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THE SPLIT COMMON FIXED POINT PROBLEM BY THE SHRINKING PROJECTION METHOD FOR FAMILIES OF NEW DEMIMETRIC MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we consider the split common fixed point problem for families of mappings in Banach spaces. Using the shrinking projection method, we prove two strong convergence theorems of finding a solution of the split common fixed point problem for families of generalized demimetric mappings in Banach spaces. We also apply these results to obtain new results for the split common fixed point problem in Banach spaces.

1. Introduction

Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let η be a real number with $\eta \in (-\infty, 1)$. A mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called η-demimetric [31] if

$$2\langle x - q, J(x - Ux)\rangle \geq (1 - \eta)\|x - Ux\|^2$$

for all $x \in C$ and $q \in F(U)$, where $F(U)$ is the set of fixed points of U and J is the duality mapping on E. We have from [31] that $F(U)$ is closed and convex. This property is important. Using this, we proved weak and strong convergence theorems for demimetric mappings in Hilbert spaces and Banach spaces; see [18, 30, 31, 32, 35]. Very recently, Kawasaki and Takahashi [14] generalized the concept of demimetric mappings as follows: Let θ be a real number with $\theta \neq 0$. A mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called generalized demimetric [14] if

$$\theta\langle x - q, J(x - Ux)\rangle \geq \|x - Ux\|^2 \tag{1.1}$$

for all $x \in C$ and $q \in F(U)$. This mapping U is called θ-generalized demimetric. The set $F(U)$ is also closed and convex; see [14].

Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [9] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Byrne, Censor, Gibali and Reich [8] considered the following

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problem: Given two set-valued mappings $G : H_1 \to 2^{H_1}$ and $B : H_2 \to 2^{H_2}$, and a bounded linear operator $A : H_1 \to H_2$, the *split common null point problem* [8] is to find a point $z \in H_1$ such that

$$z \in G^{-1}0 \cap A^{-1}(B^{-1}0),$$

where $G^{-1}0$ and $B^{-1}0$ are the null point sets of G and B, respectively. Given two mappings $T : H_1 \to H_1$ and $U : H_2 \to H_2$, and a bounded linear operator $A : H_1 \to H_2$, the *split common fixed point problem* [10, 21] is to find a point $z \in H_1$ such that $z \in F(T) \cap A^{-1}F(U)$, where $F(T)$ and $F(U)$ are the fixed point sets of T and U, respectively.

Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator [3], where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1.2}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. By using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility problem, the split common null point problem and the split common fixed point problem in Hilbert spaces; see, for instance, [1, 2, 3, 8, 10, 21, 37]. However, it is difficult to prove such results outside Hilbert spaces. Recently, Takahashi [28, 29] extended the result of (1.2) to Banach spaces. Furthermore, by using the shrinking projection method [34], Takahashi [30] proved strong convergence theorems for demimetric mappings in two Banach spaces.

In this paper, motivated by these problems and results, we consider the split common fixed point problem with families of mappings in Banach spaces. Then using the shrinking projection method, we prove two strong convergence theorems for finding a solution of the split common fixed point problem with families of mappings in Banach spaces. We also apply these results to obtain new results for the split common fixed point problem with families of mappings in Banach spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot\,,\,\cdot \rangle$ and norm $\|\cdot\|$. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [26] that

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y \rangle; \tag{2.1}$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \le \|x - y\|$ for all $x \in H$ and $y \in C$. Such a mapping P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle \tag{2.4}$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [24].

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, i.e., $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$; see [11, 23].

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.5}$$

exists. In this case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [24] and [25]. We know the following result.

Lemma 2.1 ([24])**.** *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy\rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call such a mapping P_C the metric projection of E onto C.

Lemma 2.2 ([24])**.** *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = P_C x$;
(2) $\langle z - y, J(x - z)\rangle \geq 0, \quad \forall y \in C$.

Let E be a Banach space and let B be a mapping of E into 2^{E^*}. The effective domain of B is denoted by $\text{dom}(B)$, that is, $\text{dom}(B) = \{x \in E : Bx \neq \emptyset\}$. A multi-valued mapping B on E is said to be monotone if $\langle x - y, u^* - v^*\rangle \geq 0$ for all $x, y \in \text{dom}(B)$, $u^* \in Bx$, and $v^* \in By$. A monotone operator B on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [6]; see also [25, Theorem 3.5.4].

Theorem 2.3 ([6]). *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let B be a monotone operator of E into 2^{E^*}. Then B is maximal if and only if for any $r > 0$,*

$$R(J + rB) = E^*,$$

where $R(J + rB)$ is the range of $J + rB$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let B be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rBx_r.$$

This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such $J_r, r > 0$ are called the metric resolvents of B. The set of null points of B is defined by $B^{-1}0 = \{z \in E : 0 \in Bz\}$. We know that $B^{-1}0$ is closed and convex; see [25].

Let B be a maximal monotone operator on a Hilbert space H. In a Hilbert space H, the metric resolvent J_r of B is simply called the resolvent of B. It is known that the resolvent J_r of B for $r > 0$ is firmly nonexpansive, i.e.,

$$\|J_r x - J_r y\|^2 \leq \langle x - y, J_r x - J_r y\rangle, \quad \forall x, y \in H.$$

Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let θ be a real number with $\theta \neq 0$. Then a mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called generalized demimetric [14] if it satisfies (1.1), i.e.,

$$\theta\langle x - q, J(x - Ux)\rangle \geq \|x - Ux\|^2$$

for all $x \in C$ and $q \in F(U)$, where J is the duality mapping on E.

Examples We know examples of generalized demimetric mappings.

(1) Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let t be a real number with $0 \leq t < 1$. A mapping $U : C \to H$ is called a t-strict pseudo-contraction [7] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + t\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. If U is a t-strict pseudo-contraction and $F(U) \neq \emptyset$, then U is $\frac{2}{1-t}$-generalized demimetric; see [14].

(2) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called generalized hybrid [15] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1 - \alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{2.6}$$

for all $x, y \in C$. Such a mapping U is called $(\alpha,\ \beta)$-generalized hybrid. If U is generalized hybrid and $F(U) \neq \emptyset$, then U is 2-generalized demimetric. In fact, setting $x = u \in F(U)$ and $y = x \in C$ in (2.6), we have that

$$\alpha\|u - Ux\|^2 + (1 - \alpha)\|u - Ux\|^2 \leq \beta\|u - x\|^2 + (1 - \beta)\|u - x\|^2$$

and hence $\|Ux - u\|^2 \leq \|x - u\|^2$. From

$$\|Ux - u\|^2 = \|Ux - x\|^2 + \|x - u\|^2 + 2\langle Ux - x, x - u\rangle,$$

we have that

$$2\langle x - u, x - Ux\rangle \geq \|x - Ux\|^2$$

for all $x \in C$ and $u \in F(U)$. This means that U is 2-generalized demimetric. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is nonspreading [16, 17] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Ux - Uy\|^2 \leq \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [27] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Ux - Uy\|^2 \leq \|x - y\|^2 + \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [13].

(3) Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty, closed and convex subset of E. Let P_D be the metric projection of E onto D. Then P_D is 1-generalized demimetric; see [14].

(4) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let $\lambda > 0$. Then the metric resolvent J_λ is 1-generalized demimetric; see [14].

(5) Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let T be a mapping from C into H. Suppose that T is Lipschitzian, that is, there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|$$

for all $x, y \in C$. Let $S = (L + 1)I - T$. Then S is $(-2L)$-generalized demimetric; see [14, 33].

(6) Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $\alpha > 0$. If B be an α-inverse strongly monotone mapping from C into H with $B^{-1}0 \neq \emptyset$, then $T = I + B$ is $\left(-\frac{1}{\alpha}\right)$-generalized demimetric; see [14, 33].

The following lemmas are important and crucial in the proofs of our main results.

Lemma 2.4 ([14])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. If a mapping $U : C \to E$ is θ-generalized demimetric and $\theta > 0$, then U is $\left(1 - \frac{2}{\theta}\right)$-demimetric.*

Lemma 2.5 ([14]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let θ be a real number with $\theta \neq 0$. Let T be a θ-generalized demimetric mapping of C into E. Then $F(T)$ is closed and convex.*

Lemma 2.6 ([14]). *Let E be a smooth Banach space, let C be a nonempty subset of E and let θ be a real number with $\theta \neq 0$. Let T be a θ-generalized demimetric mapping from C into E and let $k \in \mathbb{R}$ with $k \neq 0$. Then $(1-k)I + kT$ is θk-generalized demimetric from C into E.*

We also know the following lemma from [35]:

Lemma 2.7 ([35]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $k \in (-\infty, 1)$ and let T be a k-demimetric mapping of C into H such that $F(T)$ is nonempty. Let λ be a real number with $0 < \lambda \leq 1-k$ and define $S = (1-\lambda)I + \lambda T$. Then S is a quasi-nonexpansive mapping of C into H.*

For a sequence $\{C_n\}$ of nonempty, closed and convex subsets of a Banach space E, define $\text{s-Li}_n C_n$ and $\text{w-Ls}_n C_n$ as follows: $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in \text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies

$$C_0 = \operatorname*{s-Li}_n C_n = \operatorname*{w-Ls}_n C_n, \tag{2.7}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [20] and we write $C_0 = \text{M-lim}_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [20]. The following lemma was proved by Tsukada [38].

Lemma 2.8 ([38]). *Let E be a uniformly convex Banach space. Let $\{C_n\}$ be a sequence of nonempty, closed and convex subsets of E. If $C_0 = M\text{-lim}_{n\to\infty} C_n$ exists and nonempty, then for each $x \in E$, $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$, where P_{C_n} and P_{C_0} are the mertic projections of E onto C_n and C_0, respectively.*

3. Main results

In this section, using the shrinking projection method introduced by Takahashi, Takeuchi and Kubota [34], we prove two strong convergence theorems for finding a solution of the split common fixed point problem with families of generalized demimetric mappings in Banach spaces. Let E be a Banach space and let C be a nonempty, closed and convex subset of E. Let $\{U_n\}$ be a sequence of mappings of C into E such that $\cap_{n=1}^{\infty} F(U_n) \neq \emptyset$. The sequence $\{U_n\}$ is said to satisfy the condition (I) [4] if for any bounded sequence $\{z_n\}$ of C such that $\lim_{n\to\infty} \|z_n - U_n z_n\| = 0$, every weak cluster point of $\{z_n\}$ belongs to $\cap_{n=1}^{\infty} F(U_n)$.

Theorem 3.1. *Let H be a Hilbert space and let F be a smooth, strictly convex and reflexive Banach space. Let J_F be the duality mapping on F. Let $\{\theta_n\}$ and $\{\tau_n\}$ be sequences of real numbers with $\theta_n, \tau_n \neq 0$ and let $\{k_n\}$ and $\{h_n\}$ be sequences of real numbers with $\theta_n k_n > 0$ and $\tau_n h_n > 0$, respectively. Let $\{S_n\}$ be a sequence* *of θ_n-generalized demimetric mappings of H to H with $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ satisfying the condition (I) and let $\{T_n\}$ be a sequence of τ_n-generalized demimetric mappings of F to F with $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ satisfying the condition (I). Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$. Suppose that*

$$\mathbf{G} := \cap_{n=1}^{\infty} F(S_n) \cap A^{-1}(\cap_{n=1}^{\infty} F(T_n)) \neq \emptyset.$$

Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 \in H$ and $C_1 = H$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = ((1-\lambda_n)I + \lambda_n S_n)\Big(x_n - r_n h_n A^* J_F(Ax_n - T_n Ax_n)\Big), \\ y_n = (1-\alpha_n)x_n + \alpha_n z_n, \\ C_{n+1} = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, c, d, e, f, \lambda_0 \in \mathbb{R}$, $\{\alpha_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ and $\{\lambda_n\}, \{k_n\}, \{h_n\} \subset \mathbb{R}$ satisfy the following: $0 < a \leq \alpha_n \leq 1,\ 0 < b \leq |h_n| \leq c$,

$$0 < d \leq r_n \leq e < f \leq \frac{2}{\tau_n h_n \|A\|^2},\ 0 < \frac{\lambda_n}{k_n} \leq \frac{2}{\theta_n k_n} \quad \text{and} \quad 0 < \lambda_0 \leq |\lambda_n|$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a point $x_0 \in \mathbf{G}$, where $x_0 = P_{\mathbf{G}} u$.

Proof. We first show that the sequence $\{x_n\}$ is well defined. It is obvious that $\mathbf{G} \subset C_1 = H$. Suppose that $\mathbf{G} \subset C_j$ for some $j \in \mathbb{N}$. To show $\mathbf{G} \subset C_{j+1}$, let us show that $\|y_j - z\| \leq \|x_j - z\|$ for all $z \in \mathbf{G}$. Since $T_n : F \to F$ is τ_n-generalized demimetric, we have from Lemma 2.6 that $(1-h_n)I + h_n T_n$ is $\tau_n h_n$-generalized demimetric. Since $S_n : H \to H$ is θ_n-generalized demimetric, we also have from Lemma 2.6 that $(1-k_n)I + k_n S_n$ is $\theta_n k_n$-generalized demimetric. Furthermore, from Lemma 2.4 and $\theta_n k_n > 0$, we have that $(1-k_n)I + k_n S_n$ is $\Big(1 - \frac{2}{\theta_n k_n}\Big)$-demimetric in the sense of [31]. Since $0 < \frac{\lambda_n}{k_n} \leq \frac{2}{\theta_n k_n} = 1 - \Big(1 - \frac{2}{\theta_n k_n}\Big)$ and

$$(1-\lambda_n)I + \lambda_n S_n = \left(1 - \frac{\lambda_n}{k_n}\right) I + \frac{\lambda_n}{k_n}((1-k_n)I + k_n S_n),$$

we have from Lemma 2.7 that $(1-\lambda_n)I + \lambda_n S_n$ is quasi-nonexpansive. Putting $s_j = x_j - r_j h_j A^* J_F(Ax_j - T_j Ax_j)$, from $0 < d \leq r_j \leq e < f \leq \frac{2}{\tau_j h_j \|A\|^2}$, we have that for $z \in \mathbf{G}$,

$$\begin{aligned} \|z_j - z\|^2 &= \|((1-\lambda_j)I + \lambda_j S_j)s_j - ((1-\lambda_j)I + \lambda_j S_j)z\|^2 \\ &\leq \|x_j - r_j h_j A^* J_F(Ax_j - T_j Ax_j) - z\|^2 \\ &= \|x_j - z\|^2 - 2\langle x_j - z, r_j h_j A^* J_F(Ax_j - T_j Ax_j)\rangle \\ &\qquad + \|r_j h_j A^* J_F(Ax_j - T_j Ax_j)\|^2 \\ &\leq \|x_j - z\|^2 - 2r_j\langle Ax_j - Az, J_F(Ax_j - ((1-h_j)I + h_j T_j)Ax_j)\rangle \qquad (3.1) \\ &\qquad + r_j^2 h_j^2 \|A\|^2 \|J_F(Ax_j - T_j Ax_j)\|^2 \\ &\leq \|x_j - z\|^2 - \frac{2r_j}{\tau_j h_j}\|Ax_j - ((1-h_j)I + h_j T_j)Ax_j\|^2 \end{aligned}$$

$$
\begin{aligned}
&+ r_j^2 h_j^2 \|A\|^2 \|Ax_j - T_jAx_j\|^2 \\
&= \|x_j - z\|^2 - \frac{2r_j}{\tau_j h_j} h_j^2 \|Ax_j - T_jAx_j\|^2 + r_j^2 h_j^2 \|A\|^2 \|Ax_j - T_jAx_j\|^2 \\
&= \|x_j - z\|^2 + r_j h_j^2 (r_j \|A\|^2 - \frac{2}{\tau_j h_j}) \|Ax_j - T_jAx_j\|^2 \\
&\le \|x_j - z\|^2
\end{aligned}
$$

and hence

$$
\begin{aligned}
\|y_j - z\| &= \|(1-\alpha_j)x_j + \alpha_j z_j - z\| \\
&\le (1-\alpha_j)\|x_j - z\| + \alpha_j \|z_j - z\| \\
&\le (1-\alpha_j)\|x_j - z\| + \alpha_j \|x_j - z\| \\
&= \|x_j - z\|.
\end{aligned}
$$

Then $\mathbf{G} \subset C_{j+1}$. We have by mathematical induction that $\mathbf{G} \subset C_n$ for all $n \in \mathbb{N}$. Moreover, since

$$
\begin{aligned}
\{z \in H : \|y_n - z\| \le \|x_n - z\|\} &= \{z \in H : \|y_n - z\|^2 \le \|x_n - z\|^2\} \\
&= \{z \in H : \|y_n\|^2 - \|x_n\|^2 \le 2\langle y_n - x_n, z\rangle\},
\end{aligned}
$$

it is closed and convex. Applying these facts inductively, we obtain that C_n are nonempty, closed, and convex for all $n \in \mathbb{N}$, and hence $\{x_n\}$ is well defined.

Let $C_0 = \bigcap_{n=1}^{\infty} C_n$. Then since $C_0 \supset \mathbf{G} \neq \emptyset$, C_0 is nonempty. Let $w_n = P_{C_n}u$ for every $n \in \mathbb{N}$. Then, by Lemma 2.8, we have $w_n \to w_0 = P_{C_0}u$. Since a metric projection on H is nonexpansive, it follows that

$$
\begin{aligned}
\|x_n - w_0\| &\le \|x_n - w_n\| + \|w_n - w_0\| \\
&= \|P_{C_n}u_n - P_{C_n}u\| + \|w_n - w_0\| \\
&\le \|u_n - u\| + \|w_n - w_0\|
\end{aligned}
$$

and hence $x_n \to w_0$.

Since $w_0 \in C_0 \subset C_{n+1}$, we have $\|y_n - w_0\| \le \|x_n - w_0\|$ for all $n \in \mathbb{N}$. Tending $n \to \infty$, we get that $y_n \to w_0$. Then we have that

$$
\|x_n - y_n\| \le \|x_n - w_0\| + \|w_0 - y_n\| \to 0. \tag{3.2}
$$

From $y_n - x_n = (1-\alpha_n)x_n + \alpha_n z_n - x_n = \alpha_n(z_n - x_n)$, we also have that

$$
\|y_n - x_n\| = \alpha_n \|z_n - x_n\| \ge a\|z_n - x_n\|
$$

and hence

$$
\|z_n - x_n\| \to 0. \tag{3.3}
$$

On the other hand, from (3.1) we know that for $z \in \mathbf{G}$,

$$
\|z_n - z\|^2 \le \|x_n - z\|^2 + r_n h_n^2 (r_n \|A\|^2 - \frac{2}{\tau_n h_n}) \|Ax_n - T_nAx_n\|^2.
$$

Then we get that

$$
\begin{aligned}
r_n h_n^2 (\frac{2}{\tau_n h_n} - r_n \|A\|^2) \|Ax_n - T_nAx_n\|^2 &\le \|x_n - z\|^2 - \|z_n - z\|^2 \\
&= (\|x_n - z\| - \|z_n - z\|)(\|x_n - z\| + \|z_n - z\|)
\end{aligned}
$$

$$\leq \|x_n - z_n\|(\|x_n - z\| + \|z_n - z\|).$$

Since $0 < d \leq r_n \leq e < f \leq \frac{2}{\tau_n h_n \|A\|^2}$ and $\|x_n - z_n\| \to 0$, we have that

$$\lim_{n\to\infty} \|Ax_n - T_n Ax_n\| = 0. \tag{3.4}$$

Since $x_n \to w_0$ and A is bounded and linear, we have that $\{Ax_n\}$ converges strongly to Aw_0 and hence $\{Ax_n\}$ converges weakly to Aw_0. Since a family $\{T_n\}$ satisfies the condition (I) and $\lim_{n\to\infty} \|Ax_n - T_n Ax_n\| = 0$, we have that $Aw_0 \in \cap_{n=1}^{\infty} F(T_n)$ and hence $w_0 \in A^{-1} \cap_{n=1}^{\infty} F(T_n)$. We show that $w_0 \in \cap_{n=1}^{\infty} F(S_n)$. Putting $s_n = x_n - r_n h_n A^* J_F(Ax_n - T_n Ax_n)$, we have that

$$\|s_n - z_n\| = \|s_n - ((1-\lambda_n)I + \lambda_n S_n)s_n\| = \|\lambda_n(s_n - S_n s_n)\| \geq \lambda_0 \|s_n - Ss_n\|.$$

Furthemore, we have that $\|s_n - x_n\| = \|r_n h_n A^* J_F(Ax_n - T_n Ax_n)\| \to 0$. We have from $\|s_n - z_n\| \leq \|s_n - x_n\| + \|x_n - z_n\|$ and (3.3) that $\|s_n - z_n\| \to 0$. This implies that

$$\lim_{n\to\infty} \|s_n - S_n s_n\| = 0. \tag{3.5}$$

Since $\|s_n - x_n\| \to 0$, we also have that $\{s_n\}$ converges strongly to w_0 and hence $\{s_n\}$ converges weakly to w_0. Since $\{S_n\}$ satisfies the condition (I), we have $w_0 \in \cap_{n=1}^{\infty} F(S_n)$. This implies that $w_0 \in \mathbf{G}$.

Since $\mathbf{G}$ is nonempty, closed and convex, there exists $z_0 \in \mathbf{G}$ such that $z_0 = P_{\mathbf{G}} u$. From $x_{n+1} = P_{C_{n+1}} u_{n+1}$, we have that

$$\|u_{n+1} - x_{n+1}\| \leq \|u_{n+1} - y\|$$

for all $y \in C_{n+1}$. Since $z_0 \in \mathbf{G} \subset C_{n+1}$, we have that

$$\|u_{n+1} - x_{n+1}\| \leq \|u_{n+1} - z_0\|. \tag{3.6}$$

From $z_0 = P_{\mathbf{G}} u$, $w_0 \in \mathbf{G}$ and (3.9), we have that

$$\begin{aligned} \|u - z_0\| \leq \|u - w_0\| &= \lim_{n\to\infty} \|u_{n+1} - x_{n+1}\| \\ &\leq \lim_{n\to\infty} \|u_{n+1} - z_0\| = \|u - z_0\|. \end{aligned}$$

Then we get that $\|u - w_0\| = \|u - z_0\|$ and hence $z_0 = w_0$. Therefore, we have $x_n \to w_0 = z_0$. This completes the proof. □

Next, using the shrinking projection method [34], we prove a strong convergence theorem of finding a solution of the split common fixed point problem for families of generalized demimetric mappings in two Banach spaces.

Theorem 3.2. *Let E and F be uniformly convex and smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let $\{\theta_n\}$ and $\{\tau_n\}$ be sequences of real numbers with $\theta_n, \tau_n \neq 0$. Let $\{S_n\}$ be a sequence of θ_n-generalized demimetric mappings of E into E satisfying the condition (I) and $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and let $\{T_n\}$ be a sequence of τ_n-generalized demimetric mappings of F into F satisfying the condition (I) and $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{k_n\}$ and $\{h_n\}$ be sequences of real numbers with $\theta_n k_n > 0$ and $\tau_n h_n > 0$, respectively. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A.*

Suppose that $\mathbf{G} := \cap_{n=1}^{\infty} F(S_n) \cap A^{-1}(\cap_{n=1}^{\infty} F(T_n)) \neq \emptyset$. *For* $x_1 \in E$ *and* $C_1 = E$, *let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = x_n - r_n h_n J_E^{-1} A^* J_F(Ax_n - T_n Ax_n), \\ y_n = ((1-k_n)I + k_n S_n) z_n, \\ C_{n+1} = \{z \in C_n : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0\} \\ \qquad\qquad \text{and} \quad \theta_n k_n \langle z_n - z, J_E(z_n - y_n)\rangle \geq \|z_n - y_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, c, d \in \mathbb{R}$, $\{r_n\} \subset (0, \infty)$ *and* $\{k_n\}, \{h_n\} \subset \mathbb{R}$ *satisfy the following:*

$$0 < a \leq |h_n|, \quad 0 < b \leq |k_n|, \quad \theta_n k_n \leq c$$

$$\text{and} \ \ 0 < d \leq r_n \leq \frac{1}{\tau_n h_n \|A\|^2}$$

for all $n \in \mathbb{N}$. *Then* $\{x_n\}$ *converges strongly to a point* $w_1 \in \mathbf{G}$, *where* $w_1 = P_{\mathbf{G}} x_1$.

Proof. Since S_n is θ_n-generalized demimetric and T_n is τ_n-generalized demimetric, from Lemma 2.5, $F(S_n)$ and $F(T_n)$ are closed and convex. Furthermore, since A is bounded and linear, $\mathbf{G}$ is also closed and convex. It also follows that C_n are closed and convex for all $n \in \mathbb{N}$. We show that $\mathbf{G} \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that $\mathbf{G} \subset C_1 = E$. Suppose that $\mathbf{G} \subset C_j$ for some $j \in \mathbb{N}$. To show $\mathbf{G} \subset C_{j+1}$, let us show that $\langle z_j - z, J_E(x_j - z_j)\rangle \geq 0$ and $\theta_n k_n \langle z_j - z, J_E(z_j - y_j)\rangle \geq \|z_j - y_j\|^2$ for all $z \in \mathbf{G}$. Since $(1-k_j)I + k_j S_j$ is $\theta_j k_j$-generalized demimetric and $(1-h_j)I + h_j T_j$ is $\tau_j h_j$-generalized demimetric, we have that, for all $z \in \mathbf{G}$,

$$\begin{aligned} \langle z_j - z,\, J_E(x_j - z_j)\rangle &= \langle z_j - x_j + x_j - z, J_E(x_j - z_j)\rangle \\ &= \langle -r_j h_j J_E^{-1} A^* J_F(Ax_j - T_j Ax_j) \\ &\qquad + x_j - z, J_E(r_j h_j J_E^{-1} A^* J_F(Ax_j - T_j Ax_j))\rangle \\ &= \langle -r_j J_E^{-1} A^* J_F(Ax_j - ((1-h_j)I + h_j T_j)Ax_j) \\ &\qquad + x_j - z, r_j A^* J_F(Ax_j - ((1-h_j)I + h_j T_j)Ax_j)\rangle \\ &\geq -r_j^2 \|A\|^2 \|Ax_j - ((1-h_j)I + h_j T_j)Ax_j\|^2 \\ &\qquad + \langle Ax_j - Az, r_j J_F(Ax_j - ((1-h_j)I + h_j T_j)Ax_j)\rangle \\ &\geq -r_j^2 \|A\|^2 \|Ax_j - ((1-h_j)I + h_j T_j)Ax_j\|^2 \\ &\qquad + \frac{r_j}{\tau_j h_j} \|Ax_j - ((1-h_j)I + h_j T_j)Ax_j\|^2 \\ &= r_j \Big(\frac{1}{\tau_j h_j} - r_j \|A\|^2\Big) \|Ax_j - ((1-h_j)I + h_j T_j)Ax_j\|^2 \\ &\geq 0 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &\theta_j k_j \langle z_j - z, J_E(z_j - y_j)\rangle - \|z_j - y_j\|^2 \\ &\quad = \theta_j k_j \langle z_j - z, J_E(z_j - ((1-k_j)I + k_j S_j)z_j)\rangle \\ &\qquad - \|z_j - ((1-k_j)I + k_j S_j)z_j\|^2 \end{aligned} \tag{3.8}$$

$$\geq \|z_j - ((1-k_j)I + k_j S_j)z_j\|^2 - \|z_j - ((1-k_j)I + k_j S_j)z_j\|^2$$
$$= 0.$$

Then $\mathbf{G} \subset C_{j+1}$. We have by mathematical induction that $\mathbf{G} \subset C_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since $\mathbf{G}$ is nonempty, closed and convex, there exists $w_1 \in \mathbf{G}$ such that $w_1 = P_{\mathbf{G}} x_1$. From $x_n = P_{C_n} x_1$, we have that

$$\|x_1 - x_n\| \leq \|x_1 - y\|$$

for all $y \in C_n$. Since $w_1 \in \mathbf{G} \subset C_n$, we have that

$$\|x_1 - x_n\| \leq \|x_1 - w_1\|. \tag{3.9}$$

Let $C_0 = \bigcap_{n=1}^{\infty} C_n$. Since $C_0 \supset \mathbf{G} \neq \emptyset$, we have that C_0 is nonempty. Since $C_0 = \text{M-lim}_{n\to\infty} C_n$ and $x_n = P_{C_n} x_1$ for every $n \in \mathbb{N}$, by Lemma 2.8 we have that

$$x_n \to z_0 = P_{C_0} x_1. \tag{3.10}$$

We have from $x_{n+1} \in C_{n+1}$ that

$$\langle z_n - x_{n+1}, J_E(x_n - z_n)\rangle \geq 0$$

and hence

$$\langle z_n - x_n + x_n - x_{n+1}, J_E(x_n - z_n)\rangle \geq 0.$$

This implies that

$$\langle x_n - x_{n+1}, J_E(x_n - z_n)\rangle \geq \|z_n - x_n\|^2.$$

Since $\|x_n - x_{n+1}\| \to 0$ from (3.10), we get that $x_n - z_n \to 0$.

On the other hand, we know that

$$\|x_n - z_n\| = \|J_E(x_n - z_n)\| = \|r_n A^* J_F(Ax_n - ((1-h_n)I + h_n T_n)Ax_n)\|$$

and

$$\begin{aligned}
\tau_n h_n \langle x_n - z, A^* J_F(Ax_n - ((1-h_n)I + h_n T_n)Ax_n)\rangle \\
&= \tau_n h_n \langle Ax_n - Az, J_F(Ax_n - ((1-h_n)I + h_n T_n)Ax_n)\rangle \\
&\geq \|Ax_n - ((1-h_n)I + h_n T_n)Ax_n)\|^2 \\
&= h_n^2 \|Ax_n - T_n Ax_n)\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - z_n\| = 0$, we have that

$$\lim_{n\to\infty} \|A^* J_F(Ax_n - ((1-h_n)I + h_n T_n)Ax_n)\| = 0.$$

Then we get from $h_n^2 \geq a^2 > 0$ that

$$\lim_{n\to\infty} \|Ax_n - T_n Ax_n\| = 0. \tag{3.11}$$

Furthermore, we have from $x_{n+1} \in C_{n+1}$ that

$$\theta_n k_n \langle z_n - x_{n+1}, J_E(z_n - y_n)\rangle \geq \|z_n - y_n\|^2$$

and hence

$$\theta_n k_n \langle z_n - x_n + x_n - x_{n+1}, J_E(z_n - y_n)\rangle \geq \|z_n - y_n\|^2.$$

From $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - z_n\| \to 0$, we have that $\lim_{n\to\infty} \|y_n - z_n\| = 0$. Since $\|y_n - z_n\| = |k_n|\|z_n - S_n z_n\| \geq b\|z_n - S_n z_n\|$, we get that

$$\lim_{n\to\infty} \|z_n - S_n z_n\| = 0. \tag{3.12}$$

Since $x_n \to z_0$ and $\{S_n\}$ satisfies the condition (I), we have from (3.12) that $z_0 \in \cap_{n=1}^{\infty} F(S_n)$. Furthermore, since A is bounded and linear, we have that $\{Ax_n\}$ converges strongly to Az_0. Since $\{T_n\}$ satisfies the condition (I), we have $Az_0 \in \cap_{n=1}^{\infty} F(T_n)$. Therefore, $z_0 \in \mathbf{G}$.

From $w_1 = P_{\mathbf{G}} x_1$, $z_0 \in \mathbf{G}$ and (3.9), we have that

$$\|x_1 - w_1\| \leq \|x_1 - z_0\| = \lim_{n\to\infty} \|x_1 - x_n\| \leq \|x_1 - w_1\|.$$

Then we get that $\|x_1 - w_1\| = \|x_1 - z_0\|$ and hence $z_0 = w_1$. Therefore, we have $x_n \to z_0 = w_1$. This completes the proof. $\square$

4. Applications

In this section, using Theorems 3.1 and 3.2, we get new strong convergence theorems which are connected with the split common fixed point problem with families of generalized demimetric mappings in Hilbert spaces and Banach spaces. We know the following result obtained by Marino and Xu [19]; see also [36].

Lemma 4.1 ([19, 36]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \leq k < 1$ and let $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Using Lemma 4.1, we obtain the following result.

Lemma 4.2. *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $s, t \in [0, 1)$. Let $S, T : C \to H$ be s, t-strict pseudo-contractionsuchs, respectively, such that $F(S) \cap F(T) \neq \emptyset$. Let $S_1 = sI + (1-s)S$ and $T_1 = tI + (1-t)T$ and let $\{\gamma_n\}$ be a sequence of real numbers. Assume that there exist $a, b \in \mathbb{R}$ such that $0 < a \leq \gamma_n \leq b < 1$ for all $n \in \mathbb{N}$. If $T_n = \gamma_n S_1 + (1 - \gamma_n)T_1$ for all $n \in \mathbb{N}$, then $\cap_{n=1}^{\infty} F(T_n) = F(S) \cap F(T)$ and $\{T_n\}$ satisfies the condition (I).*

Proof. Since S and T are s, t-strict pseudo-contractions and $F(S) \cap F(T) \neq \emptyset$, $S_1 = sI + (1-s)S$ and $T_1 = tI + (1-t)T$ are quasi-nonexpansive mappings. Using this, we have from (2.2) that, for $z_0 \in F(S) \cap F(T)$, $z \in \cap_{n=1}^{\infty} F(T_n)$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\|z - z_0\|^2 &= \|T_n z - z_0\|^2 \\
&= \|(\gamma_n S_1 + (1-\gamma_n)T_1)z - z_0\|^2 \\
&= \|\gamma_n(S_1 z - z_0) + (1-\gamma_n)(T_1 z - z_0)\|^2 \\
&= \gamma_n\|S_1 z - z_0\|^2 + (1-\gamma_n)\|T_1 z - z_0\|^2 - \gamma_n(1-\gamma_n)\|S_1 z - T_1 z\|^2 \\
&\leq \gamma_n\|z - z_0\|^2 + (1-\gamma_n)\|z - z_0\|^2 - \gamma_n(1-\gamma_n)\|S_1 z - T_1 z\|^2 \\
&= \|z - z_0\|^2 - \gamma_n(1-\gamma_n)\|S_1 z - T_1 z\|^2.
\end{aligned}$$

This means that $\gamma_n(1-\gamma_n)\|S_1 z - T_1 z\|^2 \leq 0$. Since $0 < a \leq \gamma_n \leq b < 1$ for all $n \in \mathbb{N}$, we have $S_1 z = T_1 z$. From

$$\|S_1 z - z\| = \|\gamma_n S_1 z + (1-\gamma_n)S_1 z - z\|$$

$$= \|(\gamma_n S_1 + (1-\gamma_n)T_1)z - z\|$$
$$= \|z - z\| = 0,$$

we have that $S_1 z = z$ and hence $Sz = z$. Similarly, we have that $Tz = z$. This implies that $\cap_{n=1}^{\infty} F(T_n) \subset F(S) \cap F(T)$. It is obvious that $F(S) \cap F(T) \subset \cap_{n=1}^{\infty} F(T_n)$. Thus $\cap_{n=1}^{\infty} F(T_n) = F(S) \cap F(T)$.

Suppose that $\{z_n\}$ is a bounded sequence and $z_n - T_n z_n \to 0$. Then we have from (2.1) and (2.2) that, for $z \in \cap_{n=1}^{\infty} F(T_n)$,

$$\begin{aligned}
\|z_n - z\|^2 &= \|z_n - T_n z_n + T_n z_n - z\|^2 \\
&\leq \|T_n z_n - z\|^2 + 2\langle z_n - T_n z_n, z_n - z\rangle \\
&= \|\gamma_n S_1 z_n + (1-\gamma_n)T_1 z_n - z\|^2 + 2\langle z_n - T_n z_n, z_n - z\rangle \\
&= \gamma_n \|S_1 z_n - z\|^2 + (1-\gamma_n)\|T_1 z_n - z\|^2 \\
&\quad - \gamma_n(1-\gamma_n)\|S_1 z_n - T_1 z_n\|^2 + 2\langle z_n - T_n z_n, z_n - z\rangle \\
&\leq \gamma_n \|z_n - z\|^2 + (1-\gamma_n)\|z_n - z\|^2 \\
&\quad - \gamma_n(1-\gamma_n)\|S_1 z_n - T_1 z_n\|^2 + 2\langle z_n - T_n z_n, z_n - z\rangle \\
&= \|z_n - z\|^2 - \gamma_n(1-\gamma_n)\|S_1 z_n - T_1 z_n\|^2 + 2\langle z_n - T_n z_n, z_n - z\rangle
\end{aligned}$$

and hence

$$\gamma_n(1-\gamma_n)\|S_1 z_n - T_1 z_n\|^2 \leq 2\langle z_n - T_n z_n, z_n - z\rangle.$$

Since $z_n - T_n z_n \to 0$ and $\{z_n\}$ is bounded, we have that $S_1 z_n - T_1 z_n \to 0$. Using this, we have that

$$\begin{aligned}
(1-s)\|z_n - S z_n\| &= \|z_n - S_1 z_n\| \\
&= \|z_n - T_n z_n + T_n z_n - S_1 z_n\| \\
&\leq \|z_n - T_n z_n\| + \|T_n z - S_1 z_n\| \\
&= \|z_n - T_n z_n\| + (1-\gamma_n)\|T_1 z_n - S_1 z_n\| \\
&\to 0.
\end{aligned}$$

If a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ converges weakly to w, then we have from Lemma 4.1 and $z_n - S z_n \to 0$ that $w \in F(S)$. Similarly, $w \in F(T)$. Thus every weak cluster point $\{z_n\}$ belongs to $F(S) \cap F(T) = \cap_{n=1}^{\infty} F(T_n)$. This completes the proof. □

Using Theorem 3.1, we get the following strong convergence theorems in Hilbert spaces and Banach spaces.

Theorem 4.3. *Let H_1 and H_2 be Hilbert spaces and let $s, t \in [0,1)$. Let $S, T : C \to H$ be s, t-strict pseudo-contractions, respectively, such that $F(S) \cap F(T) \neq \emptyset$ and let $U, V : H_2 \to H_2$ be nonexpansive mappings with $F(U) \cap F(V) \neq \emptyset$. Let $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences of real numbers. Assume that there exists $s, t, u, v \in \mathbb{R}$ such that $0 < s \leq \gamma_n \leq t < 1$ and $0 < u \leq \delta_n \leq v < 1$ for all $n \in \mathbb{N}$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Suppose that $\mathbf{G} := F(S) \cap F(T) \cap A^{-1}(F(U) \cap F(V)) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that*

$u_n \to u$. For $x_1 \in H_1$ and $C_1 = H_1$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} s_n = x_n - r_n A^*(Ax_n - (\delta_n U + (1-\delta_n)V)Ax_n), \\ z_n = (\gamma_n(sI + (1-s)S) + (1-\gamma_n)(tI + (1-t)T))s_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n z_n, \\ C_{n+1} = \{z \in H_1 : \|y_n - z\| \le \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, d, e \in \mathbb{R}$, $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy the following:

$$0 < a \le \alpha_n \le 1 \quad and \quad 0 < d \le r_n \le e < \frac{1}{\|A\|^2}$$

for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to a point $x_0 \in \mathbf{G}$, where $x_0 = P_{\mathbf{G}} u$.

Proof. Since S and T are s, t-strict pseudo-contractions of H_1 into H_1, respectively, $S_n = \gamma_n(sI + (1-s)S) + (1-\gamma_n)(tI + (1-t)T)$ is a nonexpansive mapping. Since U and V are nonexpansive mappings of H_2 into H_2, $T_n = \delta_n U + (1-\delta_n)V$ is a nonexpansive mapping. From $F(S) \cap F(T) \ne \emptyset$ and $F(U) \cap F(V) \ne \emptyset$, $S_n = \gamma_n(sI + (1-s)S) + (1-\gamma_n)(tI + (1-t)T)$ and $T_n = \delta_n U + (1-\delta)V$ are quasi-nonexpansive mappings and hence they are 2-generalized demimetric mappings. Furthermore, $\{S_n\}$ and $\{T_n\}$ satisfy the condition (I) from Lemma 4.2. Putting $k_n = 1$, $h_n = 1$ and $\lambda_n = 1$ in Theorem 3.1, we obtain the desired result from Theorem 3.1. □

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A family $\mathbf{S} = \{T(t) : t \in [0,\infty)\}$ of mappings of C into itself satisfying the following conditions is said to be a one-parameter nonexpansive semigroup on C:

(1) For each $t \in [0,\infty)$, $T(t)$ is nonexpansive;
(2) $T(0) = I$;
(3) $T(t+s) = T(t)T(s)$ for every $t, s \in [0,\infty)$;
(4) for each $x \in C$, $t \mapsto T(t)x$ is continuous.

Theorem 4.4. *Let H_1 and H_2 be Hilbert spaces. Let $S, T : H_1 \to H_1$ be commutative generalized hybrid mappings with $F(S) \cap F(T) \ne \emptyset$ and define*

$$S_n = \frac{1}{(1+n)^2} \sum_{k=0}^{n} \sum_{i=0}^{n} S^k T^i$$

for all $n \in \mathbb{N}$. Let $\mathbf{S} = \{T(t) : t \in [0,\infty)\}$ be a one-parameter nonexpansive semigroup on H_2 with the common fixed point set $F(\mathbf{S}) = \cap_{t\in[0,\infty)} F(T(t)) \ne \emptyset$. Define $T_n x = \frac{1}{t_n}\int_0^{t_n} T(s)x ds$ for all $x \in H_2$ and $n \in \mathbb{N}$ with $t_n \to \infty$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \ne 0$. Suppose that $\mathbf{G} := F(S) \cap F(T) \cap A^{-1}F(\mathbf{S}) \ne \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For

$x_1 \in H_1$ and $C_1 = H_1$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = S_n\Big(x_n - r_n A^*(Ax_n - T_n A x_n)\Big), \\ y_n = (1-\alpha_n)x_n + \alpha_n z_n, \\ C_{n+1} = \{z \in H_1 : \|y_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, d, e \in \mathbb{R}$ and $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy the following:

$$0 < a \leq \alpha_n \leq 1 \quad \text{and} \quad 0 < d \leq r_n \leq e < \frac{1}{\|A\|^2}$$

for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to a point $x_0 \in \mathbf{G}$, where $x_0 = P_{\mathbf{G}} u$.

Proof. Since S and T are generalized hybrid and $F(S) \cap F(T) \neq \emptyset$, S_n is quasi-nonexpansive and $\cap_{n=0}^{\infty} F(S_n) = F(S) \cap F(T)$. We also have from [12, Lemma 3.1] that $\{S_n\}$ satisfies the condition (I). Furthermore, since T_n is a nonexpansive mapping of H_2 into itself, from (1) or (2) in Examples, T_n is 2-generalized demimetric. We also know from [24] that $\cap_{n=1}^{\infty} F(T_n) = F(\mathbf{S})$. Furthermore, let $\{z_n\}$ be a bounded sequence of H_2 such that $z_n - T_n z_n \to 0$. Then we have from [22] that $z_n - T(s)z_n \to 0$ for all $s \in [0,\infty)$. Sinve $T(s)$ is nonexpansive, every weak cluster point of $\{z_n\}$ belongs to $F(T(s))$; see [26]. Then, every weak cluster point of $\{z_n\}$ belongs to $\cap_{n=1}^{\infty} F(T_n) = F(\mathbf{S})$. This means that the family $\{T_n\}$ satisfies the condition (I). Putting $k_n = 1$, $h_n = 1$ and $\lambda_n = 1$ in Theorem 3.1, we obtain the desired result from Theorem 3.1. □

Using Theorem 3.1, we also have the following theorem for the split common null point problem in Banach spaces; see also Hojo and Takahashi [12].

Theorem 4.5. *Let H be a Hilbert space and let F be a uniformly convex and smooth Banach space. Let J_F be the duality mapping on F. Let G and B be maximal monotone operators of H and F, respectively. Let J_s and Q_t be the metric resolvents of G for $s > 0$ and B for $t > 0$, respectively. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $G^{-1}0 \cap A^{-1}(B^{-1}0) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 \in H$ and $C_1 = H$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_{s_n}\big(x_n - r_n A^* J_F(Ax_n - Q_{t_n} A x_n)\big), \\ y_n = (1-\alpha_n)x_n + \alpha_n z_n, \\ C_{n+1} = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, c, d, e \in \mathbb{R}$, $\{\alpha_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ and $\{s_n\}, \{t_n\} \subset (0,\infty)$ satisfy the following:

$$0 < a \leq \alpha_n \leq 1, \;\; s_n \geq b > 0, \;\; t_n \geq c > 0 \quad \text{and} \quad 0 < d \leq r_n\|A\|^2 \leq e < 2$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in G^{-1}0 \cap A^{-1}(B^{-1}0)$, where $z_0 = P_{G^{-1}0 \cap A^{-1}(B^{-1}0)} x_1$.

Proof. Since Q_{t_n} is the metric resolvent of B for $t_n > 0$, from (4) in Examples, Q_{t_n} is 1-generalized demimetric. We also have that if $\{z_n\}$ is a bounded sequence in F such that $z_n - Q_{t_n} z_n \to 0$, then every weak cluster point of $\{z_n\}$ belongs to $B^{-1}0 = \cap_{n=1}^{\infty} F(Q_{t_n})$. In fact, suppose that $\{z_{n_i}\}$ is a subsequence of $\{z_n\}$ such that $z_{n_i} \rightharpoonup p$. Since Q_{t_n} is the metric resolvent of B, we have that

$$J_F(z_n - Q_{t_n} z_n)/t_n \in BQ_{t_n} z_n$$

for all $n \in \mathbb{N}$; see [5, 25]. From the monotonicity of B, we have

$$0 \leq \Big\langle u - Q_{t_{n_i}} z_{n_i}, v^* - \frac{J_F(z_{n_i} - Q_{t_{n_i}} z_{n_i})}{t_{n_i}} \Big\rangle$$

for all $(u, v^*) \in B$ and $i \in \mathbb{N}$. Taking $i \to \infty$, we get that $\langle u - p, v^* \rangle \geq 0$ for all $(u, v^*) \in B$. Since B is a maximal monotone operator, we have

$$p \in B^{-1}0 = \cap_{n=1}^{\infty} F(Q_{t_n}).$$

This means that the family $\{Q_{t_n}\}$ satisfies the condition (I). On the other hand, since J_{s_n} is the metric resolvent (the resolvent) of G on a Hilbert space H, it is 1-generalized demimetric. Furthermore, as in the proof of $\{Q_{t_n}\}$, $\{J_{s_n}\}$ satisfies the condition (I). Therefore, we have the desired result from Theorem 3.1. □

Similarly, using Theorem 3.2, we have the following results.

Theorem 4.6. *Let H_1 and H_2 be Hilbert spaces and let $s, t \in [0, 1)$. Let $S, T : C \to H$ be s, t-strict pseudo-contractionsuchs, respectively, such that $F(S) \cap F(T) \neq \emptyset$ and let $U, V : H_2 \to H_2$ be nonexpansive mappings with $F(U) \cap F(V) \neq \emptyset$. Let $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences of real numbers. Assume that there exists $s, t, u, v \in \mathbb{R}$ such that $0 < s \leq \gamma_n \leq t < 1$ and $0 < u \leq \delta_n \leq v < 1$ for all $n \in \mathbb{N}$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Suppose that $\mathbf{G} := F(S) \cap F(T) \cap A^{-1}(F(U) \cap F(V)) \neq \emptyset$. For $x_1 \in H_1$ and $C_1 = H_1$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = x_n - r_n A^*(Ax_n - (\delta_n U + (1 - \delta_n)V)Ax_n), \\ y_n = (\gamma_n(sI + (1 - s)S) + (1 - \gamma_n)(tI + (1 - t)T))z_n, \\ C_{n+1} = \{z \in C_n : \langle z_n - z, x_n - z_n \rangle \geq 0 \\ \qquad\qquad \text{and} \quad 2\langle z_n - z, z_n - y_n \rangle \geq \|z_n - y_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $d \in \mathbb{R}$ and $\{r_n\} \subset (0, \infty)$ satisfy the following:

$$0 < d \leq r_n \leq \frac{1}{\|A\|^2}$$

for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to a point $w_1 \in \mathbf{G}$, where $w_1 = P_{\mathbf{G}} x_1$.

Theorem 4.7. *Let H_1 and H_2 be Hilbert spaces. Let $S, T : H_1 \to H_1$ be commutative generalized hybrid mappings with $F(S) \cap F(T) \neq \emptyset$ and define*

$$S_n = \frac{1}{(1+n)^2} \sum_{k=0}^{n} \sum_{i=0}^{n} S^k T^i$$

for all $n \in \mathbb{N}$. *Let* $\mathbf{S} = \{T(t) : t \in [0, \infty)\}$ *be a one-parameter nonexpansive semigroup on* H_2 *with the common fixed point set* $F(\mathbf{S}) = \cap_{t \in [0,\infty)} F(T(t)) \neq \emptyset$. *Define* $T_n x = \frac{1}{t_n} \int_0^{t_n} T(s) x ds$ *for all* $x \in H_2$ *and* $n \in \mathbb{N}$ *with* $t_n \to \infty$. *Let* $A : H_1 \to H_2$ *be a bounded linear operator such that* $A \neq 0$. *Suppose that* $\mathbf{G} := F(S) \cap F(T) \cap A^{-1} F(\mathbf{S}) \neq \emptyset$. *For* $x_1 \in E$ *and* $C_1 = E$, *let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = x_n - r_n A^*(Ax_n - T_n A x_n), \\ y_n = S_n z_n, \\ C_{n+1} = \{z \in C_n : \langle z_n - z, x_n - z_n \rangle \geq 0 \\ \qquad\qquad\qquad and \quad 2\langle z_n - z, z_n - y_n \rangle \geq \|z_n - y_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $d \in \mathbb{R}$ *and* $\{r_n\} \subset (0, \infty)$ *satisfy the following:*

$$0 < d \leq r_n \leq \frac{1}{\|A\|^2}$$

for all $n \in \mathbb{N}$. *Then the sequence* $\{x_n\}$ *converges strongly to a point* $w_1 \in \mathbf{G}$, *where* $w_1 = P_{\mathbf{G}} x_1$.

Using Theorem 3.2, we also have the following theorem for the split common null point problem in two Banach spaces;

Theorem 4.8. *Let* E *and* F *be uniformly convex and smooth Banach spaces and let* J_E *and* J_F *be the duality mappings on* E *and* F, *respectively. Let* G *and* B *be maximal monotone operators of* E *into* E^* *and* F *into* F^*, *respectively. Let* J_s *and* Q_t *be the metric resolvents of* G *for* $s > 0$ *and* B *for* $t > 0$, *respectively. Let* $A : E \to F$ *be a bounded linear operator such that* $A \neq 0$ *and let* A^* *be the adjoint operator of* A. *Suppose that* $\mathbf{G} := G^{-1}0 \cap A^{-1}(B^{-1}0) \neq \emptyset$. *For* $x_1 \in E$ *and* $C_1 = E$, *let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = x_n - r_n J_E^{-1} A^* J_F(Ax_n - Q_{t_n} A x_n), \\ y_n = J_{s_n} z_n, \\ C_{n+1} = \{z \in C_n : \langle z_n - z, J_E(x_n - z_n) \rangle \geq 0\} \\ \qquad\qquad\qquad and \quad \langle z_n - z, J_E(z_n - y_n) \rangle \geq \|z_n - y_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $b, c, d \in \mathbb{R}$ *and* $\{r_n\} \subset (0, \infty)$ *satisfy the following:*

$$s_n \geq b > 0, \ t_n \geq c > 0 \ \ and \ \ 0 < d \leq r_n \leq \frac{2}{\|A\|^2}$$

for all $n \in \mathbb{N}$. *Then the sequence* $\{x_n\}$ *converges strongly to a point* $w_1 \in \mathbf{G}$, *where* $w_1 = P_{\mathbf{G}} x_1$.

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