that

$$0=e^ae^b(e^a-1)+(e^a-1)=(e^ae^b+1)(e^a-1).$$

But, $e^ae^b+1>0$ and $e^a-1<0$, it says $(e^ae^b+1)(e^a-1)<0$, which contradicts the above equation. Thus, $\phi_{\ln-\text{sum}}(a,b)\neq 0$ in case (ii).

Case (iii): Suppose $a<0$ and $b<0$. Obviously, $e^{|a|}+e^{|b|}-1$ is greater than 1, and $e^ae^b=e^{a+b}$ is less than 1. Hence, $\phi_{\ln-\text{sum}}(a,b)\neq 0$ in case (iii).

Case (iv): If $a>0$ and $b<0$, then $\phi_{\ln-\text{sum}}\neq 0$ by noting the symmetry of $\phi_{\ln-\text{sum}}(a,b)$ and the arguments in case (ii).

To sum up, from all the above, we prove that $\phi_{\ln-\text{sum}}(a,b)=0\iff a\geq 0,b=0$ or $a=0,\ b\geq 0\iff a\geq 0,\ b\geq 0,\ ab=0$.

(b) Again, by using the definition of subdifferential, we calculate each case separately.

Case (1): If $a\neq 0$ and $b\neq 0$, then $\phi_{\ln-\text{sum}}$ is differentiable. Then, we have

$$\nabla\phi_{\ln-\text{sum}}(a,b)=\left(\frac{e^{|a|}a}{(e^{|a|}+e^{|b|}-1)|a|}-1,\ \frac{e^{|b|}b}{(e^{|a|}+e^{|b|}-1)|b|}-1\right). \tag{2.4}$$

Case (2): Suppose $a=0$ and $b>0$. we compute subdifferential by the definition of convex hull of all limits points of Jacobian sequence. Let $(a_k,b_k)\to(0,b)$ as

$k \to \infty$. Applying (2.4) yields

$$\lim_{k\to\infty} \nabla\phi_{\ln-\mathrm{sum}}(a_k, b_k) = \begin{cases} \left(\frac{1}{e^b} - 1, 0\right) & \text{if } \{(a_k, b_k)\} \subseteq \{(a,b) \,|\, a > 0, b > 0\}. \\ \left(\frac{-1}{e^b} - 1, 0\right) & \text{if } \{(a_k, b_k)\} \subseteq \{(a,b) \,|\, a < 0, b > 0\}. \end{cases}$$

This concludes $\partial\phi_{\ln-\mathrm{sum}}(a,b) = \{(\rho - 1, 0) \,|\, \rho \in [\frac{-1}{e^b}, \frac{1}{e^b}]\}$. The other cases exclude the case $a = b = 0$, which are similar to the above cases. Therefore, it remains to prove the case when $a = b = 0$.

For the case $a = b = 0$, let $(a_k, b_k) \to (0,0)$. Compute the limit of (2.4) as $(a_k, b_k) \to (0,0)$, we have

$$\lim_{k\to\infty} \nabla\phi_{\ln-\mathrm{sum}}(a_k, b_k) = \begin{cases} (0,0) & \text{if } \{(a_k, b_k)\} \subseteq \{(a,b) \,|\, a > 0, b > 0\}. \\ (-2,0) & \text{if } \{(a_k, b_k)\} \subseteq \{(a,b) \,|\, a < 0, b > 0\}. \\ (-2,-2) & \text{if } \{(a_k, b_k)\} \subseteq \{(a,b) \,|\, a < 0, b < 0\}. \\ (0,-2) & \text{if } \{(a_k, b_k)\} \subseteq \{(a,b) \,|\, a > 0, b < 0\}. \end{cases}$$

Hence $\partial\phi_{\ln-\mathrm{sum}}(0,0) = \mathrm{co}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix} \right\} = \{(\xi, \eta) \,|\, -2 \le \xi, \eta \le 0\}$.

□

Proposition 2.3. *Let $\phi_{\mathrm{abs-exp}} : \mathbb{R}^2 \to \mathbb{R}$ be defined as in (2.3), that is,*

$$\phi_{\mathrm{abs-exp}}(a,b) = |a| + |b| - e^a b - e^b a.$$

Then, the following hold.

(a) *The function $\phi_{\mathrm{abs-exp}}$ is an NCP function.*
(b) *The subdifferential of $\phi_{\mathrm{abs-exp}}$ is described as*

$$\partial\phi_{\mathrm{abs-exp}}(a,b) = \begin{cases} \{(\mathrm{sgn}(a) - e^a b - e^b, \mathrm{sgn}(b) - e^b a - e^a)\} & \text{if } a \ne 0,\ b \ne 0. \\ \{(0, \rho - a - e^a) \,|\, -1 \le \rho \le 1\} & \text{if } a > 0,\ b = 0. \\ \{(\rho - b - e^b, 0) \,|\, -1 \le \rho \le 1\} & \text{if } a = 0,\ b > 0. \\ \{(-2, \rho - a - e^a) \,|\, -1 \le \rho \le 1\} & \text{if } a < 0,\ b = 0. \\ \{(\rho - b - e^b, -2) \,|\, -1 \le \rho \le 1\} & \text{if } a = 0,\ b < 0. \\ \{(\xi, \eta) \,|\, -2 \le \xi, \eta \le 0\} & \text{if } a = b = 0. \end{cases}$$

Proof. (a) First, we rewrite the function $\phi_{\mathrm{abs-exp}}$ as

$$\phi_{\mathrm{abs-exp}}(a,b) = a(\mathrm{sgn}(a) - e^b) + b(\mathrm{sgn}(b) - e^a),$$

which possesses the below piecewise expression:

$$\phi_{\mathrm{abs-exp}}(a,b) = \begin{cases} 0 & \text{if} \quad a \ge 0,\ b \ge 0, \text{ and } ab = 0. \\ a(1 - e^b) + b(1 - e^a) & \text{if} \quad a > 0,\ b > 0. \\ -a(1 + e^b) + b(1 - e^a) & \text{if} \quad a < 0,\ b \ge 0. \\ -a(1 + e^b) - b(1 + e^a) & \text{if} \quad a < 0,\ b < 0. \end{cases}$$

It is noted that $\phi_{\text{abs}-\exp}(a,b)$ is negative in the second case, and positive in the third and last cases. In light of the symmetry of $\phi_{\text{abs}-\exp}(a,b)$, we obtain that $\phi_{\text{abs}-\exp}(a,b)$ is also positive on $a \geq 0$, $b < 0$. Therefore, $\phi_{\text{abs}-\exp}$ is an NCP function.

(b) We discuss a few cases in order to calculate the subdifferential of $\phi_{\text{abs}-\exp}$.

Case (1): If $a \neq 0$ and $b \neq 0$, we have

$$\nabla \phi_{\text{abs}-\exp}(a,b) = \Big(\text{sgn}(a) - e^a b - e^b,\ \text{sgn}(b) - e^b a - e^a\Big). \tag{2.5}$$

Case (2): Suppose $a > 0,\ b = 0$ and $(a_k, b_k) \to (a, 0)$ as $k \to \infty$. From expression (2.5), we know

$$\lim_{k\to\infty} \nabla \phi_{\text{abs}-\exp}(a_k,b_k) = \left\{ \begin{array}{ll} (0, 1 - a - e^a) & \text{if} \quad \{(a_k,b_k)\} \subseteq \{(a,b) \,|\, a > 0, b > 0\}. \\ (0, -1 - a - e^a) & \text{if} \quad \{(a_k,b_k)\} \subseteq \{(a,b) \,|\, a > 0, b < 0\}. \end{array} \right.$$

Then, it follows that $\partial \phi_{\text{abs}-\exp}(a,b) = \{(0, \rho - a - e^a) \,|\, -1 \leq \rho \leq 1\}$.

For the other cases except for the case when $a = b = 0$, the calculation is similar to case (2), so we omit them here.

Case (3): Suppose $a = b = 0$ and $(a_k, b_k) \to (0,0)$ as $k \to \infty$. From expression (2.5), we compute that

$$\lim_{k\to\infty} \nabla \phi_{\text{abs}-\exp}(a_k,b_k) = \begin{cases} (0,0) & \text{if } \{(a_k,b_k)\} \subseteq \{(a,b) \,|\, a > 0, b > 0\}. \\ (-2,0) & \text{if } \{(a_k,b_k)\} \subseteq \{(a,b) \,|\, a < 0, b > 0\}. \\ (-2,-2) & \text{if } \{(a_k,b_k)\} \subseteq \{(a,b) \,|\, a < 0, b < 0\}. \\ (0,-2) & \text{if } \{(a_k,b_k)\} \subseteq \{(a,b) \,|\, a > 0, b < 0\}. \end{cases}$$

This concludes $\partial \phi_{\text{abs}-\exp}(0,0) = \{(\xi, \eta) \,|\, -2 \leq \xi, \eta \leq 0\}$. □

To close this section, we point out that the function $\phi_{\text{abs}-\exp}$ is discovered in a different way from the previous two NCP functions. The main difference lies on the fact that the term $|a| + |b|$ can be viewed by the 1-norm of (a,b), and the term e^a, e^b are the monotone cofactors added to b and a.

3. NCP functions generated by invertible functions

By looking into the structures of those three newly discovered NCP functions in Section 2, we figure out a way for further generating NCP functions by employing invertible functions. From the first two NCP functions, $\phi_{\ln-\max}(a,b)$ and $\phi_{\ln-\text{sum}}(a,b)$, it is clear to see the presence of a common term $\ln(e^{|a|} + e^{|b|} - 1)$, which involves invertible functions. Therefore, we generalize the common term to

$$f(f^{-1}(|a|) + f^{-1}(|b|) - f^{-1}(0)),$$

where f is a real valued function defined on some domain and satisfies some assumptions. Accordingly, their natural extended formats become

$$\phi(a,b) = f(f^{-1}(|a|) + f^{-1}(|b|) - 1) - \max(a,b); \tag{3.1}$$

$$\phi(a,b) = f(f^{-1}(|a|) + f^{-1}(|b|) - 1) - (a+b); \tag{3.2}$$

$$\phi(a,b) = f(f^{-1}(|a|) + f^{-1}(|b|) - 1) - g(a)b - g(b)a. \tag{3.3}$$

Clearly, if $f(t) = \ln t$, then the functions (3.1) and (3.2) reduce to those functions (2.1) and (2.2). If $f^{-1}(t) = t + 1$ and $g(t) = e^t$, then the function (3.3) reduces to the function (2.3). In this Section, we provide a complete discussion on under what conditions of f, $(f^{-1})'$ and g, the above functions defined as in (3.1), (3.2) and (3.3) will be NCP functions.

Proposition 3.1. *Suppose f is a real valued function defined on $\mathbb{R}$ with $f(1) = 0$ and $f|_I$ denotes the restricted function of f on $I \subseteq \mathbb{R}$. If $f|_I$ satisfies one of the following conditions:*

(a) *$f|_I : [1,\infty) \to [0,\infty)$ is invertible, or*
(b) *$f|_I : (-\infty,1] \to [0,\infty)$ is invertible,*

then $\phi(a,b) = f(f^{-1}(|a|) + f^{-1}(|b|) - f^{-1}(0)) - \max(a,b)$ is an NCP function.

Proof. (a) Without loss of ambiguity, we still use f instead of $f|_I$ in our analysis. Since $f : [1,\infty) \to [0,\infty)$ is invertible and $f(1) = 0$, f is strictly monotone increasing on $[1,\infty)$. In addition, f^{-1} is also strictly monotone increasing. To verify that ϕ is an NCP function, we need to show that ϕ satisfies condition (1.3).

"$\Rightarrow$" Suppose $\phi(a,b) = 0$, we consider the two regions on the a,b-plane which are $a \geq b$ and $b \geq a$.

Case (1): If $a \geq b$, then $f(f^{-1}(|a|) + f^{-1}(|b|) - f^{-1}(0)) = \max(a,b) = a$. Since $f^{-1}(|a|) + f^{-1}(|b|) - f^{-1}(0) \geq 1$ and f is strictly monotone increasing on $[1,\infty)$, so $a = f(f^{-1}(|a|) + f^{-1}(|b|) - f^{-1}(0)) \geq f(1) = 0$. Then, $f^{-1}(|b|) - 1 = f^{-1}(a) - f^{-1}(|a|) = 0$ since a is nonnegative. This says $b = 0,\ a \geq 0$.

Case (2): If $b \geq a$, by the symmetric form of the function, we obtain $a = 0,\ b \geq 0$. Therefore, $a \geq 0,\ b = 0$ or $b = 0,\ a \geq 0$. This is equivalent to $a \geq 0,\ b \geq 0,\ ab = 0$.

"$\Leftarrow$" Conversely, suppose that $a \geq 0,\ b \geq 0,\ ab = 0$. Then, we have $a \geq 0,\ b = 0$ or $b = 0,\ a \geq 0$. If $a \geq 0,\ b = 0$, it is clear that $\phi(a,b) = |a| - a = 0$. If $b = 0,\ a \geq 0$, it is also clear that $\phi(a,b) = |b| - b = 0$.

(b) Since $f : (-\infty,1] \to [0,\infty)$ is invertible and $f(1) = 0$, f is strictly monotone decreasing. In addition, f^{-1} is also strictly monotone decreasing. Next, we show that ϕ is an NCP function.

"$\Rightarrow$" Suppose $\phi(a,b) = 0$, we consider the two regions on the a,b-plane which are $a \geq b$ and $b \geq a$.

Case (1): If $a \geq b$, then $f(f^{-1}(|a|) + f^{-1}(|b|) - f^{-1}(0)) = a$. Since $f^{-1}(|a|) + f^{-1}(|b|) - f^{-1}(0) \leq 1$ and f is strictly monotone decreasing on $(-\infty,1]$, $a =$

$f(f^{-1}(|a|)+f^{-1}(|b|)-f^{-1}(0)) \geq f(1) = 0$. Then, $f^{-1}(|b|)-1 = f^{-1}(a)-f^{-1}(|a|) = 0$ since a is nonnegative. Hence $b = 0,\ a \geq 0$.

Case (2): If $b \geq a$, by the symmetric form of the function, we obtain that $a = 0,\ b \geq 0$. Therefore, $a \geq 0,\ b = 0$ or $b = 0,\ a \geq 0$. This is equivalent to $a \geq 0,\ b \geq 0,\ ab = 0$.

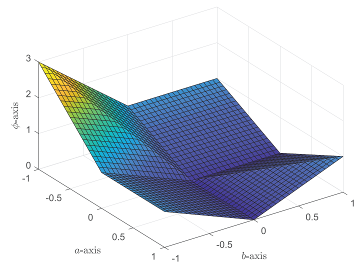
"$\Leftarrow$" Conversely, suppose that $a \geq 0,\ b \geq 0,\ ab = 0$. Then, we have $a \geq 0,\ b = 0$ or $b = 0,\ a \geq 0$. For $a \geq 0,\ b = 0$, it is clear that $\phi(a,b) = |a| - a = 0$. For $b = 0,\ a \geq 0$, it is also trivial that $\phi(a,b) = |b| - b = 0$. $\square$

Example 3.2. Here are examples of f satisfying condition in Proposition 3.1(a).

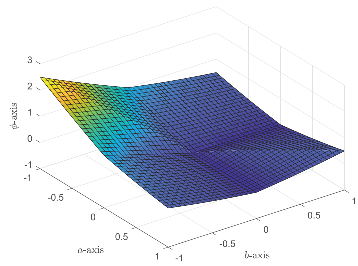
(1) $f_1(t) = (t-1)\big|_{[1,\infty)}$.
(2) $f_2(t) = \ln(t)\big|_{[1,\infty)}$.
(3) $f_3(t) = (t-1)^{1/2}\big|_{[1,\infty)}$.
(4) $f_4(t) = (t-1)^{1/5}\big|_{[1,\infty)}$.

Then, their corresponding NCP functions are as below.

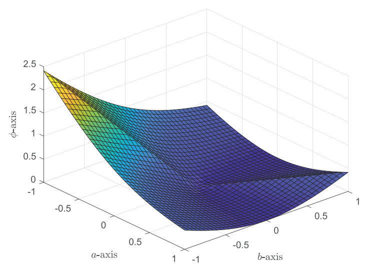
(1) $\phi_{f_1}(a,b) = |a| + |b| - \max(a,b)$.
(2) $\phi_{f_2}(a,b) = \ln(e^{|a|} + e^{|b|} - 1) - \max(a,b)$.
(3) $\phi_{f_3}(a,b) = \|(a,b)\|_2 - \max(a,b)$.
(4) $\phi_{f_4}(a,b) = \|(a,b)\|_5 - \max(a,b)$.



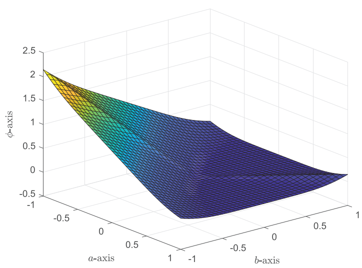
(a) Graph of ϕ_{f_1} in Example 3.2



(b) Graph of ϕ_{f_2} in Example 3.2



(c) Graph of ϕ_{f_3} in Example 3.2



(d) Graph of ϕ_{f_4} in Example 3.2

FIGURE 1. Graphs of NCP functions shown in Example 3.2

For the case of Proposition 3.1(b), since every strictly monotone decreasing function can be produced by adding a negative sign to the strictly monotone increasing function, the examples for Proposition 3.1(b) can be obtained immediately from Example 3.2.

Proposition 3.3. *Suppose f is a continuously differentiable real valued function with $f(1) = 0$. If f satisfies the following conditions:*

(i) $f : [1, \infty) \to [0, \infty)$ *is invertible, and*

(ii) $(f^{-1})'$ *is strictly monotone on* $[0, \infty)$,

then $\phi(a, b) = f(f^{-1}(|a|) + f^{-1}(|b|) - f^{-1}(0)) - (a + b)$ *is an NCP function.*

Proof. To verify that ϕ is an NCP function, we need to show that ϕ is equal to zero only on the nonnegative sides of the a, b-axes. To proceed, we have to check the four quadrants on the a, b-plane. For the analysis of the second, third, and the fourth quadrant, we only use the monotonicity of f^{-1}. We need the monotonicity of $(f^{-1})'$ only when analyzing the first quadrant.

Case (1): Suppose $a > 0$ and $b > 0$. If $(f^{-1})'$ is strictly monotone increasing on $[0, \infty)$, then we have

$$\begin{aligned} f^{-1}(a + b) - f^{-1}(b) &= \int_b^{a+b} (f^{-1})'(x)dx \\ &> \int_0^a (f^{-1})'(x)dx \\ &= f^{-1}(a) - f^{-1}(0) \\ &= f^{-1}(a) - 1. \end{aligned}$$

Thus, $1 < f^{-1}(a) + f^{-1}(b) - 1 < f^{-1}(a+b)$. Since f is strictly monotone increasing on $[1, \infty)$, so

$$f(f^{-1}(a) + f^{-1}(b) - 1) < f(f^{-1}(a + b)) = a + b.$$

Thus,

$$\phi(a, b) = f(f^{-1}(a) + f^{-1}(b) - 1) - (a + b) < 0.$$

If $(f^{-1})'$ is strictly monotone decreasing on $[0, \infty)$, then we have

$$\begin{aligned} f^{-1}(a + b) - f^{-1}(b) &= \int_b^{a+b} (f^{-1})'(x)dx \\ &< \int_0^a (f^{-1})'(x)dx \\ &= f^{-1}(a) - f^{-1}(0) \\ &= f^{-1}(a) - 1. \end{aligned}$$

Thus, $1 < f^{-1}(a+b) < f^{-1}(a) + f^{-1}(b) - 1$. Since f is strictly monotone increasing on $[1, \infty)$, we see

$$f(f^{-1}(a) + f^{-1}(b) - 1) > f(f^{-1}(a + b)) = a + b.$$

Then, it is clear that

$$\phi(a,b) = f(f^{-1}(a) + f^{-1}(b) - 1) - (a+b) > 0.$$

Case (2): Suppose $a < 0$ and $b > 0$. Under this case, if $a+b > 0$, since f^{-1} is strictly monotone increasing on $[0,\infty)$, so we have $f^{-1}(|b|) - f^{-1}(0) > f^{-1}(a+b) - f^{-1}(|a|)$. Thus,

$$f^{-1}(|a|) + f^{-1}(|b|) - 1 > f^{-1}(a+b) > 1.$$

Since f is strictly monotone increasing on $[1,\infty)$, we have

$$f(f^{-1}(|a|) + f^{-1}(|b|) - 1) > f(f^{-1}(a+b)) = a+b.$$

If $a+b \le 0$, we still have $f(f^{-1}(|a|) + f^{-1}(|b|) - 1) > a+b$ because f is positive on $(1,\infty)$ and $f^{-1}(|a|) + f^{-1}(|b|) - 1 > 1$. Thus, there holds

$$\phi(a,b) = f(f^{-1}(|a|) + f^{-1}(|b|) - 1) - (a+b) > 0.$$

Case (3): Suppose $a < 0$ and $b < 0$. Since $f^{-1}(|a|) + f^{-1}(|b|) - 1 > 1$, $f(1) = 0$, and f is strictly monotone increasing on $[1,\infty)$, $f(f^{-1}(|a|) + f^{-1}(|b|) - 1) > 0$. Thus, we have

$$\phi(a,b) = f(f^{-1}(|a|) + f^{-1}(|b|) - 1) - (a+b) > 0.$$

Case (4): Suppose $a > 0$ and $b < 0$. This case is the symmetric case of $a < 0, b > 0$. Thus, there holds

$$\phi(a,b) = f(f^{-1}(|a|) + f^{-1}(|b|) - 1) - a + b > 0.$$

Case (5): Suppose $a \ge 0,\ b = 0$ or $a = 0,\ b \ge 0$. In this case, ϕ is zero.

Case (6): Suppose $a < 0,\ b = 0$ or $a = 0,\ b < 0$. In this case, ϕ is positive.

In summary, ϕ is zero only on the nonnegative sides of a, b-axes. $\square$

By similar arguments as in Proposition 3.3, we can conclude the results when f is strictly monotone decreasing.

Proposition 3.4. *Suppose f is a continuously differentiable real valued function with $f(1) = 0$. If f satisfies the following conditions:*

(i) $f : (-\infty, 1] \to [0,\infty)$ *is invertible, and*

(ii) $(f^{-1})'$ *is strictly monotone on* $[0,\infty)$,

then $\phi(a,b) = f(f^{-1}(|a|) + f^{-1}(|b|) - f^{-1}(0)) - (a+b)$ *is an NCP function.*

Example 3.5. Here are examples of f satisfying the conditions in Proposition 3.3.

(1) $f_1(t) = \ln(t)\big|_{[1,\infty)}$.

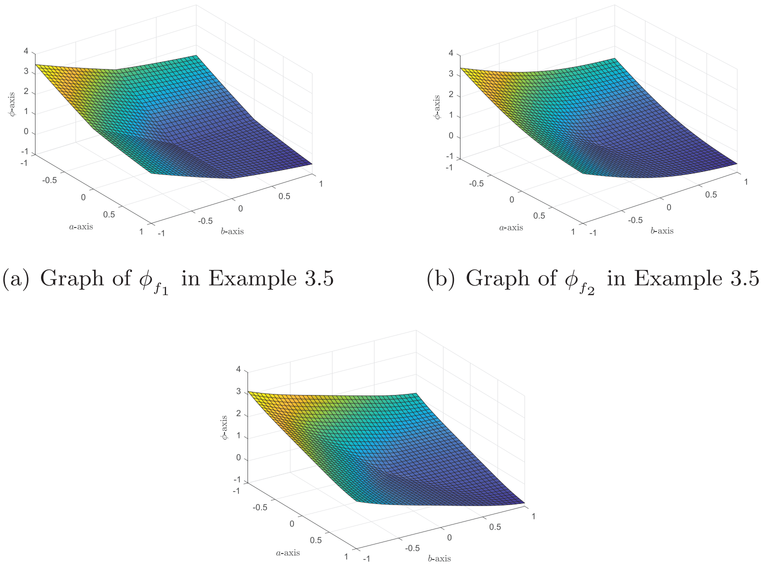
(2) $f_2(t) = (t-1)^{1/2}\big|_{[1,\infty)}$.

(3) $f_3(t) = (t-1)^{1/5}\big|_{[1,\infty)}$.

Then, their corresponding NCP functions are as below.

(1) $\phi_{f_1}(a,b) = \ln(e^{|a|} + e^{|b|} - 1) - (a+b)$.

(2) $\phi_{f_2}(a,b) = \|(a,b)\|_2 - (a+b)$.
(3) $\phi_{f_3}(a,b) = \|(a,b)\|_5 - (a+b)$.



(a) Graph of ϕ_{f_1} in Example 3.5

(b) Graph of ϕ_{f_2} in Example 3.5

(c) Graph of ϕ_{f_3} in Example 3.5

FIGURE 2. Graphs of NCP functions shown in Example 3.5

Proposition 3.3 provides the sufficient condition of f and $(f^{-1})'$ making ϕ to be an NCP function. However, it is not the necessary condition. There are some choices of f such that $(f^{-1})'$ is neither strictly monotone increasing nor decreasing. Below are two counterexamples of f providing that ϕ is also an NCP function.

Example 3.6. Let f be a real valued function defined by

$$f(t) = \begin{cases} -\sqrt{38-2t}+6, & \text{if} \quad 1 \le t \le 18.5. \\ \sqrt{2t-36}+4, & \text{if} \quad 18.5 \le t \le 20. \\ \frac{t}{2}-4, & \text{if} \quad 20 \le t. \end{cases}$$

Then, we compute that

$$f^{-1}(t) = \begin{cases} -\frac{t^2}{2}+6t+1, & \text{if} \quad 0 \le t \le 5. \\ \frac{t^2}{2}-4t+26, & \text{if} \quad 5 \le t \le 6. \\ 2t+8, & \text{if} \quad 6 \le t. \end{cases}$$

and

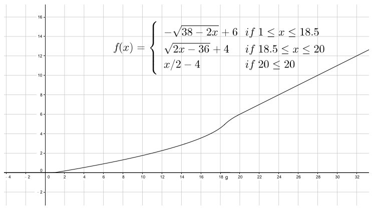
$$(f^{-1})'(t) = \begin{cases} -t+6, & \text{if} \quad 0 \le t \le 5. \\ t-4, & \text{if} \quad 5 \le t \le 6. \\ 2, & \text{if} \quad 6 \le t. \end{cases}$$

Consider $\phi(a,b) = f(f^{-1}(|a|) + f^{-1}(|b|) - f^{-1}(0)) - (a+b)$, we see that ϕ is zero on the nonnegative sides of the a,b-axes and positive on the negative sides of the

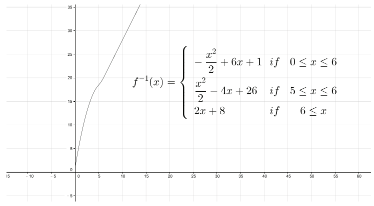
a, b-axes. In addition, by using the monotonicity of f^{-1}, ϕ is positive on the second, third, and fourth quadrant due to Proposition 3.3. Thus, we only have to check the value of ϕ on the first quadrant. From the expression of $(f^{-1})'$, we can draw a diagram of the function and easily find that $\int_0^a (f^{-1})'(t)dt > \int_b^{a+b} (f^{-1})'(t)dt$ for all $a, b > 0$. This implies

$$\begin{aligned} & f^{-1}(a) - 1 > f^{-1}(a+b) - f^{-1}(b) \\ \Longrightarrow \quad & f^{-1}(a) + f^{-1}(b) - 1 > f^{-1}(a+b) > 1 \\ \Longrightarrow \quad & f(f^{-1}(a) + f^{-1}(b) - 1) > f(f^{-1}(a+b)) = a+b \\ \Longrightarrow \quad & f(f^{-1}(a) + f^{-1}(b) - 1) - (a+b) > 0, \end{aligned}$$

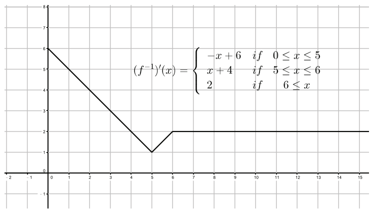
which says $\phi(a, b) > 0$ on the first quadrant. Hence, ϕ is an NCP function.



(a) Graph of f in Example 3.6



(b) Graph of f^{-1} in Example 3.6



(c) Graph of $(f^{-1})'$ in Example 3.6

FIGURE 3. Graph of f, f^{-1} and $(f^{-1})'$ in Example 3.6

Example 3.7. Let f be a real valued function defined by

$$f(t) = \begin{cases} \sqrt{2t-1} - 1, & \text{if} \quad 1 \le t \le 18.5. \\ -\sqrt{-2t+73} + 11, & \text{if} \quad 18.5 \le t \le 24. \\ \sqrt{2t-23} + 1, & \text{if} \quad 24 \le t. \end{cases}$$

Then, we compute that

$$f^{-1}(t) = \begin{cases} \frac{t^2}{2} + t + 1, & \text{if} \quad 0 \le t \le 5. \\ -\frac{t^2}{2} + 11t - 24, & \text{if} \quad 5 \le t \le 6. \\ \frac{t^2}{2} - t + 12, & \text{if} \quad 6 \le t. \end{cases}$$

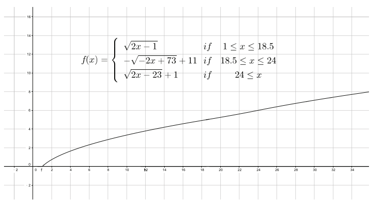
and

$$(f^{-1})'(t) = \begin{cases} t + 1, & \text{if} \quad 0 \le t \le 5. \\ -t + 11, & \text{if} \quad 5 \le t \le 6. \\ t - 1, & \text{if} \quad 6 \le t. \end{cases}$$

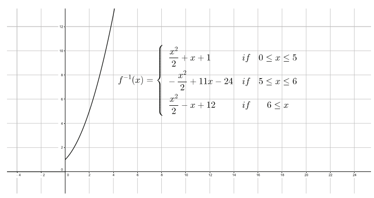
Consider $\phi(a,b) = f(f^{-1}(|a|) + f^{-1}(|b|) - f^{-1}(0)) - (a+b)$, we see that ϕ is zero on the nonnegative sides of the a,b-axes and positive on the negative sides of the a,b-axes. In addition, by using the monotonicity of f^{-1}, ϕ is positive on the second, third, and fourth quadrant due to Proposition 3.3. Thus, we only have to check the value of ϕ on the first quadrant. From the expression of $(f^{-1})'$, we can draw a diagram of the function and easily find that $\int_0^a (f^{-1})'(x)dx < \int_b^{a+b} (f^{-1})'(x)dx$ $\forall a,b > 0$. This implies

$$\begin{aligned} & f^{-1}(a) - 1 < f^{-1}(a+b) - f^{-1}(b) \\ \Longrightarrow\ & 1 < f^{-1}(a) + f^{-1}(b) - 1 < f^{-1}(a+b) \\ \Longrightarrow\ & f(f^{-1}(a) + f^{-1}(b) - 1) < f(f^{-1}(a+b)) = a+b \\ \Longrightarrow\ & f(f^{-1}(a) + f^{-1}(b) - 1) - (a+b) < 0, \end{aligned}$$

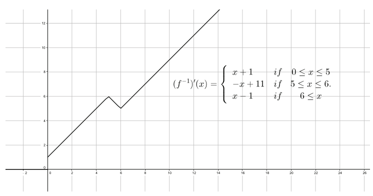
which says $\phi(a,b) < 0$ on the first quadrant. Hence, ϕ is an NCP function.



(a) Graph of f in Example 3.7

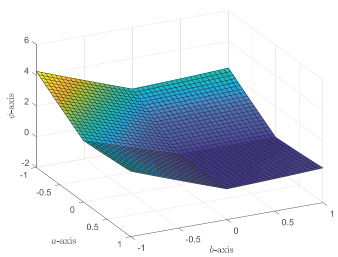


(b) Graph of f^{-1} in Example 3.7

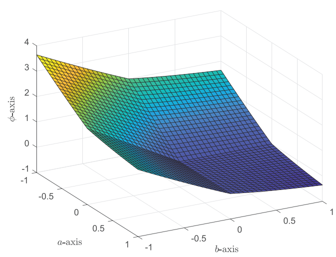


(c) Graph of $(f^{-1})'$ in Example 3.7

FIGURE 4. Graphs of f, f^{-1} and $(f^{-1})'$ in Example 3.7



(a) Graph of ϕ in Example 3.6



(b) Graph of ϕ in Example 3.7

FIGURE 5. Graphs of two ϕ functions in Example 3.6 and Example 3.7

There is a possibility to further extend the NCP function of Proposition 3.3. We find that we can add another functions satisfying some conditions in front of the terms negative a and negative b, a class of totally new NCP functions will be formulated. We elaborate it in the next proposition.

Proposition 3.8. *Suppose f is a continuously differentiable real valued function with $f(1) = 0$ and g is a real valued function with $g(0) = 1$. If f and g satisfy the following conditions:*

(i) $f : [1, \infty) \to [0, \infty)$ *is invertible;*

(ii) $(f^{-1})'$ *is strictly monotone increasing;*

(iii) $g(0) = 1$, $g(t) \geq 1\ \forall t > 0$, *and* $1 \geq g(t) > \frac{-1}{2}\ \forall t < 0$.

Then, $\phi(a, b) = f(f^{-1}(|a|)+f^{-1}(|b|)-f^{-1}(0))-(g(b)a+g(a)b)$ *is an NCP function.*

Proof. To show that ϕ is an NCP function, we have to verify that ϕ is zero only on the nonnegative sides of the a, b-axes. To this end, we check all the regions of the a, b-plane.

Case (1): Suppose $a > 0$ and $b > 0$. By Proposition 3.3, we have

$$f(f^{-1}(|a|) + f^{-1}(|b|) - 1) < a + b \leq g(b)a + g(a)b,$$

which yields

$$f(f^{-1}(|a|) + f^{-1}(|b|) - 1) - (g(b)a + g(a)b) < 0.$$

Case (2): Suppose $a < 0$ and $b > 0$. By Proposition 3.3, we have

$$f(f^{-1}(|a|) + f^{-1}(|b|) - 1) > a + b \geq g(b)a + g(a)b,$$

which implies

$$f(f^{-1}(|a|) + f^{-1}(|b|) - 1) - (g(b)a + g(a)b) > 0.$$

Case (3): Suppose $a < 0$ and $b < 0$. Since f^{-1} and $(f^{-1})'$ are both strictly monotone increasing on $[0, \infty)$, we know that f^{-1} is strictly convex on $[0, \infty)$. This indicates that

$$f^{-1}(|a|) + f^{-1}(|b|) - 1 > 2f^{-1}\left(\frac{|a| + |b|}{2}\right) - 1 > f^{-1}\left(\frac{|a| + |b|}{2}\right) > 1.$$

In addition, using f being strictly monotone increasing on $[1, \infty)$, it gives

$$\begin{aligned} f(f^{-1}(|a|) + f^{-1}(|b|) - 1) &> f\left(2f^{-1}\left(\frac{|a| + |b|}{2}\right) - 1\right) \\ &> f\left(f^{-1}\left(\frac{|a| + |b|}{2}\right)\right) = \frac{|a| + |b|}{2}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} f(f^{-1}(|a|) + f^{-1}(|b|) - 1) - g(b)a - g(a)b &> \frac{|a| + |b|}{2} - (g(b)a + g(a)b) \\ &= \frac{a}{2}(\mathrm{sgn}(a) - 2g(b)) + \frac{b}{2}(\mathrm{sgn}(b) - 2g(a)) \\ &> 0. \end{aligned}$$

Case (4): Suppose $a > 0$ and $b < 0$. This case is the symmetric case of $a < 0,\ b > 0$. Due to

$$f(f^{-1}(|a|) + f^{-1}(|b|) - 1) > a + b \geq g(b)a + g(a)b,$$

it is clear to see

$$f(f^{-1}(|a|) + f^{-1}(|b|) - 1) - (g(b)a + g(a)b) > 0.$$

Case (5): Suppose $a \geq 0, b = 0$ or $a = 0, b \geq 0$. In this case, ϕ is zero.

Case (6): Suppose $a < 0, b = 0$ or $a = 0, b < 0$. In this case, ϕ is positive.

From all the above, ϕ is zero only on the nonnegative sides of the a, b-axes. Hence, ϕ is an NCP function. $\square$

Example 3.9. Here are examples of f and g satisfying those conditions in Proposition 3.8.

(a) Three examples for function f:
  - (a) $f_1(t) = \ln(t)\big|_{[1,\infty)}$.
  - (b) $f_2(t) = (t-1)^{1/2}\big|_{[1,\infty)}$.
  - (c) $f_3(t) = (t-1)^{1/5}\big|_{[1,\infty)}$.

(b) Three examples for function g:
  - (a) $g_1(t) = e^t$
  - (b) $g_2(t) = \frac{4-e^{-t}}{1+2e^{-t}}$
  - (c) $g_3(t) = \frac{\sqrt{t^2+4}+t}{2}$

Then, applying those f and g functions in Example 3.9, we can generate the following nine NCP functions.

$$\begin{aligned}
\phi_1(a,b) &= \ln(e^{|a|} + e^{|b|} - 1) - e^b a - e^a b,\\
\phi_2(a,b) &= \ln(e^{|a|} + e^{|b|} - 1) - \frac{4-e^{-b}}{1+2e^{-b}}a - \frac{4-e^{-a}}{1+2e^{-a}}b,\\
\phi_3(a,b) &= \ln(e^{|a|} + e^{|b|} - 1) - \frac{\sqrt{b^2+4}+b}{2}a - \frac{\sqrt{a^2+4}+a}{2}b,\\
\phi_4(a,b) &= \|(a,b)\|_2 - e^b a - e^a b,\\
\phi_5(a,b) &= \|(a,b)\|_2 - \frac{4-e^{-b}}{1+2e^{-b}}a - \frac{4-e^{-a}}{1+2e^{-a}}b,\\
\phi_6(a,b) &= \|(a,b)\|_2 - \frac{\sqrt{b^2+4}+b}{2}a - \frac{\sqrt{a^2+4}+a}{2}b,\\
\phi_7(a,b) &= \|(a,b)\|_5 - e^b a - e^a b,\\
\phi_8(a,b) &= \|(a,b)\|_5 - \frac{4-e^{-b}}{1+2e^{-b}}a - \frac{4-e^{-a}}{1+2e^{-a}}b,\\
\phi_9(a,b) &= \|(a,b)\|_5 - \frac{\sqrt{b^2+4}+b}{2}a - \frac{\sqrt{a^2+4}+a}{2}b.
\end{aligned}$$

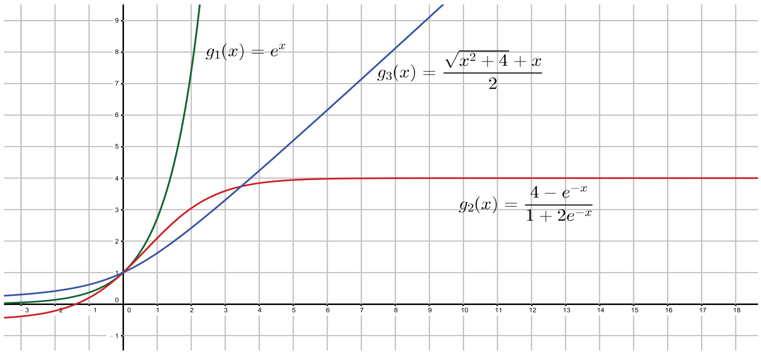
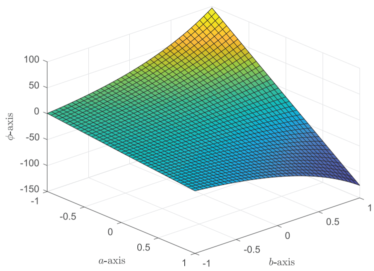
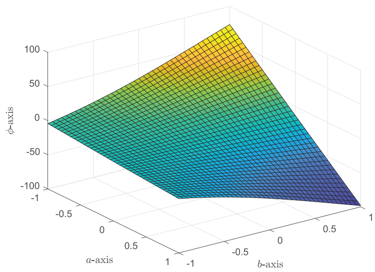


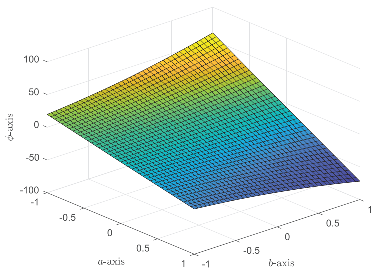
FIGURE 6. Graph of g functions given in Example 3.9



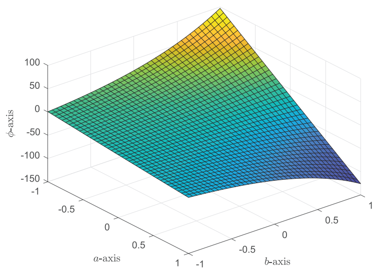
(a) Graph of ϕ_1 from Example 3.9



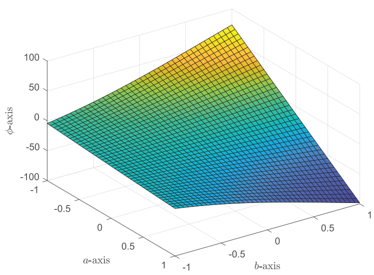
(b) Graph of ϕ_2 from Example 3.9



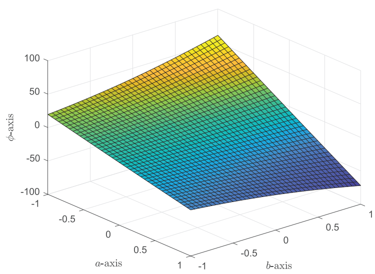
(c) Graph of ϕ_3 from Example 3.9



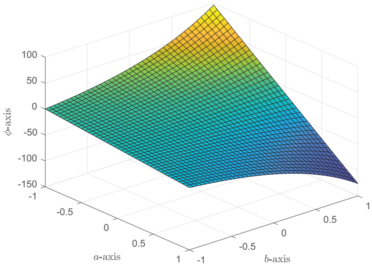
(d) Graph of ϕ_4 from Example 3.9



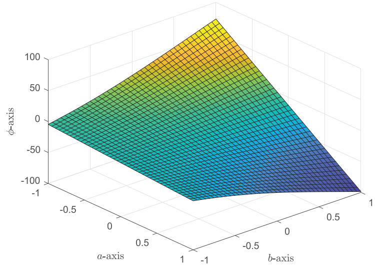
(e) Graph of ϕ_5 from Example 3.9



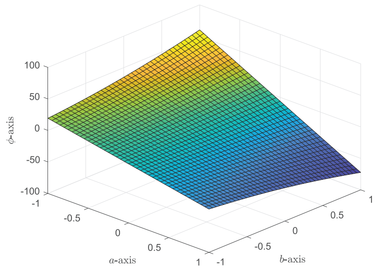
(f) Graph of ϕ_6 from Example 3.9



(g) Graph of ϕ_7 from Example 3.9



(h) Graph of ϕ_8 from Example 3.9



(i) Graph of ϕ_9 from Example 3.9

FIGURE 7. Graphs of generated NCP functions by Proposition 3.8

The next proposition is a counterpart to Proposition 3.4 whose arguments are quite similar to those for Proposition 3.8 by using the monotonicity and concavity of f^{-1} on $[0,\infty)$. We omit the proof.

Proposition 3.10. *Suppose f is a continuously differentiable real valued function with $f(1)=0$ and g is a real valued function with $g(0)=1$. If f and g satisfy the following conditions:*

(i) $f:(-\infty,1]\to[0,\infty)$ *is invertible;*
(ii) $(f^{-1})'$ *is strictly monotone decreasing;*
(iii) $g(0)=1$, $g(t)\geq 1\ \forall t>0$, *and* $1\geq g(t)>\frac{-1}{2}\ \forall t<0$.

Then, $\phi(a,b)=f(f^{-1}(|a|)+f^{-1}(|b|)-f^{-1}(0))-(g(b)a+g(a)b)$ *is an NCP function.*

In Proposition 3.8, if we use $f(t)=t-1$, then we have $f(f^{-1}(|a|)+f^{-1}(|b|)-1)=|a|+|b|$. In this case, $(f^{-1})'$ is not strictly monotone increasing. Nonetheless, we find that if the function g satisfies the strict inequality condition rather than the equality condition on $(0,\infty)$, then ϕ will be an NCP function. Thus, we take it as an extended case of Proposition 3.8. The following type of NCP functions was discovered in [2], which is connected to Proposition 3.8 in this paper.

Remark 3.11. Suppose that $\phi(a,b)=|a|+|b|-(g(a)b+g(b)a)$, where

$$g:\mathbb{R}\to\mathbb{R},\ g(0)=1,\ g(t)>1\ \forall t>0,\ \text{and } 1\geq g(t)>-1\ \forall t<0.$$

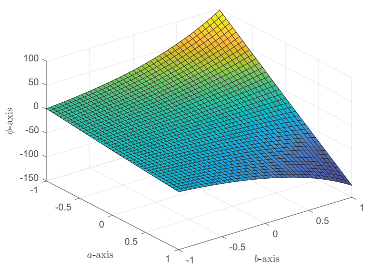
Then, ϕ is an NCP function. Note that the third condition $1 \geq g(t) > -1,\ \forall t < 0$ is a bit weaker than $1 \geq g(t) > -\frac{1}{2},\ \forall t < 0$, appeared in Proposition 3.8 and Proposition 3.10. This is because that our analysis involve both f and g, whereas the one in [2] only use g.

Example 3.12. For Remark 3.11, we point out that the same examples of g as in Example 3.9 also satisfy the conditions in Remark 3.11.

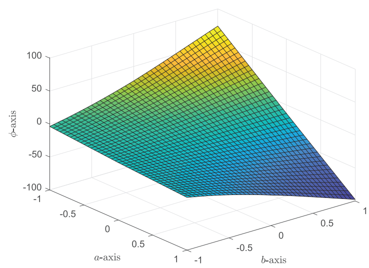
(1) $g_1(t) = e^t$.
(2) $g_2(t) = \frac{4-e^{-t}}{1+2e^{-t}}$.
(3) $g_3(t) = \frac{\sqrt{x^2+4}+t}{2}$.

Then, their corresponding NCP functions are as below.

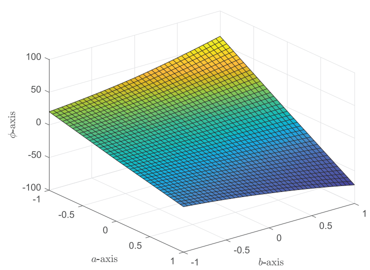
(1) $\phi_{g_1}(a,b) = |a| + |b| - e^b a - e^a b$.
(2) $\phi_{g_2}(a,b) = |a| + |b| - \frac{4-e^{-b}}{1+2e^{-b}} a - \frac{4-e^{-a}}{1+2e^{-a}} b$.
(3) $\phi_{g_3}(a,b) = |a| + |b| - \frac{\sqrt{b^2+4}+b}{2} a - \frac{\sqrt{a^2+4}+a}{2} b$.



(a) Graph of ϕ_{g_1} Example 3.12



(b) Graph of ϕ_{g_2} Example 3.12



(c) Graph of ϕ_{g_3} Example 3.12

FIGURE 8. Graphs of NCP functions generated in Example 3.12

4. Conclusion

In this paper, we present a novel idea to construct NCP functions by using certain invertible functions. This can produce a very large pool of new NCP functions. In general, the NCP functions we discover are neither convex nor differentiable on the whole domain. However, it will be interesting to study whether they are semismooth or not. Numerical side lies in clarifying what types of generated NCP functions can be employed to real algorithms. We leave them as our future research directions.

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Manuscript received 26 August 2020
revised 16 October 2020

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