Linear and Nonlinear Analysis
Yokohama Publishers
ISSN 2188-8167

Volume 4, Number 3, 2018, 405–412

CONVERGENCE TO A FIXED POINT OF A BALANCED MAPPING BY THE MANN ALGORITHM IN A HADAMARD SPACE

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Abstract. In this paper, we propose a notion of a balanced mapping for a finite family of nonexpansive mappings. We prove a Δ-convergence theorem for the Mann algorithm of the balanced mapping in a Hadamard space.

1. Introduction

The fixed point approximation has been studied by a large number of researchers. In 1953, Mann [5] introduced an iteration procedure for approximating fixed points of a nonexpansive mapping in a Hilbert space. This iteration is often called the Mann algorithm. Later, many researchers studied this algorithm approximating a common fixed point of nonexpansive mappings on Hilbert or Banach spaces. On the other hand, we know that the class of complete CAT(0) spaces, called Hadamard spaces, includes Hilbert spaces. In 2008, Dhompongsa and Panyanak [2] proved a convergence theorem of the Mann algorithm to a fixed point on Hadamard spaces.

Theorem 1.1 (Dhompongsa and Panyanak [2])**.** *Let C be a bounded closed convex subset of a Hadamard space and $T : C \to C$ a nonexpansive mapping. For an initial point $x_1 \in C$, define an iterative sequence $\{x_n\}$ by*

$$x_{n+1} = (1 - t_n)x_n \oplus t_n T x_n$$

for $n \in \mathbb{N}$, where $\{t_n\}$ is a sequence in $[0, 1]$, with the restrictions that $\sum_{n=0}^{\infty} t_n = \infty$ and $\limsup_{n\to\infty} t_n < 1$. Then $\{x_n\}$ Δ-converges to a fixed point of T.

Further, in complete CAT(1) spaces, Kimura, Saejung, and Yotkaew [3] proved a Δ-convergence theorem to common fixed point for a family of nonexpansive mappings including the following scheme:

$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n((1 - \beta_n)Sx_n \oplus \beta_n T x_n).$$

It should be noted that, in CAT(κ) spaces for $\kappa \in \mathbb{R}$, a weighted average of more than two points may not be unique according to the order of the convex combination.

In this paper, we propose a new definition about convex combination of mappings. We define a balanced mapping by a minimizer of a certain function generated by

2010 *Mathematics Subject Classification.* 47H09.
Key words and phrases. Fixed point, approximation, iterative scheme, balanced mapping, Hadamard space.
This work was supported by JSPS KAKENHI Grant Number 15K05007.

nonexpansive mappings and show that it is well-defined. We apply this mapping for a Δ-convergence theorem by the Mann algorithm to a common fixed point of the mappings.

2. Preliminaries

Let (X, d) be a metric space. For $x, y \in X$, a mapping $c : [0, l] \to X$ is called a geodesic with endpoints x and y if c satisfies $c(0) = x$, $c(l) = y$ and $d(c(u), c(v)) = |u - v|$ for $u, v \in [0, l]$. If a geodesic with endpoints x and y exists for any $x, y \in X$, then we call X a geodesic space. Moreover, if a geodesic uniquely exists for each $x, y \in X$, then we call X a uniquely geodesic space.

Let X be a uniquely geodesic space. For $x, y \in X$, the image of a geodesic c with endpoints x and y is called a geodesic segment joining x and y, and is denoted by $[x, y]$. A geodesic triangle $\Delta(x, y, z) \subset X$ with vertices x, y, z in X is the union of geodesic segments joining each pair of vertices. A comparison triangle $\bar{\Delta}(\bar{x}, \bar{y}, \bar{z}) \subset \mathbb{R}^2$ for $\Delta(x, y, z)$ is a triangle such that $d(x, y) = \|\bar{x} - \bar{y}\|$, $d(y, z) = \|\bar{y} - \bar{z}\|$, and $d(z, x) = \|\bar{z} - \bar{x}\|$. If for any $p, q \in \Delta(x, y, z)$ and their comparison points $\bar{p}, \bar{q} \in \bar{\Delta}(\bar{x}, \bar{y}, \bar{z})$, the inequality

$$d(p, q) \leq \|\bar{p} - \bar{q}\|$$

is satisfied for all triangle in X, then X is called a CAT(0) space, and this inequality is called the CAT(0) inequality. A Hadamard space is defined as a complete CAT(0) space.

Let X be a Hadamard space. For $t \in [0, 1]$ and $x, y \in X$, there exists unique $z \in [x, y]$ such that $d(x, z) = (1 - t)d(x, y)$ and $d(y, z) = td(x, y)$. We denote this z by $tx \oplus (1 - t)y$. From the CAT(0) inequality, we obtain the following lemma.

Lemma 2.1. *Let X be a Hadamard space, $x, y, z \in X$ and $t \in [0, 1]$. Then*

$$d(z, tx \oplus (1 - t)y)^2 \leq td(z, x)^2 + (1 - t)d(z, y)^2 - t(1 - t)d(x, y)^2.$$

Lemma 2.2 ([1, Corollary 1.2.5])**.** *Let X be a Hadamard space. Then for any four points $x_0, x_1, y_0, y_1 \in X$,*

$$d(x_0, y_1)^2 + d(x_1, y_0)^2 \leq d(x_0, y_0)^2 + d(x_1, y_1)^2 + 2d(x_0, x_1)d(y_0, y_1).$$

Let X be a Hadamard space and $\{x_n\} \subset X$ be a bounded sequence. For any $x \in X$, we put

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n), \quad r(\{x_n\}) = \inf_{x\in X} r(x, \{x_n\}).$$

If $x \in X$ satisfies that $r(x, \{x_n\}) = r(\{x_n\})$, we call x an asymptotic center of $\{x_n\}$. Furthermore, if for any subsequence of $\{x_n\}$, each asymptotic center is a unique point x, we say that $\{x_n\}$ is Δ-convergent to x. It means that for $\{x_n\} \subset X$ Δ-converging to x and all $y \in X$ with $x \neq y$, it follows

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

We know that any bounded sequence $\{x_n\} \subset X$ has a Δ-convergent subsequence [4]. Let T be a mapping from X into itself. A mapping T from X into itself is called a nonexpansive mapping if it satisfies the inequality $d(Tx, Ty) \leq d(x, y)$ for any $x, y \in X$. A point $z \in X$ is called a fixed point of T if $Tz = z$. We denote the set of all fixed points of T by $F(T)$.

3. Main results

In this section, we propose a notion of a balanced mapping for a finite family of nonexpansive mappings. We prove a Δ-convergence theorem for the Mann algorithm of the balanced mapping in a Hadamard space. First we prove a balanced mapping is well defined. Then we show its nonexpansivenesss and properties of the set of its fixed points.

Lemma 3.1. *Let X be a Hadamard space. Let $\{T_1, T_2, \dots, T_N\}$ be nonexpansive mappings from X to X. Let $\{\alpha^1, \alpha^2, \dots, \alpha^N\} \subset [0, 1]$ such that $\sum_{k=1}^N \alpha^k = 1$. Then the set*

$$\operatorname*{argmin}_{y \in X} \sum_{k=1}^{N} \alpha^k d(T_k x, y)^2$$

consists of one point.

Proof. Let $d = \inf_{y \in X} \sum_{k=1}^N \alpha^k d(T_k x, y)^2$ and $\{y_n\} \subset X$ a sequence such that

$$\lim_{n \to \infty} \sum_{k=1}^{N} \alpha^k d(T_k x, y_n)^2 = d.$$

For $m, n \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{k=1}^{N} \alpha^k d(T_k x, \frac{1}{2} y_n \oplus \frac{1}{2} y_m)^2 \leq \frac{1}{2} \sum_{k=1}^{N} \alpha^k d(T_k x, y_n)^2 &+ \frac{1}{2} \sum_{k=1}^{N} \alpha^k d(T_k x, y_m)^2 \\ &- \frac{1}{4} d(y_n, y_m)^2. \end{aligned}$$

Thus we have

$$\begin{aligned} \frac{1}{4} d(y_n, y_m)^2 &\leq \frac{1}{2} \sum_{k=1}^{N} \alpha^k d(T_k x, y_n)^2 + \frac{1}{2} \sum_{k=1}^{N} \alpha^k d(T_k x, y_m)^2 \\ &\quad - \sum_{k=1}^{N} \alpha^k d(T_k x, \frac{1}{2} y_n \oplus \frac{1}{2} y_m)^2 \\ &\leq \frac{1}{2} \sum_{k=1}^{N} \alpha^k d(T_k x, y_n)^2 + \frac{1}{2} \sum_{k=1}^{N} \alpha^k d(T_k x, y_m)^2 - d. \end{aligned}$$

Hence we obtain $\{y_n\}$ is Cauchy. By the completeness of the Hadamard space, there exists $u = \lim_{n \to \infty} y_n$. By the continuity of the metric, we get $\sum_{k=1}^N \alpha^k d(T_k x, u)^2 = \inf_{y \in X} \sum_{k=1}^N \alpha^k d(T_k x, y)^2$.

Let $u, v \in \operatorname{argmin}_{y\in X} \sum_{k=1}^{N} \alpha^k d(T_k x, y)^2$. By definition, we have

$$\begin{aligned}\sum_{k=1}^{N} \alpha^k d(T_k x, u)^2 &\leq \sum_{k=1}^{N} \alpha^k d(T_k x, \frac{1}{2}u \oplus \frac{1}{2}v)^2 \\ &\leq \frac{1}{2}\sum_{k=1}^{N} \alpha^k d(T_k x, u)^2 + \frac{1}{2}\sum_{k=1}^{N} \alpha^k d(T_k x, v)^2 - \frac{1}{4}d(u,v)^2.\end{aligned}$$

Therefore we obtain

$$\frac{1}{4}d(u,v)^2 \leq -\frac{1}{2}\sum_{k=1}^{N} \alpha^k d(T_k x, u)^2 + \frac{1}{2}\sum_{k=1}^{N} \alpha^k d(T_k x, v)^2.$$

Similary we obtain

$$\frac{1}{4}d(u,v)^2 \leq \frac{1}{2}\sum_{k=1}^{N} \alpha^k d(T_k x, u)^2 - \frac{1}{2}\sum_{k=1}^{N} \alpha^k d(T_k x, v)^2.$$

Adding these inequalities, we obtain $d(u,v) = 0$, and hence $u = v$. $\square$

From the result above, we can define a new mapping $U : X \to X$ by $Ux = \operatorname{argmin}_{y\in X} \sum_{k=1}^{N} \alpha^k d(T_k x, y)^2$ for $x \in X$, and we call it a balanced mapping generated by $\{T_1, T_2, \dots, T_N\}$ and $\{\alpha^1, \alpha^2, \dots, \alpha^N\}$.

Lemma 3.2. *Let X be a Hadamard space. Let $\{T_1, T_2, \dots, T_N\}$ be nonexpansive mappings from X to X. Let $\{\alpha^1, \alpha^2, \dots, \alpha^N\} \subset [0,1]$ such that $\sum_{k=1}^{N}\alpha^k = 1$. Define a mapping $U : X \to X$ by*

$$Ux = \operatorname*{argmin}_{y\in X} \sum_{k=1}^{N} \alpha^k d(T_k x, y)^2$$

for $x \in X$. Then U is a nonexpansive mapping.

Proof. Let $t \in\,]0,1[$ and $x, y \in X$. We may assume that $Ux \neq Uy$.

$$\begin{aligned}&\sum_{k=1}^{N} \alpha^k d(T_k x, Ux)^2 \\ &\leq \sum_{k=1}^{N} \alpha^k d(T_k x, tUx \oplus (1-t)Uy)^2 \\ &\leq t\sum_{k=1}^{N} \alpha^k d(T_k x, Ux)^2 + (1-t)\sum_{k=1}^{N} \alpha^k d(T_k x, Uy)^2 - t(1-t)\sum_{k=1}^{N} \alpha^k d(Ux, Uy)^2 \\ &= t\sum_{k=1}^{N} \alpha^k d(T_k x, Ux)^2 + (1-t)\sum_{k=1}^{N} \alpha^k d(T_k x, Uy)^2 - t(1-t)d(Ux, Uy)^2,\end{aligned}$$

and thus

$$(1-t)\sum_{k=1}^{N} \alpha^k d(T_k x, Ux)^2 \leq (1-t)\sum_{k=1}^{N} \alpha^k d(T_k x, Uy)^2 - t(1-t)d(Ux, Uy)^2.$$

Dividing by $1-t$ and taking the limit as $t \to 1$, we have

$$\sum_{k=1}^{N} \alpha^k d(T_k x, Ux)^2 \le \sum_{k=1}^{N} \alpha^k d(T_k x, y)^2 - d(Ux, Uy)^2. \tag{3.1}$$

Thus we have

$$d(Ux, Uy)^2 \le \sum_{k=1}^{N} \alpha^k d(T_k x, Uy)^2 - \sum_{k=1}^{N} \alpha^k d(T_k x, Ux)^2. \tag{3.2}$$

Similarly, we obtain

$$d(Uy, Ux)^2 \le -\sum_{k=1}^{N} \alpha^k d(T_k y, Uy)^2 + \sum_{k=1}^{N} \alpha^k d(T_k y, Ux)^2. \tag{3.3}$$

By Lemma 2.2, adding (3.2) and (3.3), we have

$$\begin{aligned} 2d(Ux, Uy)^2 &\le -\sum_{k=1}^{N} \alpha^k d(T_k x, Ux)^2 - \sum_{k=1}^{N} \alpha^k d(T_k y, Uy)^2 \\ &\quad + \sum_{k=1}^{N} \alpha^k d(T_k x, Uy)^2 + \sum_{k=1}^{N} \alpha^k d(T_k y, Ux)^2 \\ &\le 2\sum_{k=1}^{N} \alpha^k d(T_k x, T_k y) d(Ux, Uy). \end{aligned}$$

Since T_k is nonexpansive,

$$2d(Ux, Uy) \le 2\sum_{k=1}^{N} \alpha^k d(T_k x, T_k y) \le 2\sum_{k=1}^{N} \alpha^k d(x, y) = 2d(x, y).$$

Therefore we have $d(Ux, Uy) \le d(x, y)$. □

Lemma 3.3. *Let X be a Hadamard space. Let $\{T_1, T_2, \dots, T_N\}$ be nonexpansive mappings from X to X such that $\bigcap_{k=1}^{N} F(T_k) \neq \emptyset$. Let $\{\alpha^1, \alpha^2, \dots, \alpha^N\} \subset]0, 1]$ such that $\sum_{k=1}^{N} \alpha^k = 1$. Define a mapping $U : X \to X$ by*

$$Ux = \operatorname*{argmin}_{y \in X} \sum_{k=1}^{N} \alpha^k d(T_k x, y)^2$$

for $x \in X$. Then $F(U) = \bigcap_{k=1}^{N} F(T_k)$.

Proof. For $z \in \bigcap_{k=1}^N F(T_k)$, we have

$$
\begin{aligned}
Uz &= \operatorname*{argmin}_{y \in X} \sum_{k=1}^N \alpha^k d(T_k z, y)^2 \\
&= \operatorname*{argmin}_{y \in X} \sum_{k=1}^N \alpha^k d(z, y)^2 \\
&= \operatorname*{argmin}_{y \in X} d(z, y)^2 \\
&= z.
\end{aligned}
$$

Thus $z \in F(U)$ and hence $\bigcap_{k=1}^N F(T_k) \subset F(U)$.

On the other hand, let $z \in F(U)$. By (3.1), for $w \in \bigcap_{k=1}^N F(T_k)$ and $t \in]0,1[$, we have

$$
\begin{aligned}
\sum_{k=1}^N \alpha^k d(T_k z, z)^2 &\leq \sum_{k=1}^N \alpha^k d(T_k z, w)^2 - d(z,w)^2 \\
&\leq \sum_{k=1}^N \alpha^k d(z, w)^2 - d(z,w)^2 \\
&= d(z,w)^2 - d(z,w)^2 \\
&= 0.
\end{aligned}
$$

Since $\alpha^k > 0$, we obtain $T_k z = z$ for $k = 1, 2, \dots, N$, and hence $F(U) \subset \bigcap_{k=1}^N F(T_k)$, which is the desired result. $\square$

In the main theorem, we consider a sequence $\{U_n\}$ of balanced mappings. From Lemma 3.3, we have $F(U_n) = \bigcap_{k=1}^N F(T_k)$ for all $n \in \mathbb{N}$ if $\{T_k\}$ has a common fixed point.

Theorem 3.4. *Let X be a Hadamard space. Let $\{T_1, T_2, \dots, T_N\}$ be nonexpansive mappings from X to X such that $\bigcap_{k=1}^N F(T_k) \neq \emptyset$. Let $\{\alpha_n^1, \alpha_n^2, \dots, \alpha_n^N\} \subset [a,1] \subset]0,1]$ such that $\sum_{k=1}^N \alpha_n^k = 1$ for every $n \in \mathbb{N}$. For $n \in \mathbb{N}$, define a mapping $U_n : X \to X$ by*

$$
U_n x = \operatorname*{argmin}_{y \in X} \sum_{k=1}^N \alpha_n^k d(T_k x, y)^2
$$

for $x \in X$. Let $\{\delta_n\} \subset [c,d] \subset]0,1[$, $x_1 \in X$ and let $\{x_n\}$ be a sequence in X generated by

$$
x_{n+1} = \delta_n x_n \oplus (1 - \delta_n) U_n x_n
$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to an element of $\bigcap_{k=1}^N F(T_k)$.

Proof. First we note that $F(U_n) = \bigcap_{k=1}^{N} F(T_k)$ for all $n \in \mathbb{N}$. Let $z \in \bigcap_{k=1}^{N} F(T_k)$. From Lemma 3.3,

$$\begin{aligned} d(x_{n+1}, z)^2 &= d(\delta_n x_n \oplus (1-\delta_n) U_n x_n, z)^2 \\ &\le \delta_n d(x_n, z)^2 + (1-\delta_n) d(U_n x_n, z)^2 - \delta_n (1-\delta_n) d(U_n x_n, x_n)^2 \\ &\le d(x_n, z)^2 - \delta_n (1-\delta_n) d(U_n x_n, x_n)^2 \\ &\le d(x_n, z)^2. \end{aligned}$$

Thus there exists $m = \lim_{n\to\infty} d(x_n, z)$. Since $0 < c(1-d) < \delta_n(1-\delta_n)$, we obtain $\lim_{n\to\infty} d(U_n x_n, x_n) = 0$.

We show $\lim_{n\to\infty} d(T_k x_n, x_n) = 0$ for all $k = 1, \ldots, N$. Since $\{x_n\}$ is bounded, it follows that

$$\begin{aligned} m = \lim_{n\to\infty} d(x_n, z) &\le \lim_{n\to\infty} (d(x_n, U_n x_n) + d(U_n x_n, z)) \\ &= \lim_{n\to\infty} d(U_n x_n, z) \\ &= \lim_{n\to\infty} d(U_n x_n, U_n z) \\ &\le \lim_{n\to\infty} d(x_n, z) = m. \end{aligned}$$

Thus $\lim_{n\to\infty} d(x_n, z) = \lim_{n\to\infty} d(U_n x_n, z) = m$. By (3.1), it holds that

$$\begin{aligned} \sum_{k=1}^{N} \alpha_n^k d(T_k x_n, U_n x_n)^2 &\le \sum_{k=1}^{N} \alpha_n^k d(T_k x_n, z)^2 - d(z, U_n x_n)^2 \\ &\le \sum_{k=1}^{N} \alpha_n^k d(x_n, z)^2 - d(z, U_n x_n)^2 \\ &= d(x_n, z)^2 - d(z, U_n x_n)^2. \end{aligned}$$

Since $0 < a < \alpha_n^k$, we get $\lim_{n\to\infty} d(T_k x_n, U_n x_n) = 0$ and $\lim_{n\to\infty} d(T_k x_n, x_n) = 0$ for $k = 1, 2, \ldots, N$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which Δ-converges to a point $x_0 \in X$. Assume $x_0 \notin F(T_1)$. Then we have

$$\begin{aligned} \limsup_{i\to\infty} d(x_{n_i}, x_0) &< \limsup_{i\to\infty} d(x_{n_i}, T_1 x_0) \\ &\le \limsup_{i\to\infty} (d(x_{n_i}, T_1 x_{n_i}) + d(T_1 x_{n_i}, T_1 x_0)) \\ &\le \limsup_{i\to\infty} d(x_{n_i}, x_0). \end{aligned}$$

We get a contradiction and $x_0 \in F(T_1)$. Similarly, we can show $x_0 \in F(T_k)$ for all $k = 1, 2 \ldots, N$.

Suppose that there are two subsequences $\{u_i\}$ and $\{v_i\}$ of $\{x_n\}$ which Δ-converge to u_0 and v_0, respectively. Then it follows that $u_0, v_0 \in \bigcap_{k=1}^{N} F(T_k)$ and thus both

$\{d(x_n, u_0)\}$ and $\{d(x_n, v_0)\}$ have limits. Suppose that $u_0 \neq v_0$. Then, we have

$$\begin{aligned}\lim_{n\to\infty} d(x_n, u_0) &= \lim_{i\to\infty} d(u_i, u_0)\\ &< \lim_{i\to\infty} d(u_i, v_0)\\ &= \lim_{n\to\infty} d(x_n, v_0)\\ &= \lim_{i\to\infty} d(v_i, v_0)\\ &< \lim_{i\to\infty} d(v_i, u_0)\\ &= \lim_{n\to\infty} d(x_n, u_0).\end{aligned}$$

We get a contradiction and thus $u_0 = v_0$. This shows that $\{x_n\}$ Δ-converges to $x_0 \in \bigcap_{k=1}^N F(T_k)$. □

Remark. Let H be a Hilbert space and $x_1, \dots, x_N \in H$. Then we can show $\operatorname{argmin}_{y\in H} \sum_{k=1}^N \alpha^k \|x_k - y\|^2$ consists of one point and it coincides with $\sum_{k=1}^N \alpha^k x_k$. Indeed, by the parallelogram law, we have

$$\begin{aligned}\sum_{k=1}^N \alpha^k \|x_k - y\|^2 &= \left\|\sum_{k=1}^N \alpha^k (x_k - y)\right\|^2 + \sum_{k=2}^N \sum_{j=1}^{k-1} \alpha^k \alpha^j \|x_k - x_j\|^2\\ &= \left\|\sum_{k=1}^N \alpha^k x_k - y\right\|^2 + \sum_{k=2}^N \sum_{j=1}^{k-1} \alpha^k \alpha^j \|x_k - x_j\|^2.\end{aligned}$$

Therefore, it attains the minimum when $y = \sum_{k=1}^N \alpha^k x_k$.

References

[1] M. Bačák, *Convex Analysis and Optimization in Hadamard Spaces*, De Gruyter, Würzburg, 2014.
[2] S. Dhompongsa and B. Panyanak, *On Δ-convergence theorems in CAT(0) spaces*, Comput. Math. Appl. **56** (2008), 2572–2579.
[3] Y. Kimura, S. Saejung and P. Yotkaew, *The Mann algorithm in a complete geodesic space with bounded above*, Fixed Point Theory Appl. **2013** (2013), 13pages.
[4] W. A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal. **68** (2008), 3689–3696.
[5] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

Manuscript received 30 July 2018
revised 11 October 2018

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