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A NEW RESOLVENT OPERATOR APPROACH FOR SOLVING A GENERAL VARIATIONAL INCLUSION PROBLEM INVOLVING XOR OPERATION WITH CONVERGENCE AND STABILITY ANALYSIS

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Abstract. This work is focused on the construction of a new resolvent operator using XOR operation and it is shown that it is single-valued, comparison and Lipschitz continuous. We apply this new resolvent operator approach to solve a general variational inclusion problem which also involves XOR operation. Finally, we prove some results for existence of solution, convergence of iterative sequences generated by Ishikawa type iterative algorithm and stability analysis for general variational inclusion problem.

1. Introduction

The term "resolvent" was a term coined by Fredholm in the 19^{th} century when he initiated a study of integral equations arising from the study of partial differential equations. The resolvent captures the spectral properties of an operator in the analytic structure of the functional. The resolvent operator technique is important to study the existence of solution and to develop iterative procedures for several types of variational inequalities and their generalizations.

To study wide class of nonlinear problems arising in many diverse fields of pure mathematics, applied and basic sciences, the techniques based on variational inequalities theory are very effective, see [8, 9, 10, 14, 18]. A useful and important generalization of variational inequalities is a variational inclusion problem introduced and studied by Hassouni and Moudafi [19], which is applicable to solve many problems related to optimization and control, nonlinear programming, engineering, elasticity theory, economics and game theory etc., see [1, 2, 4, 5, 7, 11, 13].

Further, Adly [1], Chang [8], Ahmad and Ansari [4], Fang and Huang [15, 16], Chang et al. [9], Fang et al. [17] and others studied the properties of many kinds of resolvent operators related to generalized variational inequalities (inclusions) etc..

Many problems related to ordered variational inequalities and ordered equations were studied by H-G Li together with his co-authors, see [24, 25]. In a slight different direction some work is done by I. Ahmad et al., see [3, 6].

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Motivated by the applications of above mentioned works, in this paper, we introduced a new resolvent operator involving XOR operation and prove some of its properties. Using this approach, we solve a variational inclusion problem involving XOR operation. We prove an existence result, a convergence result as well as a result for stability analysis.

2. PRELIMINARIES

Throughout this paper, we suppose that $\mathcal{H}_p$ is a real ordered positive Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, d is the metric induced by the norm $\|\cdot\|$ and $2^{\mathcal{H}_p}$ is the family of all nonempty subsets of $\mathcal{H}_p$.

For the presentation of the results, let us demonstrate some known definitions and results.

Definition 2.1 ([12, 27])**.** A nonempty subset C of $\mathcal{H}_p$ is called

- (i) a normal cone if there exists a constant $N > 0$ such that for $0 \leq x \leq y$, we have $||x|| \leq N||y||$.
- (ii) for any $x, y \in \mathcal{H}_p$, $x \leq y$ if and only if $y - x \in C$.
- (iii) x and y are said to be comparative to each other if and only if, we have either $x \leq y$ or $y \leq x$ and is denoted by $x \propto y$.

Definition 2.2 ([27])**.** For arbitrary elements $x, y \in \mathcal{H}_p$, $lub\{x,y\}$ and $glb\{x,y\}$ mean least upper bound and greatest upper bound of the set $\{x,y\}$. Suppose $lub\{x,y\}$ and $glb\{x,y\}$ exist, some binary operations are defined as follows:

- (i) $x \vee y = lub\{x,y\}$;
- (ii) $x \wedge y = glb\{x,y\}$;
- (iii) $x \oplus y = (x-y) \vee (y-x)$;
- (iv) $x \odot y = (x-y) \wedge (y-x)$.

The operations $\vee, \wedge$, $\oplus$ and $\odot$ are called OR, AND, XOR and XNOR operations, respectively.

Lemma 2.3 ([12])**.** *If $x \propto y$, then $lub\{x,y\}$ and $glb\{x,y\}$ exist, $x - y \propto y - x$ and $0 \leq (x-y) \vee (y-x)$.*

Lemma 2.4 ([12])**.** *For any natural number n, $x \propto y_n$ and $y_n \to y^*$ as $n \to \infty$, then $x \propto y^*$.*

Proposition 2.5 ([12, 22, 24])**.** *Let $\oplus$ be an XOR operation and $\odot$ be an XNOR operation. Then the following relations hold:*

- (i) $x \odot x = 0$, $x \odot y = y \odot x = -(x \oplus y) = -(y \oplus x)$;
- (ii) *if* $x \propto 0$, *then* $-x \oplus 0 \leq x \leq x \oplus 0$;
- (iii) $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$;
- (iv) $0 \leq x \oplus y$, *if* $x \propto y$;
- (v) *if* $x \propto y$, *then* $x \oplus y = 0$ *if and only if* $x = y$;
- (vi) $(x+y) \odot (u+v) \geq (x \odot u) + (y \odot v)$;
- (vii) $(x+y) \odot (u+v) \geq (x \odot v) + (y \odot u)$;
- $(viii)$ *if* x, y *and* w *are comparative to each other, then* $(x \oplus y) \leq x \oplus w + w \oplus y$;
- (ix) $\alpha x \oplus \beta x = |\alpha - \beta| x = (\alpha \oplus \beta) x$, *if* $x \propto 0$, $\forall\ x, y, u, v \in \mathcal{H}_p$ *and* $\alpha, \beta, \lambda \in \mathbb{R}$.

Proposition 2.6 ([12]). *Let C be a normal cone in $\mathcal{H}_p$ with normal constant N, then for each $x, y \in \mathcal{H}_p$, the following relations hold:*

(*i*) $\|0 \oplus 0\| = \|0\| = 0$;
(*ii*) $\|x \vee y\| \leq \|x\| \vee \|y\| \leq \|x\| + \|y\|$;
(*iii*) $\|x \oplus y\| \leq \|x - y\| \leq N|x \oplus y\|$;
(*iv*) *if* $x \propto y$, *then* $\|x \oplus y\| = \|x - y\|$.

Definition 2.7 ([22]). Let $A : \mathcal{H}_p \to \mathcal{H}_p$ be a single-valued mapping.

(*i*) A is said to be comparison mapping, if for each $x, y \in \mathcal{H}_p$, $x \propto y$ then $A(x) \propto A(y)$, $x \propto A(x)$ and $y \propto A(y)$.
(*ii*) A is said to be strongly comparison mapping, if A is a comparison mapping and $A(x) \propto A(y)$ if and only if $x \propto y$, for any $x, y \in \mathcal{H}_p$.

Definition 2.8 ([20]). A mapping $A : \mathcal{H}_p \to \mathcal{H}_p$ is said to be β-ordered compression mapping, if A is a comparison mapping and

$$A(x) \oplus A(y) \leq \beta(x \oplus y), \text{ for } 0 < \beta < 1.$$

Definition 2.9 ([22, 25]). Let $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued mapping. Then

(*i*) M is said to be a comparison mapping, if for any $v_x \in M(x)$, $x \propto v_x$, and if $x \propto y$, then for any $v_x \in M(x)$ and any $v_y \in M(y)$, $v_x \propto v_y$, $\forall\, x, y \in \mathcal{H}_p$;
(*ii*) a comparison mapping M is said to be α-non-ordinary difference mapping, if for each $x, y \in \mathcal{H}_p$, $v_x \in M(x)$ and $v_y \in M(y)$ such that
$$(v_x \oplus v_y) \oplus \alpha(x \oplus y) = 0;$$
(*iii*) a comparison mapping M is said to be θ-ordered rectangular, if there exists a constant $\theta > 0$, for any $x, y \in \mathcal{H}_p$, there exist $v_x \in M(x)$ and $v_y \in M(y)$ such that
$$\langle v_x \odot v_y, -(x \oplus y)\rangle \geq \theta\|x \oplus y\|^2,$$
holds.

Now, we introduce some new definitions of XOR-ordered strongly compression mapping, XOR-NODSM mapping and a resolvent operator associated with XOR-NODSM mapping.

Definition 2.10. A mapping $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is said to be λ-XOR-ordered strongly monotone compression mapping, if $x \propto y$, then there exists a constant $\lambda > 0$ such that

$$\lambda(v_x \oplus v_y) \geq x \oplus y, \forall x, y \in \mathcal{H}_p, v_x \in M(x), v_y \in M(y).$$

Definition 2.11. Let $A : \mathcal{H}_p \to \mathcal{H}_p$ be a strongly comparison and β-ordered compression mapping. Then, a comparison set-valued mapping $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is said to be (α, λ)-XOR-NODSM if M is a α-non-ordinary difference mapping and λ-XOR-ordered strongly monotone mapping and $[A \oplus \lambda M](\mathcal{H}_p) = \mathcal{H}_p$, for $\lambda, \beta, \alpha > 0$.

Definition 2.12. Let $A : \mathcal{H}_p \to \mathcal{H}_p$ be a strongly comparison and β-ordered compression mapping. Suppose that $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is a set-valued, (α, λ)-XOR-NODSM mapping. The resolvent operator $\mathcal{J}^A_{\lambda,M} : \mathcal{H}_p \to \mathcal{H}_p$ associated with A and M is defined by

$$\mathcal{J}^A_{\lambda,M}(x) = [A \oplus \lambda M]^{-1}(x), \forall x \in \mathcal{H}_p, \tag{2.1}$$

where $\lambda > 0$ is a constant.

Definition 2.13 ([26])**.** Let $S, T : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings, $x_0 \in \mathcal{H}_p$ and let

$$x_{n+1} = S(T, x_n)$$

defines an iterative sequence which yields a sequence of points $\{x_n\}$ in $\mathcal{H}_p$. Suppose that $F(T) = \{x \in \mathcal{H}_p : Tx = x\} \neq \emptyset$ and $\{x_n\}$ converges to a fixed point x^* of T. Let $\{u_n\} \subset \mathcal{H}_p$ and

$$\vartheta_n = \|u_{n+1} - S(T, u_n)\|.$$

If $\lim_{n\to\infty} \vartheta_n = 0$, which implies that $u_n \to x^*$, then the iterative sequence $\{x_n\}$ is said to be T-stable or stable with respect to T.

Lemma 2.14 ([28])**.** *Let $\{\chi_n\}$ be a nonnegative real sequence and $\{\zeta_n\}$ be a real sequence in $[0,1]$ such that $\sum_{n=0}^{\infty} \zeta_n = \infty$. If there exists a positive integer m such that*

$$\chi_n \leq (1 - \zeta_n)\chi_n + \zeta_n \eta_n, \ \forall n \geq m, \tag{2.2}$$

where $\eta_n \geq 0$, for all $n \geq 0$ and $\eta_n \to 0$ $(n \to 0)$, then $\lim_{n\to\infty} \chi_n = 0$.

Now, we show that the resolvent operator defined by (2.1) is single-valued, a comparison mapping as well as Lipschitz continuous.

Proposition 2.15. *Let $A : \mathcal{H}_p \to \mathcal{H}_p$ be a β-ordered compression mapping and $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be the set-valued θ-ordered rectangular mapping with $\theta\lambda > \beta$. Then the resolvent operator $\mathcal{J}^A_{\lambda,M} : \mathcal{H}_p \to \mathcal{H}_p$ is a single-valued, for all $\lambda > 0$.*

Proof. For any given $u \in \mathcal{H}_p$ and a constant $\lambda > 0$, let $x, y \in [A \oplus \lambda M]^{-1}(u)$. Then, let

$$v_x = \frac{1}{\lambda}(u \oplus A(x)) \in M(x),$$

and

$$v_y = \frac{1}{\lambda}(u \oplus A(y)) \in M(y).$$

Using (i) and (ii) of Proposition 2.5, we have

$$\begin{aligned} v_x \odot v_y &= \frac{1}{\lambda}(u \oplus A(x)) \odot \frac{1}{\lambda}(u \oplus A(y)) \\ &= \frac{1}{\lambda}[(u \oplus A(x)) \odot (u \oplus A(y))] \\ &= -\frac{1}{\lambda}[(u \oplus A(x)) \oplus (u \oplus A(y))] \\ &= -\frac{1}{\lambda}[(u \oplus u) \oplus (A(x) \oplus A(y))] \\ &= -\frac{1}{\lambda}[0 \oplus (A(x) \oplus A(y))] \\ &\leq -\frac{1}{\lambda}[A(x) \oplus A(y)]. \end{aligned}$$

Thus, we have

$$v_x \odot v_y \leq -\frac{1}{\lambda}[A(x) \oplus A(y)]. \tag{2.3}$$

Since M is θ-ordered rectangular mapping, A is β-ordered compression mapping and using (2.3), we have

$$\begin{aligned} \theta\|x \oplus y\|^2 &\leq \langle v_x \odot v_y, -(x \oplus y)\rangle \\ &\leq \langle -\frac{1}{\lambda}[A(x) \oplus A(y)], -(x \oplus y)\rangle \\ &\leq \frac{1}{\lambda}\langle A(x) \oplus A(y), x \oplus y\rangle \\ &\leq \frac{1}{\lambda}\langle \beta(x \oplus y), x \oplus y\rangle \\ &= \frac{\beta}{\lambda}\|x \oplus y\|^2, \end{aligned}$$

i.e.,

$$\begin{aligned} \theta\|x \oplus y\|^2 &\leq \frac{\beta}{\lambda}\|x \oplus y\|^2, \\ \Big(\theta - \frac{\beta}{\lambda}\Big)\|x \oplus y\|^2 &\leq 0, \text{ for } \theta\lambda > \beta, \end{aligned}$$

which implies that

$$\|x \oplus y\| = 0 \Rightarrow x \oplus y = 0.$$

Therefore, $x = y$. Hence the resolvent operator $\mathcal{J}^A_{\lambda,M}$ is single-valued, for $\theta\lambda > \beta$. □

Proposition 2.16. *Let $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a (α, λ)-XOR-NODSM set-valued mapping with respect to $\mathcal{J}^A_{\lambda,M}$. Let $A : \mathcal{H}_p \to \mathcal{H}_p$ be a strongly comparison mapping with respect to $\mathcal{J}^A_{\lambda,M}$. Then the resolvent operator $\mathcal{J}^A_{\lambda,M} : \mathcal{H}_p \to \mathcal{H}_p$ is a comparison mapping.*

Proof. Let M be a (α, λ)-XOR-NODSM set-valued mapping with respect to $\mathcal{J}^A_{\lambda,M}$. That is, M is α-non-ordinary difference and λ-XOR-ordered strongly monotone comparison mapping with respect to $\mathcal{J}^A_{\lambda,M}$ so that $x \propto \mathcal{J}^A_{\lambda,M}(x)$. For any $x, y \in \mathcal{H}_p$, let $x \propto y$ and

$$v_{x^*} = \frac{1}{\lambda}(x \oplus A(\mathcal{J}^A_{\lambda,M}(x))) \in M(\mathcal{J}^A_{\lambda,M}(x)) \tag{2.4}$$

and

$$v_{y^*} = \frac{1}{\lambda}(y \oplus A(\mathcal{J}^A_{\lambda,M}(y)) \in M(\mathcal{J}^A_{\lambda,M}(y)). \tag{2.5}$$

Since M is λ-XOR-ordered strongly monotone mapping, using (2.4) and (2.5), we have

$$\begin{aligned}
(x \oplus y) &\leq \lambda(v_{x^*} \oplus v_{y^*}) \\
(x \oplus y) &\leq \Big(x \oplus A(\mathcal{J}^A_{\lambda,M}(x))\Big) \oplus \Big(y \oplus A(\mathcal{J}^A_{\lambda,M}(y))\Big) \\
(x \oplus y) &\leq (x \oplus y) \oplus \Big(A(\mathcal{J}^A_{\lambda,M}(x)) \oplus A(\mathcal{J}^A_{\lambda,M}(y))\Big) \\
0 &\leq A(\mathcal{J}^A_{\lambda,M}(x)) \oplus A(\mathcal{J}^A_{\lambda,M}(y)) \\
0 &\leq \Big[A(\mathcal{J}^A_{\lambda,M}(x)) - A(\mathcal{J}^A_{\lambda,M}(y))\Big] \vee \Big[A(\mathcal{J}^A_{\lambda,M}(y)) - A(\mathcal{J}^A_{\lambda,M}(x))\Big] \\
0 &\leq \Big[A(\mathcal{J}^A_{\lambda,M}(x)) - A(\mathcal{J}^A_{\lambda,M}(y))\Big] \text{ or } 0 \leq \Big[A(\mathcal{J}^A_{\lambda,M}(y)) - A(\mathcal{J}^A_{\lambda,M}(x))\Big].
\end{aligned}$$

Thus, we have

$$A(\mathcal{J}^A_{\lambda,M}(x)) \geq A(\mathcal{J}^A_{\lambda,M}(y)) \text{ or } A(\mathcal{J}^A_{\lambda,M}(y)) \geq A(\mathcal{J}^A_{\lambda,M}(x)),$$

which implies that

$$A(\mathcal{J}^A_{\lambda,M}(x)) \propto A(\mathcal{J}^A_{\lambda,M}(y)).$$

Since A is strongly comparison mapping with respect to $\mathcal{J}^A_{\lambda,M}$. Therefore, $\mathcal{J}^A_{\lambda,M}(x) \propto \mathcal{J}^A_{\lambda,M}(y)$. That is, the resolvent operator $\mathcal{J}^A_{\lambda,M}$ is a comparison mapping. □

Proposition 2.17. *Let $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a (α, λ)-XOR-NODSM set-valued mapping with respect to $\mathcal{J}^A_{\lambda,M}$. Let $A : \mathcal{H}_p \to \mathcal{H}_p$ be a comparison and β-ordered compression mapping with respect to $\mathcal{J}^A_{\lambda,M}$, for $\alpha\lambda > \beta$. Then the following condition holds:*

$$\mathcal{J}^A_{\lambda,M}(x) \oplus \mathcal{J}^A_{\lambda,M}(y) \leq \frac{1}{(\alpha\lambda \oplus \beta)}(x \oplus y),$$

$\forall x, y \in \mathcal{H}_p,$ *i.e., the resolvent operator is Lipschitz continuos.*

Proof. Let $x, y \in \mathcal{H}_p$, $u_x = \mathcal{J}^A_{\lambda,M}(x)$, $u_y = \mathcal{J}^A_{\lambda,M}(y)$, and let

$$v_{x^{**}} = \frac{1}{\lambda}\big(x \oplus A(u_x)\big) \in M(u_x) \text{ and } v_{y^{**}} = \frac{1}{\lambda}\big(y \oplus A(u_y)\big) \in M(u_y).$$

As M be an (α, λ)-XOR-NODSM set-valued mapping with respect to $\mathcal{J}^A_{\lambda,M}$ and A is β-ordered compression mapping with respect to $\mathcal{J}^A_{\lambda,M}$. It follows that M is also an α-non-ordinary difference mapping with respect to $\mathcal{J}^A_{\lambda,M}$, we have

$$(v_{x^{**}} \oplus v_{y^{**}}) \oplus \alpha(u_x \oplus u_y) = 0, \tag{2.6}$$

and

$$\begin{aligned}
v_{x^{**}} \oplus v_{y^{**}} &= \frac{1}{\lambda}[(x \oplus A(u_x)) \oplus (y \oplus A(u_y))] \\
&= \frac{1}{\lambda}[(x \oplus y) \oplus (A(u_x) \oplus A(u_y))] \\
&\leq \frac{1}{\lambda}[(x \oplus y) \oplus \beta(u_x \oplus u_y)].
\end{aligned}$$

From (2.6), we have

$$\begin{aligned}\alpha(u_x \oplus u_y) &= v_{x^{**}} \oplus v_{y^{**}}\\ &\leq \frac{1}{\lambda}[(x \oplus y) \oplus \beta(u_x \oplus u_y)],\end{aligned}$$

i.e.,

$$\alpha\lambda(u_x \oplus u_y) \leq [(x \oplus y) \oplus \beta(u_x \oplus u_y)].$$

Now,

$$\begin{aligned}\big(\alpha\lambda(u_x \oplus u_y)\big) \oplus \big(\beta(u_x \oplus u_y)\big) &\leq (x \oplus y) \oplus 0 = x \oplus y\\ \big(\alpha\lambda \oplus \beta\big)\big(u_x \oplus u_y\big) &\leq x \oplus y.\end{aligned}$$

It follows that $u_x \oplus u_y \leq \Big(\frac{1}{(\alpha\lambda\oplus\beta)}\Big)(x \oplus y)$ and consequently, we have

$$\mathcal{J}^A_{\lambda,M}(x) \oplus \mathcal{J}^A_{\lambda,M}(y) \leq \frac{1}{(\alpha\lambda \oplus \beta)}(x \oplus y).$$

□

3. Formulation of the problem and existence result

Let $C \subseteq \mathcal{H}_p$ be a normal cone with constant N. Let $P : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mapping and $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued mapping. We consider the following problem:

Find $x \in \mathcal{H}_p$ such that

$$0 \in P(x) \oplus M(x). \tag{3.1}$$

We call problem (3.1) as general ordered variational inclusion problem involving XOR operation (in short, GOVIP).

Below we list some special cases of problem (3.1).

(*i*) If M is single-valued and $M(x) = F(x, g(x))$, then GOVIP (3.1) becomes the problem of finding $x \in \mathcal{H}_p$ such that

$$P(x) \oplus F(x, g(x)) \geq 0. \tag{3.2}$$

Problem (3.2) is introduced and studied by Li [21].

(*ii*) If $P = 0$, then GOVIP (3.1) reduces to the problem of finding $x \in \mathcal{H}_p$ such that

$$0 \in M(x). \tag{3.3}$$

Problem (3.3) is introduced and studied by Li [22].

(*iii*) If M is single-valued and $M(x) = F(g(x))$, then problem (3.3) becomes the problem of finding $x \in \mathcal{H}_p$ such that

$$F(g(x)) \geq 0. \tag{3.4}$$

Problem (3.4) is introduced and studied by Li [20].

Hence, we claim that our problem is much more general that many existing problems in the literature. The following lemma is a fixed point formulation of GOVIP (3.1).

Lemma 3.1. *The GOVIP* (3.1) *admits a solution* $x \in \mathcal{H}_p$ *if and only if it satisfies the following equation:*

$$x = \mathcal{J}^{A}_{\lambda,M}[\lambda P(x) \oplus A(x)], \tag{3.5}$$

where $\lambda > 0$ *is constant.*

Proof. Proof is a direct consequence of the definition of resolvent operator (2.1). □

Theorem 3.2. *Let* $P, A : \mathcal{H}_p \to \mathcal{H}_p$ *be the single-valued mappings such that* P *is comparison,* τ*-ordered compression mapping and* A *is comparison and* β*-ordered compression mapping, respectively. Suppose that* $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ *is a* (α, λ)*-XOR-NODSM set-valued mapping. In addition, if* $M \propto A$, $A \propto P$ *and for all* $\lambda, \alpha > 0$, *such that the following conditions are satisfied:*

$$\begin{cases} |(|\lambda|\tau \oplus \beta)| < \frac{|\alpha\lambda \oplus \beta|}{N}; \\ \alpha\lambda > \beta, \end{cases} \tag{3.6}$$

then, GOVIP (3.1) *admits a solution* $x^* \in \mathcal{H}_p$, *which is a fixed point of* $\mathcal{J}^{A}_{\lambda,M}[\lambda P(x^*) \oplus A(x^*)]$.

Proof. By Proposition 2.16, if $x_1 \propto x_2$, then

$$\mathcal{J}^{A}_{\lambda,M}[\lambda P(.) \oplus A(.)](x_1) \propto \mathcal{J}^{A}_{\lambda,M}[\lambda P(.) \oplus A(.)](x_2).$$

Since P is τ-ordered compression, A is β-ordered compression mapping and using Proposition 2.17, we have

$$\begin{aligned}
0 &\le \mathcal{J}^{A}_{\lambda,M}[\lambda P(.) \oplus A(.)](x_1) \oplus \mathcal{J}^{A}_{\lambda,M}[\lambda P(.) \oplus A(.)](x_2) \\
&\le \frac{1}{(\alpha\lambda \oplus \beta)}\Big([\lambda P(.) \oplus A(.)](x_1) \oplus [\lambda P(.) \oplus A(.)](x_2)\Big) \\
&= \frac{1}{(\alpha\lambda \oplus \beta)}\Big([\lambda P(x_1) \oplus A(x_1)] \oplus [\lambda P(x_2) \oplus A(x_2)]\Big) \\
&= \frac{1}{(\alpha\lambda \oplus \beta)}\Big([|\lambda|(P(x_1) \oplus P(x_2))] \oplus [A(x_1) \oplus A(x_2)]\Big) \\
&\le \frac{1}{(\alpha\lambda \oplus \beta)}\Big([|\lambda|\tau(x_1 \oplus x_2)] \oplus [\beta(x_1 \oplus x_2)]\Big) \\
&= \frac{1}{(\alpha\lambda \oplus \beta)}\Big([|\lambda|\tau \oplus \beta](x_1 \oplus x_2)\Big) \\
&= \frac{(|\lambda|\tau \oplus \beta)}{(\alpha\lambda \oplus \beta)}(x_1 \oplus x_2),
\end{aligned}$$

i.e.,

$$0 \le \mathcal{J}^{A}_{\lambda,M}[\lambda P(.) \oplus A(.)](x_1) \oplus \mathcal{J}^{A}_{\lambda,M}[\lambda P(.) \oplus A(.)](x_2) \le \Psi(x_1 \oplus x_2), \tag{3.7}$$

$$\text{where } \Psi = \frac{(|\lambda|\tau \oplus \beta)}{(\alpha\lambda \oplus \beta)}.$$

Using (3.7), Definition 2.1 and Proposition 2.6, we have

$$\Big\|\mathcal{J}^{A}_{\lambda,M}[\lambda P(.) \oplus A(.)](x_1) - \mathcal{J}^{A}_{\lambda,M}[\lambda P(.) \oplus A(.)](x_2)\Big\| \le N|\Psi|\|x_1 - x_2\|. \tag{3.8}$$

It is clear from (3.6), that $|\Psi| < \frac{1}{N}$. It follows that from (3.8) that $\mathcal{J}^A_{\lambda,M}[\lambda P(.)\oplus A(.)]$ is contraction mapping. Therefore, there exists a unique $x^* \in \mathcal{H}_p$ such that

$$x^* = \mathcal{J}^A_{\lambda,M}[\lambda P(x^*) \oplus A(x^*)].$$

By Lemma 3.1, x^* is a unique solution of GOVIP (3.1), which is a fixed point of $\mathcal{J}^A_{\lambda,M}[\lambda P(x^*) \oplus A(x^*)]$. □

4. Convergence and stability analysis

First we establish an Ishikawa type iterative algorithm based on Lemma 3.1 for finding the approximate solution of GOVIP (3.1), and then we prove a convergence result and a result for stability of the iterative sequence generalized by the proposed algorithm.

Iterative Algorithm 4.1. Let $A, P : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings and $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued mapping. Given any $x_0 \in \mathcal{H}_p$, compute the sequence $\{x_n\}$ defined by the following iterative scheme:

$$\begin{cases} x_{n+1} = (1-a_n)x_n + a_n\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(y_n) \oplus A(y_n)]\Big) + a_n\alpha_n, \\ y_n = (1-b_n)x_n + b_n\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x_n) \oplus A(x_n)]\Big) + b_n\beta_n. \end{cases} \tag{4.1}$$

Let $\{u_n\}$ be any sequence in $\mathcal{H}_p$ and define $\{\vartheta_n\}$ by

$$\begin{cases} \vartheta_n = \Big\|u_{n+1} - \Big[(1-a_n)u_n + a_n\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(t_n) \oplus A(t_n)]\Big) + a_n\alpha_n\Big]\Big\|, \\ t_n = (1-b_n)u_n + b_n\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(u_n) \oplus A(u_n)]\Big) + b_n\beta_n, \end{cases} \tag{4.2}$$

where $0 \le a_n, b_n \le 1$, $\sum_{n=0}^{\infty} a_n = \infty, \forall\ n \ge 0$, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $\mathcal{H}_p$ introduced to take into account the possible inexact computation provided that $\alpha_n \oplus 0 = \alpha_n$ and $\beta_n \oplus 0 = \beta_n$, $\forall\ n \ge 0$.

Remark 4.1. If $b_n = 0, \forall\ n \ge 0$, then Algorithm 4.1 becomes Mann type iterative algorithm. Also, we remark that for suitable choices of operators involved in Algorithm 4.1, we can easily obtain many more algorithms studied by several authors for solving ordered variational inclusion problems, see e.g. [3, 6, 22, 23, 24, 25].

Theorem 4.2. *Let M, A and P be the same as in Theorem 3.2 such that all the conditions of Theorem* 3.2 *are satisfied. Additionally if the following conditions are satisfied:*

$$\begin{cases} |(|\lambda|\tau \oplus \beta)| < |\alpha\lambda \oplus \beta|\ \ \min\{\frac{1}{N}, \frac{1}{2}\}; \\ \alpha\lambda > \beta, \end{cases} \tag{4.3}$$

and $\lim_{n\to\infty} \|\alpha_n \vee (-\alpha_n)\| = \lim_{n\to\infty} \|\beta_n \vee (-\beta_n)\| = 0$, *then*

- (*I*) *the sequence $\{x_n\}$ generated by Algorithm* 4.1 *converges strongly to the unique solution x^* of GOVIP (3.1).*
- (*II*) *Moreover, if* $0 < \pi \le a_n$, *then* $\lim_{n\to\infty} u_n = x^*$ *if and only if* $\lim_{n\to\infty} \vartheta_n = 0$, *where ϑ_n is defined in (4.2) i.e., the sequence $\{x_n\}$ generated by (4.1) is stable with respect to $\mathcal{J}^A_{\lambda,M}$.*

Proof. (**I**). First, we show that the sequence $\{x_n\}$ converges strongly to the unique solution x^* of GOVIP (3.1). Theorem 3.2 implies that x^* is a unique solution of GOVIP (3.1). Then, we have

$$\begin{cases} x^* = (1-a_n)x^* + a_n\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x^*) \oplus A(x^*)]\Big) \\ \ = (1-b_n)x^* + b_n\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x^*) \oplus A(x^*)]\Big). \end{cases} \tag{4.4}$$

Using Algorithm 4.1, (4.4), Proposition 2.5 and Proposition 2.17, it follows that

$$\begin{aligned} 0 &\le x_{n+1} \oplus x^* \\ &= \Big[(1-a_n)x_n + a_n\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(y_n) \oplus A(y_n)]\Big) + a_n\alpha_n\Big] \\ &\quad \oplus\Big[(1-a_n)x^* + a_n\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x^*) \oplus A(x^*)]\Big) + a_n 0\Big] \\ &\le (1-a_n)(x_n \oplus x^*) + a_n(\alpha_n \oplus 0) \\ &\quad + a_n\Big[\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(y_n) \oplus A(y_n)]\Big) \oplus \Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x^*) \oplus A(x^*)]\Big)\Big] \\ &\le (1-a_n)(x_n \oplus x^*) + \Psi a_n(y_n \oplus x^*) + a_n(\alpha_n \oplus 0), \end{aligned} \tag{4.5}$$

$$\text{where } \Psi = \frac{(|\lambda|\tau \oplus \beta)}{(\alpha\lambda \oplus \beta)}.$$

We evaluate,

$$\begin{aligned} 0 &\le y_n \oplus x^* \\ &= \Big[(1-b_n)x_n + b_n\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x_n) \oplus A(x_n)]\Big) + b_n\beta_n\Big] \\ &\quad \oplus\Big[(1-b_n)x^* + b_n\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x^*) \oplus A(x^*)]\Big) + b_n 0\Big] \\ &\le (1-b_n)(x_n \oplus x^*) + b_n(\beta_n \oplus 0) \\ &\quad + b_n\Big[\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x_n) \oplus A(x_n)]\Big) \oplus \Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x^*) \oplus A(x^*)]\Big)\Big] \\ &\le (1-b_n)(x_n \oplus x^*) + \Psi b_n(x_n \oplus x^*) + b_n(\beta_n \oplus 0). \end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6), it follows that

$$\begin{aligned} 0 &\le x_{n+1} \oplus x^* \\ &\le (1-a_n)(x_n \oplus x^*) + \Psi a_n\big[(1-b_n)(x_n \oplus x^*) + \Psi b_n(x_n \oplus x^*) \\ &\quad + b_n(\beta_n \oplus 0)\big] + a_n(\alpha_n \oplus 0) \\ &\le (1 - a_n(1-2\Psi))(x_n \oplus x^*) + a_n\big[\Psi b_n(\beta_n \oplus 0) + (\alpha_n \oplus 0)\big]. \end{aligned} \tag{4.7}$$

Using definition of normal cone and Proposition 2.6, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\le N(1 - a_n(1-2\Psi))\|x_n - x^*\| \\ &\quad + N a_n(1-2\Psi)\left(\frac{\Psi b_n\|\beta_n \vee (-\beta_n)\| + \|\alpha_n \vee (-\alpha_n)\|}{(1-2\Psi)}\right). \end{aligned} \tag{4.8}$$

By setting $\eta_n = \frac{\Psi b_n \|\beta_n \vee (-\beta_n)\| + \|\alpha_n \vee (-\alpha_n)\|}{(1-2\Psi)}$, $\chi_n = \|x_n - x^*\|$, $\zeta_n = Na_n(1-2\Psi)$, inequality (4.8) can be rewritten as

$$\chi_n \leq (1-\zeta_n)\chi_n + \zeta_n \eta_n. \tag{4.9}$$

From Lemma 2.14 and using the hypothesis $\lim_{n\to\infty} \|\alpha_n \vee (-\alpha_n)\| = \lim_{n\to\infty} \|\beta_n \vee (-\beta_n)\| = 0$, we deduce that $\chi_n \to 0$, as $n \to \infty$, and so $\{x_n\}$ converges strongly to a unique solution x^* of GOVIP (3.1).

(II). Let $H(x^*) = \mathcal{J}^A_{\lambda,M}[\lambda P(x^*) \oplus A(x^*)]$. Using Algorithm 4.1 and Proposition 2.5, we obtain

$$\begin{aligned}
0 &\leq u_{n+1} \oplus x^* \\
&\leq u_{n+1} \oplus \big((1-a_n)u_n + a_n H(t_n) + a_n\alpha_n\big) \\
&\quad + \big((1-a_n)u_n + a_n H(t_n) + a_n\alpha_n\big) \oplus \big((1-a_n)x^* + a_n H(x^*)\big) \\
&\leq \big[u_{n+1} \oplus \big((1-a_n)u_n + a_n H(t_n) + a_n\alpha_n\big)\big] \\
&\quad + (1-a_n)(u_n \oplus x^*) + a_n(H(t_n) \oplus H(x^*)) + a_n(\alpha_n \oplus 0) \\
&\leq \big[u_{n+1} \oplus \big((1-a_n)u_n + a_n H(t_n) + a_n\alpha_n\big)\big] \\
&\quad + (1-a_n)(u_n \oplus x^*) + a_n\Psi(t_n \oplus x^*) + a_n(\alpha_n \oplus 0),
\end{aligned} \tag{4.10}$$

$$\text{where } \Psi = \frac{(|\lambda|\tau \oplus \beta)}{(\alpha\lambda \oplus \beta)}.$$

From (4.2) and (4.10), we have

$$\begin{aligned}
0 &\leq t_n \oplus x^* \\
&\leq \big((1-b_n)u_n + b_n H(u_n) + b_n\beta_n\big) \oplus \big((1-b_n)x^* + b_n H(x^*)\big) \\
&\leq (1-b_n)(u_n \oplus x^*) + b_n(H(u_n) \oplus H(x^*)) + b_n(\beta_n \oplus 0) \\
&\leq (1-b_n)(u_n \oplus x^*) + b_n\Psi(u_n \oplus x^*) + b_n(\beta_n \oplus 0).
\end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11), we obtain

$$\begin{aligned}
0 &\leq u_{n+1} \oplus x^* \\
&\leq \big[u_{n+1} \oplus \big((1-a_n)u_n + a_n H(t_n) + a_n\alpha_n\big)\big] \\
&\quad + (1-a_n)(u_n \oplus x^*) + \Psi a_n\big[(1-b_n)(u_n \oplus x^*) + b_n\Psi(u_n \oplus x^*) \\
&\quad + b_n(\beta_n \oplus 0)] + a_n(\alpha_n \oplus 0) \\
&\leq \big[u_{n+1} \oplus \big((1-a_n)u_n + a_n H(t_n) + a_n\alpha_n\big)\big] \\
&\quad + (1-a_n(1-2\Psi))(u_n \oplus x^*) + a_n\big[\Psi b_n(\beta_n \oplus 0) + (\alpha_n \oplus 0)\big].
\end{aligned} \tag{4.12}$$

Using the definition of normal cone and Proposition 2.6, (4.12) becomes

$$\begin{aligned}
\|u_{n+1} - x^*\| &\leq N\|u_{n+1} - \big((1-a_n)u_n + a_n H(t_n) + a_n\alpha_n\big)\| \\
&\quad N(1-a_n(1-2\Psi))\|u_n - x^*\| + Na_n\big[\Psi b_n\|\beta_n \vee (-\beta_n)\| \\
&\quad + \|\alpha_n \vee (-\alpha_n)\|\big] \\
&\leq N\vartheta_n + N(1-a_n(1-2\Psi))\|u_n - x^*\| \\
&\quad + Na_n\big[\Psi b_n\|\beta_n \vee (-\beta_n)\| + \|\alpha_n \vee (-\alpha_n)\|\big].
\end{aligned} \tag{4.13}$$

Since $0 < \pi \leq a_n$, (4.13) becomes

$$\begin{aligned}\|u_{n+1} - x^*\| &\leq N(1 - a_n(1 - 2\Psi))\|u_n - x^*\| + a_n N(1 - 2\Psi)\Bigg[\frac{\vartheta_n}{\pi(1 - 2\Psi)} \\ &\quad + \left(\frac{\Psi b_n\|\beta_n \vee (-\beta_n)\| + \|\alpha_n \vee (-\alpha_n)\|}{(1 - 2\Psi)}\right)\Bigg]. \end{aligned} \tag{4.14}$$

Assume that $\lim_{n\to\infty} \vartheta_n = 0$, then $\lim_{n\to\infty} u_n = x^*$, where $\lim_{n\to\infty} \|\alpha_n \vee (-\alpha_n)\| = \lim_{n\to\infty} \|\beta_n \vee (-\beta_n)\| = 0$.

Conversely, suppose that $\lim_{n\to\infty} u_n = x^*$. From (4.9) and $\lim_{n\to\infty} \|\alpha_n \vee (-\alpha_n)\| = \lim_{n\to\infty} \|\beta_n \vee (-\beta_n)\| = 0$, we have

$$\begin{aligned} 0 &\leq u_{n+1} \oplus \big[(1 - a_n)u_n + a_n H(t_n) + a_n\alpha_n\big] \\ &\leq (u_{n+1} \oplus x^*) + \big[((1 - a_n)u_n + a_n H(t_n) + a_n\alpha_n) \oplus x^*\big] \\ &\leq (u_{n+1} \oplus x^*) + \big[((1 - a_n)u_n + a_n H(t_n) + a_n\alpha_n) \\ &\quad \oplus((1 - a_n) + a_n H(t_n)x^*)\big] \\ &\leq (u_{n+1} \oplus x^*) + (1 - a_n)(u_n \oplus x^*) + a_n(H(t_n) \oplus H(x^*)) + a_n(\alpha_n \oplus 0) \\ &\leq (u_{n+1} \oplus x^*) + (1 - a_n)(u_n \oplus x^*) + a_n\Psi(t_n \oplus x^*) + a_n(\alpha_n \oplus 0) \\ &\leq (u_{n+1} \oplus x^*) + (1 - a_n(1 - 2\Psi))(u_n \oplus x^*) \\ &\quad + a_n\big[\Psi b_n(\beta_n \oplus 0) + (\alpha_n \oplus 0)\big]. \end{aligned} \tag{4.15}$$

Again applying the definition of normal cone and Proposition 2.6, it follows that

$$\begin{aligned} \vartheta_n &= \|u_{n+1} - \big[(1 - a_n)u_n + a_n H(t_n) + a_n\alpha_n\big]\| \\ &\leq N\|u_{n+1} - x^*\| + N(1 - a_n(1 - 2\Psi))\|u_n - x^*\| \\ &\quad + a_n N\big[\Psi b_n\|\beta_n \vee (-\beta_n)\| + \|\alpha_n \vee (-\alpha_n)\|\big], \end{aligned} \tag{4.16}$$

which implies that

$$\lim_{n\to\infty} \vartheta_n = 0. \tag{4.17}$$

Hence, the iterative sequence $\{x_n\}$ generated by (4.2) is stable with respect to $\mathcal{J}^A_{\lambda,M}$. □

5. Numerical example

In this section, we construct a numerical example to illustrate our Algorithm 4.1 and to justify Proposition 2.17 and Theorem 4.2.

Example 5.1. Let $\mathcal{H} = [0, \infty)$ with the usual inner product and $C = [0, 1]$ be a normal cone with normal constant $N = 1$. Let $A, P : \mathcal{H}_p \to \mathcal{H}_p$ be the mappings defined by

$$A(x) = \frac{x}{3} + \frac{1}{2} \text{ and } P(x) = \frac{x}{2} + \frac{1}{4},\ \forall x \in \mathcal{H}_p.$$

For each $x, y \in \mathcal{H}_p,\ x \propto y$. We calculate

$$\begin{aligned}
A(x) \oplus A(y) &= \Big(\frac{x}{3} + \frac{1}{2}\Big) \oplus \Big(\frac{y}{3} + \frac{1}{2}\Big) \\
&= \Big[\Big(\frac{x}{3} + \frac{1}{2}\Big) - \Big(\frac{y}{3} + \frac{1}{2}\Big)\Big] \vee \Big[\Big(\frac{y}{3} + \frac{1}{2}\Big) \\
&\quad -\Big(\frac{x}{3} + \frac{1}{2}\Big)\Big] \\
&= \Big[\frac{x}{3} - \frac{y}{3}\Big] \vee \Big[\frac{y}{3} - \frac{x}{3}\Big] \\
&= \frac{1}{3}\big[(x-y) \vee (y-x)\big] \\
&= \frac{1}{3}(x \oplus y) \\
&\leq \frac{1}{2}(x \oplus y),
\end{aligned}$$

i.e.,

$$A(x) \oplus A(y) \leq \frac{1}{2}(x \oplus y),\ \forall x, y \in \mathcal{H}_p.$$

Hence, A is $\frac{1}{2}$-ordered compression mapping. Similarly, it is easy to check that P is $\frac{3}{4}$-ordered compression mapping.

Suppose that $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is a set-valued mapping defined by

$$M(x) = \Big\{3x + \frac{1}{4}\Big\}, \forall x \in \mathcal{H}_p.$$

Let $v_x = 3x + \frac{1}{4} \in M(x)$ and $v_y = 3y + \frac{1}{4} \in M(y)$, we evaluate

$$\begin{aligned}
v_x \oplus v_y &= \Big(3x + \frac{1}{4}\Big) \oplus \Big(3y + \frac{1}{4}\Big) \\
&= \Big[\Big(3x + \frac{1}{4}\Big) - \Big(3y + \frac{1}{4}\Big)\Big] \vee \Big[\Big(3y + \frac{1}{4}\Big) - \Big(3x + \frac{1}{4}\Big)\Big] \\
&= 3\big[(x-y) \vee (y-x)\big] \\
&= 3(x \oplus y) \geq (x \oplus y),
\end{aligned}$$

and also,

$$(v_x \oplus v_y) \oplus 3(x \oplus y) = 0.$$

Thus, M is a 2-XOR-ordered strongly monotone and 3-non-ordinary difference comparison mapping. It is clear that for $\lambda = 2$, $[A \oplus \lambda M](\mathcal{H}_p) = \mathcal{H}_p$. Hence, M is an $(3, 2)$-XOR-NODSM set-valued mapping.

The resolvent operator defined by (2.1) associated with A and M is given by

$$\mathcal{J}^{A}_{\lambda,M}(x) = \frac{2x}{17}, \forall x \in \mathcal{H}_p. \tag{5.1}$$

It is easy to examine that the resolvent operator defined above is comparison and single-valued mapping. We evaluate

$$\begin{aligned}
\mathcal{J}^A_{\lambda,M}(x) \oplus \mathcal{J}^A_{\lambda,M}(y) &= \Big[\frac{2x}{17} \oplus \frac{2y}{17}\Big] \\
&= \Big[\Big(\frac{2x}{17} - \frac{2y}{17}\Big) \vee \Big(\frac{2y}{17} - \frac{2x}{17}\Big)\Big] \\
&= \frac{2}{17}\Big[(x-y) \vee (y-x)\Big] \\
&= \frac{2}{17}(x \oplus y) \\
&\leq \frac{2}{11}(x \oplus y),
\end{aligned}$$

i.e.,

$$\mathcal{J}^A_{\lambda,M}(x) \oplus \mathcal{J}^A_{\lambda,M}(y) \leq \frac{2}{11}(x \oplus y), \forall x, y \in \mathcal{H}_p.$$

Hence, all the conditions of Proposition 2.17 are satisfied.

For $\lambda = 2$, we calculate

$$\begin{aligned}
\mathcal{J}^A_{\lambda,M}[\lambda P(x) \oplus A(x)] &= \frac{2[\lambda P(x) \oplus A(x)]}{17} \\
&= \frac{2}{17}\Big[2\Big(\frac{x}{2} + \frac{1}{4}\Big) \oplus \Big(\frac{x}{3} + \frac{1}{2}\Big)\Big] \\
&= \frac{2}{17}\Big[\Big(x + \frac{1}{2}\Big) \oplus \Big(\frac{x}{3} + \frac{1}{2}\Big)\Big] \\
&= \frac{2}{17}\Big[\Big(\Big(x + \frac{1}{2}\Big) - \Big(\frac{x}{3} + \frac{1}{2}\Big)\Big) \vee \Big(\Big(\frac{x}{3} + \frac{1}{2}\Big) \\
&\quad - \Big(x + \frac{1}{2}\Big)\Big)\Big] \\
&= \frac{2}{17}\Big[\Big(\frac{2x}{3}\Big) \vee \Big(-\frac{2x}{3}\Big)\Big] \\
&= \frac{2}{17}\sup\Big\{\frac{2x}{3}, -\frac{2x}{3}\Big\} \\
&= \frac{4x}{51}.
\end{aligned}$$

Clearly, 0 is a fixed point of $\mathcal{J}^A_{\lambda,M}[\lambda P(.) \oplus A(.)]$.

Let $a_n = \frac{1}{n}$, $b_n = \frac{n}{n+1}$, $\alpha_n = \frac{n}{6n^2+1}$ and $\beta_n = \frac{1}{n^3}$. It is easy to show that the sequences $\{a_n\}, \{b_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ satisfying the conditions $0 \leq a_n, b_n \leq 1$, $\sum_{n=0}^{\infty} a_n = \infty$, $\alpha_n \oplus 0 = \alpha_n$ and $\beta_n \oplus 0 = \beta_n$.

Now, we can estimate the sequences $\{x_n\}$ and $\{y_n\}$ by the following schemes:

$$\begin{aligned}
x_{n+1} &= \Big(1 - \frac{1}{n}\Big)x_n + \frac{7}{52(2n+1)}y_n + \frac{1}{(6n^2+1)}, \\
y_n &= \frac{1}{n+1}x_n + \frac{7n}{52(n+1)}x_n + \frac{1}{n^2(n+1)}.
\end{aligned}$$

It is also verified that condition (4.3) is satisfied. Thus, all the assumptions of Theorem 4.2 are fulfilled. Hence, the sequence $\{x_n\}$ converges strongly to the unique solution $x^* = 0$ of GOVIP (3.1).

All codes are written in MATLAB Version 7.13, we have the following different initial values $x_0 = 5$ and $x_0 = 10$ which shows that the sequence $\{x_n\}$ converges to $x^* = 0$, shown in Figure 1 and Figure 2.

TABLE 1. The values of x_n with initial values $x_0 = 5$ and $x_0 = 10$

No. of Iteration	For $x_0 = 5$ x_n	For $x_0 = 10$ x_n
n=1	5	10
n=2	1.86406	3.6579156
n=3	0.40432	0.78350
n=4	0.06366	0.12038
n=5	0.00871	0.01538
n=6	0.00152	0.00219
n=7	5.15234e-04	5.64551e-04
n=8	2.84366e-04	2.87758e-04
n=9	1.83785e-04	1.83991e-04
n=10	1.26786e-04	1.26793e-04
n=11	9.12714e-05	9.12726e-05
n=12	6.79151e-05	6.79152e-05
n=13	5.19099e-05	5.19099e-05
n=14	4.05704e-05	4.05708e-05
n=15	3.23115e-05	3.23111e-05
n=16	2.61523e-05	2.61524e-05
n=17	2.14654e-05	2.14656e-05
n=18	1.78351e-05	1.78352e-05
n=19	0	0
n=20	0	0

6. CONCLUSION

In this article, we study a general ordered variational inclusion problem based on XOR operator in a real ordered positive Hilbert space and prove the existence of solution. We establish an Ishikawa-type iterative algorithms with error terms for this class of general ordered variational inclusion problem which is more general than Mann-type and many other iterative schemes studies by several author's, see e.g., [3, 6, 22, 23, 24, 25]. We prove that the iterative sequence generated by the suggested iterative algorithm converges to a unique solution GONVIP (3.1). Stability analysis is also discussed. In the last of this paper, we construct a numerical example in support of Proposition 2.17 and Theorem 4.2.

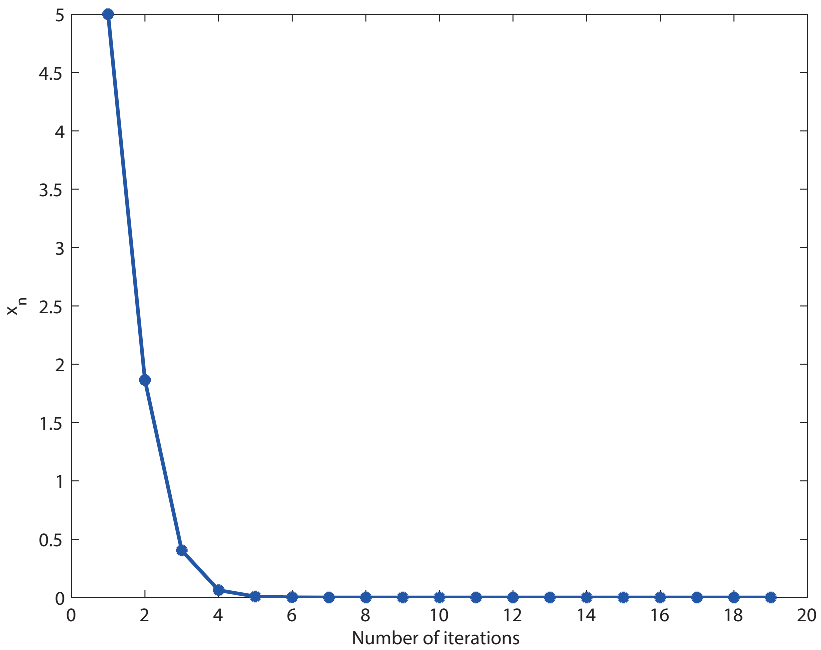

Figure 1: The convergence of x_n with initial value $x_0 = 5$

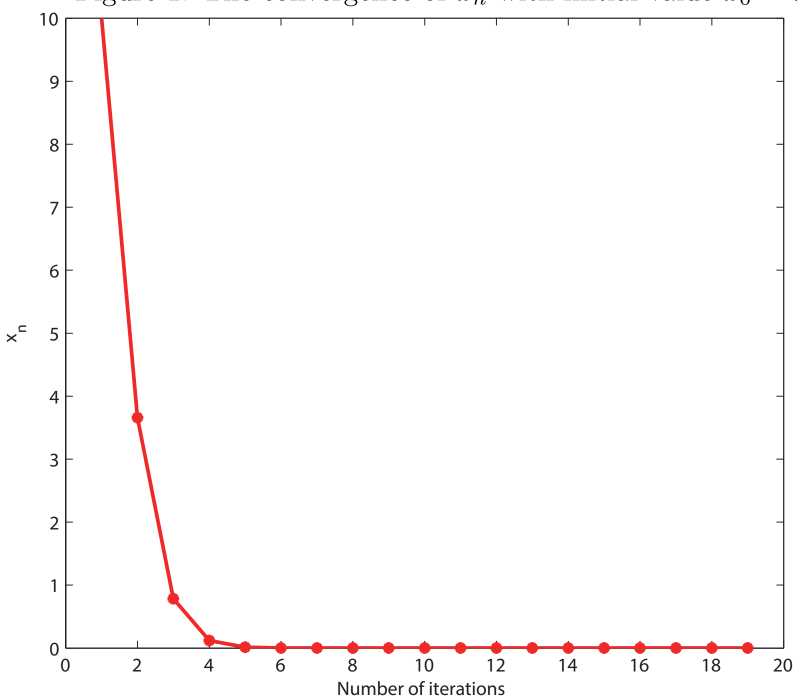

Figure 2: The convergence of x_n with initial value $x_0 = 10$

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