



SOLUTIONS OF GENERAL VARIATIONAL INEQUALITY PROBLEMS IN BANACH SPACES

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ABSTRACT. In this paper, we introduce a new iterative process which converges strongly to a solution of general variational inequality problems for η -inverse strongly accretive mappings in the set of common fixed point of finite family of strictly pseudocontractive mappings in Banach spaces.

1. INTRODUCTION

Let E be a real normed linear space with dual E^* . Let $J_q : E \rightarrow 2^{E^*}$ be a generalized duality mapping defined by

$$(1.1) \quad J_q(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}, 1 < q < \infty,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E^* . In particular, $J = J_2$ is called the normalized duality mapping. It is well known that J_q is single-valued, if E is smooth and

$$J_q(x) = \|x\|^{q-2} J(x), x \neq 0.$$

A mapping A with domain $D(A) \subseteq E$ and range $R(A)$ in E is called α -strongly accretive if there exists an $\alpha \in (0, 1)$ and $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Ax - Ay, j_q(x - y) \rangle \geq \alpha \|x - y\|^q.$$

A is called η -inverse strongly accretive if there exists an $\eta \in (0, 1)$ and $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Ax - Ay, j_q(x - y) \rangle \geq \eta \|Ax - Ay\|^q, \forall x, y \in D(A).$$

Let C be a nonempty closed convex subset of E and $A : C \rightarrow E$ be a nonlinear mapping. The general variational inequality problem is to find $x^* \in C$ such that

$$(1.2) \quad \langle Ax^*, j(x - x^*) \rangle \geq 0, \forall x \in C, j(x - x^*) \in J(x - x^*),$$

studied by Aoyama *et al.* [7] and the set of solution of general variational inequality problems is denoted by $S(A, C)$. If $E = H$ is a real Hilbert space, the general variational inequality problem is reduced to finding $x^* \in C$ such that

$$(1.3) \quad \langle Ax^*, x - x^* \rangle \geq 0, \forall x \in C$$

which was studied by Lions and Stampacchia [20].

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In 1976, Korpelevic [17] introduced the following well-known extragradient method

$$(1.4) \quad \begin{aligned} y_n &= P_C(x_n - \gamma Ax_n) \\ x_{n+1} &= P_C(x_n - \gamma Ay_n), n \geq 0, \end{aligned}$$

where P_C is the metric projection from \mathbb{R}^n onto its subset C for some $\gamma > 0$ and $A : C \rightarrow \mathbb{R}^n$ is an accretive operator. He proved that the sequence $\{x_n\}$ converges to a solution of the variational inequality (1.3).

Yao *et al.* [34] presented the following modified Korpelevich method for solving (1.3)

$$(1.5) \quad \begin{aligned} y_n &= P_C(x_n - \gamma Ax_n - \alpha_n x_n) \\ x_{n+1} &= P_C(x_n - \gamma Ay_n + \mu(y_n - x_n)), n \geq 0. \end{aligned}$$

Aoyama *et al.* [7] introduced the iterative algorithm

$$(1.6) \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Q_C(x_n - \gamma_n Ax_n), n \geq 0,$$

where Q_C is a sunny nonexpansive retraction from E onto C and $\{\alpha_n\} \subset (0, 1)$, $\{\gamma_n\} \subset (0, \infty)$ are two real number sequences. Motivated by (1.6), Yao and Maruster [33] presented a modification of (1.6) as follows:

$$(1.7) \quad x_{n+1} = \beta_n x_n + (1 - \beta_n)Q_C((1 - \alpha_n)(x_n - \gamma_n Ax_n)), n \geq 0.$$

Motivated and inspired by the above algorithms and recent works [1, 2, 3, 4, 5, 6, 14, 15, 16, 19, 27, 37], in this paper we suggest an extragradient type method via the sunny nonexpansive retraction for solving the general variational inequality problems (1.2) in Banach spaces. It is shown that the presented algorithms converges strongly to a special solutions of the general variational inequality problems (1.2).

2. PRELIMINARIES

Let E be a real Banach space. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \rightarrow [0, \infty)$ defined by

$$\rho_E(\tau) = \sup \left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| = 1, \|y\| = \tau \right\}.$$

If $\rho_E(\tau) > 0$ for all 0 , then E is said to be smooth. If there exists a constant $c > 0$ and a real number $1 < q \leq 2$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be q -uniformly smooth. If E is a real q -uniformly smooth Banach space, then by [28] the following geometric inequality holds:

$$(2.1) \quad \|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x) \rangle + 2\|Ky\|^q, \text{ for } j_q(x) \in J_q(x),$$

for $x, y \in E$ and K is the q -uniformly smoothness constant of E and J_q satisfying the equation (1.1). It is well known that

$$L_p(l_p) \text{ or } W_m^p \text{ is } \begin{cases} p - \text{uniformly smooth if } 1 < p < 2 \\ 2 - \text{uniformly smooth if } p \geq 2. \end{cases}$$

The Banach space E is said to be uniformly convex if given $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in E$ with $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$,

$$\left\| \frac{1}{2}(x + y) \right\| \leq 1 - \delta.$$

It is well known that L_p, l_p and Sobolev spaces $W_m^p (1 < p < \infty)$ are uniformly convex.

Let $C \subseteq E$ be a closed convex and $Q : E \rightarrow C$ be a mapping. Then Q is said to be sunny if

$$Q(Q(x) + t(x - Q(x))) = Q(x).$$

Moreover $Q(x) + t(x - Q(x)) \in C$ for $x \in C$ and $t > 0$. A mapping $Q : E \rightarrow C$ is said to be a retraction if $Q^2 = Q$. If a mapping Q is a retraction, then $Q(z) = z$ for every $z \in R(Q)$. A subset C of E is said to be sunny nonexpansive retract of E if there exists a sunny nonexpansive retraction of E onto C and it is said to be a nonexpansive retract of E if there exists a nonexpansive retraction of E onto C . If $E = H$, the metric projection P_C is a sunny nonexpansive retraction from H to any closed convex subset of H . Moreover if C is a nonempty closed convex subset of an uniformly convex and uniformly smooth real Banach spaces E and T is a nonexpansive mapping of C into itself with $F(T) := \{x \in C : Tx = x\} \neq \emptyset$ (the set of all fixed points of T) then the set $F(T)$ is a sunny nonexpansive retract of C .

Lemma 2.1 ([10]). *Let E be a smooth Banach space and let K be a nonempty subset of E . Let $Q : E \rightarrow K$ be a retraction and let J be the normalized duality mapping on E . Then the following are equivalent:*

- (i) Q is sunny nonexpansive;
- (ii) $\langle x - Q(x), j(y - Q(x)) \rangle \leq 0, \forall x \in E$ and $y \in K$.

Lemma 2.2 ([7]). *Let C be a nonempty closed convex subset of a smooth Banach space E . Let Q_C be a sunny nonexpansive retraction from E onto C and let A be an accretive operator of C into E . Then for all $\gamma > 0$,*

$$S(A, C) = F(Q_C(I - \gamma A)),$$

where

$$S(A, C) = \{x^* \in C : \langle Ax^*, j(x - x^*) \rangle \geq 0, \forall x \in C\}.$$

Lemma 2.3 ([10]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n \rightarrow x$ weakly and $x_n - Tx_n \rightarrow 0$ strongly then x is a fixed point of T .*

Lemma 2.4 ([22]). *Let E be a real Banach space. Then for any given $x, y \in E$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \forall j(x + y) \in J(x + y).$$

Lemma 2.5 ([26]). *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X . Let $A : C \rightarrow X$ be an α -inverse strongly accretive mapping. Then we have*

$$\|(I - \gamma A)x - (I - \gamma A)y\|^2 \leq \|x - y\|^2 + 2\gamma(K^2\gamma - \alpha)\|Ax - Ay\|^2.$$

In Particular, if $0 \leq \gamma \leq \frac{\alpha}{K^2}$ then $I - \gamma A$ is nonexpansive.

Lemma 2.6. *Let C be a nonempty closed convex subset of a real q -uniformly smooth Banach space E for $1 < q \leq 2$. Let $T : C \rightarrow E$ be a γ -strictly pseudo-contractive mapping. Then for $0 < \mu < \mu_0 = \min\{1, \frac{\alpha}{K^2}\}$ where K is satisfying the inequality (2.1), the mapping $T_\mu(x) = (1 - \mu)x + \mu Tx$ is nonexpansive and $F(T_\mu) = F(T)$.*

Lemma 2.7. *Let C be a nonempty closed convex subset of a real q -uniformly smooth Banach space E for $1 < q \leq 2$. Let $A : C \rightarrow E$ be an η -inverse strongly accretive mapping. Then for $0 < \gamma < \frac{\alpha}{K^2}$, the mapping*

$$A_\mu x = (x - \gamma Ax)$$

is nonexpansive.

Lemma 2.8 ([21]). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{\eta_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for $i \in N$. Then there exists a nondecreasing sequence $\{m_k\} \subset N$ such that $m_k \rightarrow \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in N$:*

$$a_{m_k} \leq a_{m_k+1}, \quad a_k \leq a_{m_k+1}.$$

In fact $m_k = \max\{j \leq K : a_j < a_{j+1}\}$.

Lemma 2.9 ([29]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relations:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\delta_n, \quad n \geq n_0$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions:

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} \delta_n \leq 0.$$

Then, we have

$$\lim_{n \rightarrow \infty} a_n = 0.$$

3. MAIN RESULTS

In this section, we presented our Korpelevich like algorithm and consequently, we will show its strong convergence.

Assumption A:

- (A1) E is a uniformly convex and 2-uniformly smooth Banach space with a weakly sequentially continuous duality mapping;
- (A2) C is a nonempty closed convex subset of E ;
- (A3) $A : C \rightarrow E$ is an α -strongly accretive and L -Lipschitz continuous mapping with $S(A, C) \neq \emptyset$;
- (A4) Q_C is a sunny nonexpansive retraction from E onto C .

Parametric Restrictions:

- (P1) λ, μ and γ are three positive constant satisfying
 - (i) $\gamma \in (0, 1), \lambda \in [a, b]$ for some a, b with $0 < a < b < \frac{\alpha}{K^2 L^2}$;
 - (ii) $\frac{\lambda}{\mu} < \frac{\alpha}{K^2 L^2}$ where K is the smooth constant of E ;
- (P2) $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Algorithm 3.1. For given $x_0 \in C$ define a sequence $\{x_n\}$ iteratively by

$$(3.1) \quad \begin{aligned} y_n &= Q_C[(1 - \alpha_n)x_n - \lambda A x_n], \\ x_{n+1} &= (1 - \gamma)x_n + \gamma Q_C[x_n - \lambda A y_n + \mu(y_n - x_n)] + \gamma e_n, \end{aligned}$$

where $n = 0, 1, \dots$, and e_n is an error to take into account of a possible inexact computation of a sunny nonexpansive retraction.

Theorem 3.2. *The sequence $\{x_n\}$ generated by (3.1) converges strongly to $Q'(0)$, where Q' is a sunny nonexpansive retraction of E onto $S(A, C)$.*

Proof. Let $p \in S(A, C)$. First from Lemma 2.1, we have

$$p = Q_C[p - \lambda A p], \forall \lambda > 0.$$

In particular

$$p = Q_C[p - \lambda A p] = Q_C \left[\alpha_n p + (1 - \alpha_n) \left(p - \frac{\lambda}{1 - \alpha_n} A p \right) \right], \forall n \geq 0.$$

Since $A : C \rightarrow E$ is an α -strongly accretive and L -Lipschitzian continuous mapping, it must be $\frac{\alpha}{L^2}$ -inverse strongly accretive mapping. Thus by Lemma 2.5 we have

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + 2\lambda \left(K^2 \lambda - \frac{\alpha}{L^2} \right) \|Ax - Ay\|^2,$$

since $\alpha_n \rightarrow 0$ and $\lambda \in [a, b] \subset (0, \frac{\alpha}{K^2 L^2})$, we get $\alpha_n < 1 - \frac{K^2 L^2 \lambda}{\alpha}$ for enough large n . Without loss of generality, we may assume that for all $n \in N, \alpha_n < 1 - \frac{K^2 L^2 \lambda}{\alpha}$, that is, $\frac{\lambda}{1 - \alpha_n} \in (0, \frac{\alpha}{K^2 L^2})$. Hence $I - \frac{\lambda}{1 - \alpha_n} A$ is nonexpansive.

From (3.1) we have

$$\begin{aligned}
 \|y_n - p\| &= \left\| Q_C[(1 - \alpha_n)x_n - \lambda Ax_n] - Q_C \left[\alpha_n p + (1 - \alpha_n) \left(p - \frac{\lambda}{1 - \alpha_n} Ap \right) \right] \right\| \\
 &\leq \left\| \alpha_n(-p) + (1 - \alpha_n) \left[\left(x_n - \frac{\lambda}{1 - \alpha_n} Ax_n \right) - \left(p - \frac{\lambda}{1 - \alpha_n} Ap \right) \right] \right\| \\
 &\leq \alpha_n \|p\| + (1 - \alpha_n) \left\| \left(I - \frac{\lambda}{1 - \alpha_n} A \right) x_n - \left(I - \frac{\lambda}{1 - \alpha_n} A \right) p \right\| \\
 (3.2) \quad &\leq \alpha_n \|p\| + (1 - \alpha_n) \|x_n - p\|.
 \end{aligned}$$

By (3.1) and (3.2) we have

$$\begin{aligned}
 \|x_{n+1} - p\| &\leq (1 - \gamma) \|x_n - p\| + \gamma \|Q_C[x_n - \lambda Ay_n + \mu(y_n - x_n)] - p\| + \gamma \|e_n\| \\
 &\leq (1 - \gamma) \|x_n - p\| + \gamma \left\| Q_C \left[(1 - \mu)x_n + \mu \left(y_n - \frac{\lambda}{\mu} Ay_n \right) \right] \right. \\
 &\quad \left. - Q_C \left[(1 - \mu)p + \mu \left(p - \frac{\lambda}{\mu} Ap \right) \right] \right\| + \gamma \|e_n\| \\
 &\leq (1 - \gamma) \|x_n - p\| \\
 &\quad + \gamma \left\| (1 - \mu)(x_n - p) + \mu \left[\left(y_n - \frac{\lambda}{\mu} Ay_n \right) - \left(p - \frac{\lambda}{\mu} Ap \right) \right] \right\| + \gamma \|e_n\| \\
 &\leq (1 - \gamma) \|x_n - p\| \\
 &\quad + \gamma(1 - \mu) \|x_n - p\| + \mu\gamma \left\| \left(y_n - \frac{\lambda}{\mu} Ay_n \right) - \left(p - \frac{\lambda}{\mu} Ap \right) \right\| + \gamma \|e_n\| \\
 &\leq (1 - \mu\gamma) \|x_n - p\| + \gamma\mu \|y_n - p\| + \gamma \|e_n\| \\
 &\leq (1 - \mu\gamma) \|x_n - p\| + \gamma\mu\alpha_n \|p\| + \mu\gamma(1 - \alpha_n) \|x_n - p\| + \gamma \|e_n\| \\
 &\leq (1 - \mu\gamma\alpha_n) \|x_n - p\| + \gamma\mu\alpha_n \|p\| + \gamma \|e_n\| \\
 &\leq \max\{\|x_n - p\|, \|p\|, \|e_n\|\} \\
 &\quad \vdots \\
 (3.3) \quad &\leq \max\{\|x_0 - p\|, \|p\|, \|e_0\|\}.
 \end{aligned}$$

Hence $\{x_n\}$ is bounded.

Set $z_n = Q_C[x_n - \lambda Ay_n + \mu(y_n - x_n)]$. From (3.1) we have

$$x_{n+1} = (1 - \gamma)x_n + \gamma z_n + \gamma e_n, \quad \forall n \geq 0.$$

Then, we have

$$\begin{aligned}
& \|y_n - y_{n-1}\| \\
&= \|Q_C[(1 - \alpha_n)x_n - \lambda Ax_n] - Q_C[(1 - \alpha_{n-1})x_{n-1} - \lambda Ax_{n-1}]\| \\
&\leq \left\| (1 - \alpha_n) \left(x_n - \frac{\lambda}{1 - \alpha_n} Ax_n \right) - (1 - \alpha_{n-1}) \left(x_{n-1} - \frac{\lambda}{1 - \alpha_{n-1}} Ax_{n-1} \right) \right\| \\
&\leq (1 - \alpha_n) \left\| \left(x_n - \frac{\lambda}{1 - \alpha_n} Ax_n \right) - \left(x_{n-1} - \frac{\lambda}{1 - \alpha_n} Ax_{n-1} \right) \right\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \|x_{n-1}\| \\
&\leq (1 - \alpha_n) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|x_{n-1}\|,
\end{aligned}$$

and thus

$$\begin{aligned}
& \|z_n - z_{n-1}\| \\
&= \|Q_C[x_n - \lambda Ay_n + \mu(y_n - x_n)] - Q_C[x_{n-1} - \lambda Ay_{n-1} + \mu(y_{n-1} - x_{n-1})]\| \\
&\leq (1 - \mu) \|x_n - x_{n-1}\| + \mu \left\| \left(y_n - \frac{\lambda}{\mu} Ay_n \right) - \left(y_{n-1} - \frac{\lambda}{\mu} Ay_{n-1} \right) \right\| \\
&\leq (1 - \mu) \|x_n - x_{n-1}\| + \mu \|y_n - y_{n-1}\| \\
&\leq (1 - \mu\alpha_n) \|x_n - x_{n-1}\| + \mu |\alpha_n - \alpha_{n-1}| \|x_{n-1}\|.
\end{aligned}$$

It follows that

$$\limsup_{n \rightarrow \infty} (\|z_n - z_{n-1}\| - \|x_n - x_{n-1}\|) \leq 0.$$

This together with Lemma 2.9 implies that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.$$

From (3.2) we have

$$\begin{aligned}
& \|y_n - p\|^2 \\
&\leq \left\| \alpha_n(-p) + (1 - \alpha_n) \left[\left(x_n - \frac{\lambda}{1 - \alpha_n} Ax_n \right) - \left(p - \frac{\lambda}{1 - \alpha_n} Ap \right) \right] \right\|^2 \\
&\leq \alpha_n \|p\|^2 + (1 - \alpha_n) \left\| \left(x_n - \frac{\lambda}{1 - \alpha_n} Ax_n \right) - \left(p - \frac{\lambda}{1 - \alpha_n} Ap \right) \right\|^2 \\
(3.4) \quad &\leq \alpha_n \|p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 + 2\lambda \left(\frac{K^2 \lambda}{1 - \alpha_n} - \frac{\alpha}{L^2} \right) \|Ax_n - Ap\|^2.
\end{aligned}$$

From (3.1), (3.3) and (3.4), we obtain

$$\begin{aligned}
& \|x_{n+1} - p\|^2 \\
& \leq (1 - \gamma)\|x_n - p\|^2 \\
& \quad + \gamma \left\| (1 - \mu)(x_n - p) + \mu \left[\left(y_n - \frac{\lambda}{\mu} Ay_n \right) - \left(p - \frac{\lambda}{\mu} Ap \right) \right] \right\|^2 + \gamma \|e_n\|^2 \\
& \leq (1 - \gamma)\|x_n - p\|^2 + \gamma(1 - \mu)\|x_n - p\|^2 \\
& \quad + \mu\gamma \left\| \left(y_n - \frac{\lambda}{\mu} Ay_n \right) - \left(p - \frac{\lambda}{\mu} Ap \right) \right\|^2 + \gamma \|e_n\|^2 \\
& \leq (1 - \mu\gamma)\|x_n - p\|^2 + \gamma\mu\|y_n - p\|^2 \\
& \quad + \frac{2\lambda}{\mu} \left(\frac{K^2\lambda}{\mu} - \frac{\alpha}{L^2} \right) \|Ay_n - Ap\|^2 + \gamma \|e_n\|^2 \\
& \leq \gamma\mu \left[\alpha_n \|p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 + 2\lambda \left(\frac{K^2\lambda}{1 - \alpha_n} - \frac{\alpha}{L^2} \right) \|Ax_n - Ap\|^2 \right] \\
& \quad + (1 - \gamma\mu)\|x_n - p\|^2 + 2\lambda\gamma \left(\frac{K^2\lambda}{\mu} - \frac{\alpha}{L^2} \right) \|Ay_n - Ap\|^2 + \gamma \|e_n\|^2 \\
& \leq \gamma\mu\alpha_n\|p\|^2 + (1 - \gamma\mu\alpha_n)\|x_n - p\|^2 \\
& \quad + 2\lambda\mu\gamma \left(\frac{K^2\lambda}{1 - \alpha_n} - \frac{\alpha}{L^2} \right) \|Ax_n - Ap\|^2 \\
& \quad + 2\lambda\gamma\mu \left(\frac{K^2\lambda}{\mu} - \frac{\alpha}{L^2} \right) \|Ay_n - Ap\|^2 + \gamma \|e_n\|^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
0 & \leq -2\lambda\mu\gamma \left(\frac{K^2\lambda}{1 - \alpha_n} - \frac{\alpha}{L^2} \right) \|Ax_n - Ap\|^2 \\
& \quad - 2\lambda\gamma\mu \left(\frac{K^2\lambda}{\mu} - \frac{\alpha}{L^2} \right) \|Ay_n - Ap\|^2 \\
& \leq \alpha_n\gamma\mu\|p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \gamma\|e_n\|^2 \\
& \leq \alpha_n\gamma\mu\|p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)(\|x_n - p\| - \|x_{n+1} - p\|) \\
& \quad + \gamma\|e_n\|^2 \\
& \leq \alpha_n\gamma\mu\|p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + \gamma\|e_n\|^2.
\end{aligned}$$

Since $\alpha_n \rightarrow 0$, $\|e_n\| \rightarrow 0$ and $\|x_n - x_{n+1}\| \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} \|Ax_n - Ap\| = \lim_{n \rightarrow \infty} \|Ay_n - Ap\| = 0.$$

It follows that

$$\lim_{n \rightarrow \infty} \|Ay_n - Ax_n\| = 0.$$

Since A is α -strongly accretive, we obtain

$$\|Ay_n - Ax_n\| \geq \alpha\|y_n - x_n\|$$

which implies that

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0,$$

that is,

$$\lim_{n \rightarrow \infty} \|Q_C[(1 - \alpha_n)x_n - \lambda Ax_n] - x_n\| = 0.$$

It follows that

$$(3.5) \quad \lim_{n \rightarrow \infty} \|Q_C[x_n - \lambda Ax_n] - x_n\| = 0.$$

Now, we show that

$$(3.6) \quad \limsup_{n \rightarrow \infty} \langle Q'(0), j(x_n - Q'(0)) \rangle \geq 0.$$

To show that (3.6), since $\{x_n\}$ is bounded, we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to z such that

$$(3.7) \quad \limsup_{n \rightarrow \infty} \langle Q'(0), j(x_n - Q'(0)) \rangle = \limsup_{i \rightarrow \infty} \langle Q'(0), j(x_{n_i} - Q'(0)) \rangle.$$

We first prove that $z \in S(A, C)$. It follows that

$$(3.8) \quad \lim_{i \rightarrow \infty} \|Q_C(I - \lambda A)x_{n_i} - x_{n_i}\| = 0.$$

By Lemma 2.3 and (3.8) we have

$$z \in F(Q_C(I - \lambda A)),$$

it follows from Lemma 2.2 that $z \in S(A, C)$.

Now from (3.7) and Lemma 2.1, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle Q'(0), j(x_n - Q'(0)) \rangle &= \limsup_{i \rightarrow \infty} \langle Q'(0), j(x_{n_i} - Q'(0)) \rangle \\ &= \langle Q'(0), j(z - Q'(0)) \rangle \\ &\geq 0. \end{aligned}$$

Note that $\|x_n - y_n\| \rightarrow 0$, we deduce that

$$\limsup_{n \rightarrow \infty} \langle Q'(0), j(y_n - Q'(0)) \rangle \geq 0.$$

Since $y_n = Q_C[(I - \alpha_n)(x_n - \frac{\lambda}{1 - \alpha_n} Ax_n)]$ and

$$Q'(0) = Q_C[\alpha_n Q'(0) + (1 - \alpha_n)(Q'(0) - \frac{\lambda}{1 - \alpha_n} A Q'(0))], \forall n \geq 0,$$

we can deduce from Lemma 2.1 that

$$\begin{aligned} \left\langle Q_C \left[(1 - \alpha_n) \left(x_n - \frac{\lambda}{1 - \alpha_n} Ax_n \right) \right] \right. \\ \left. - \left[(1 - \alpha_n) \left(x_n - \frac{\lambda}{1 - \alpha_n} Ax_n \right) \right], j(y_n - Q'(0)) \right\rangle \leq 0 \end{aligned}$$

and

$$\left\langle \left[\alpha_n Q'(0) + (1 - \alpha_n) \left(Q'(0) - \frac{\lambda}{1 - \alpha_n} A Q'(0) \right) \right] - Q_C \left[\alpha_n Q'(0) + (1 - \alpha_n) \left(Q'(0) - \frac{\lambda}{1 - \alpha_n} A Q'(0) \right) \right], j(y_n - Q'(0)) \right\rangle \leq 0.$$

Therefore we have

$$\begin{aligned} & \|y_n - Q'(0)\|^2 \\ &= \left\| Q_C \left[(1 - \alpha_n) \left(x_n - \frac{\lambda}{1 - \alpha_n} A x_n \right) \right] - Q_C \left[\alpha_n Q'(0) + (1 - \alpha_n) \left(Q'(0) - \frac{\lambda}{1 - \alpha_n} A Q'(0) \right) \right] \right\|^2 \\ &\leq \left\langle \alpha_n (-Q'(0)) + (1 - \alpha_n) \left[\left(x_n - \frac{\lambda}{1 - \alpha_n} A x_n \right) - \left(Q'(0) - \frac{\lambda}{1 - \alpha_n} A Q'(0) \right) \right], j(y_n - Q'(0)) \right\rangle \\ &\leq -\alpha_n \langle Q'(0), j(y_n - Q'(0)) \rangle \\ &\quad + (1 - \alpha_n) \left\| \left(x_n - \frac{\lambda}{1 - \alpha_n} A x_n \right) - \left(Q'(0) - \frac{\lambda}{1 - \alpha_n} A Q'(0) \right) \right\| \|y_n - Q'(0)\| \\ &\leq -\alpha_n \langle Q'(0), j(y_n - Q'(0)) \rangle + (1 - \alpha_n) \|x_n - Q'(0)\| \|y_n - Q'(0)\| \\ &\leq -\alpha_n \langle Q'(0), j(y_n - Q'(0)) \rangle + \frac{1 - \alpha_n}{2} (\|x_n - Q'(0)\|^2 + \|y_n - Q'(0)\|^2), \end{aligned}$$

which implies that

$$(3.9) \quad \|y_n - Q'(0)\|^2 \leq (1 - \alpha_n) \|x_n - Q'(0)\|^2 + 2\alpha_n \langle -Q'(0), j(y_n - Q'(0)) \rangle.$$

Finally, we will prove that the sequence $x_n \rightarrow Q'(0)$. As a matter of fact from (3.1) and (3.9) we have

$$\begin{aligned} & \|x_{n+1} - Q'(0)\|^2 \leq (1 - \gamma) \|x_n - Q'(0)\|^2 \\ & \quad + \gamma \left\| (1 - \mu)(x_n - Q'(0)) + \mu \left[\left(y_n - \frac{\lambda}{\mu} A y_n \right) - \left(Q'(0) - \frac{\lambda}{\mu} A Q'(0) \right) \right] \right\|^2 + \gamma \|e_n\|^2 \\ & \leq (1 - \gamma\mu) \|x_n - Q'(0)\|^2 + \gamma\mu \left\| \left(y_n - \frac{\lambda}{\mu} A y_n \right) - \left(Q'(0) - \frac{\lambda}{\mu} A Q'(0) \right) \right\|^2 + \gamma \|e_n\|^2 \\ & \leq (1 - \gamma\mu) \|x_n - Q'(0)\|^2 + \gamma\mu \|y_n - Q'(0)\|^2 + \gamma \|e_n\|^2 \\ & \leq (1 - \gamma\mu\alpha_n) \|x_n - Q'(0)\|^2 + 2\gamma\mu\alpha_n \langle -Q'(0), j(y_n - Q'(0)) \rangle + \gamma \|e_n\|^2 \\ & \leq (1 - \gamma\mu\alpha_n) \|x_n - Q'(0)\|^2 + 2\gamma\mu\alpha_n \left\{ \langle -Q'(0), j(y_n - Q'(0)) \rangle + \frac{\|e_n\|^2}{2\mu\alpha_n} \right\}. \end{aligned}$$

Applying Lemma 2.9 to the last inequality we conclude that $\{x_n\}$ converges strongly to $Q'(0)$. This completes the proof \square

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