Linear and Nonlinear Analysis
Yokohama Publishers
ISSN 2188-8167
Volume 4, Number 3, 2018, 393–404

SECOND-ORDER MIXED TYPE DUALITY FOR MINIMAX FRACTIONAL PROGRAMMING IN COMPLEX SPACES

TONE-YAU HUANG

Abstract. The purpose of this paper is to formulate the second-order mixed type dual model for minimax fractional programming in complex spaces, and then we could know that the second-order mixed type dual includes second-order Mond-Weir type dual and second-order Wolfe type dual as the special cases. Under the second-order generalized Θ-bonvexity assumptions, we will derive the weak, strong and strict converse duality theorems.

1. Introduction

Consider the following complex fractional minimax programming problem (please see Lai *et al.* [12]):

$$\text{(P)} \quad \min_{\zeta\in X}\ \sup_{\eta\in Y}\ \frac{\text{Re } f(\zeta,\eta)}{\text{Re } g(\zeta,\eta)}$$
$$\text{subject to} \quad \zeta \in X = \{\zeta\in\mathbb{C}^{2n} \mid -h(\zeta)\in S\},$$

where Y is a compact subset in $\mathbb{C}^{2m}$, S is a polyhedral cone in $\mathbb{C}^p$, for $\zeta=(z,\bar{z})\in\mathbb{C}^{2n}$, $\eta=(w,\overline{w})\in\mathbb{C}^{2m}$, functions $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are continuous functions; for each $\eta\in Y$, $f(\cdot,\eta)$, $g(\cdot,\eta)$ and $h(\cdot)$ are analytic on the $Q=\{\zeta=(z,\bar{z})|z\in\mathbb{C}^n\}\subset\mathbb{C}^{2n}$. Without loss of generality, we could assume that $\text{Re}f(\zeta,\eta)\geq 0$ and $\text{Re}g(\zeta,\eta)>0$.

Lai et. al. [12] established the necessary optimality conditions theorem of (P) as follows.

Theorem 1.1 ([12, Theorem 3.1]). *Let $\zeta_0=(z_0,\overline{z_0})$ be a (P)-optimal with optimal value γ^*. Suppose that the problem (P) satisfies the constraint qualification at ζ_0. Then there exist a positive integer k, scalars $\lambda_i>0$ with $\sum_{i=1}^k\lambda_i=1$, vectors $\eta_i\in Y(\zeta_0)$ for $i=1,\dots,k$ and non-zero vector $\mu\in S^*\subset\mathbb{C}^p$ such that*

$$\sum_{i=1}^k \lambda_i\left\{\left[\overline{\nabla_z f(\zeta_0,\eta_i)}+\nabla_{\overline{z}}f(\zeta_0,\eta_i)\right]-\gamma^*\left[\overline{\nabla_z g(\zeta_0,\eta_i)}+\nabla_{\overline{z}}g(\zeta_0,\eta_i)\right]\right\} + \mu^T\overline{\nabla_z h(\zeta_0)}+\mu^H\nabla_{\overline{z}}h(\zeta_0)=0, \tag{1.1}$$

$$\text{Re }[f(\zeta_0,\eta_i)-\gamma^* g(\zeta_0,\eta_i)]=0,\quad i=1,2,\dots,k, \tag{1.2}$$

$$\text{Re }\langle\mu,h(\zeta_0)\rangle=0. \tag{1.3}$$

2010 *Mathematics Subject Classification.* 90C46, 90C47.
Key words and phrases. complex minimax fractional programming, second-order duality problem, duality theorems .
This research was supported by MOST 107-2115-M-035-004-, Taiwan.

When the necessary optimality conditions for the programming problem was established, after then we could consider some duality models for this programming problem. In this article, we are interesting in establish the second-order duality models for complex minimax programming problems.

In real variables case of second-order duality for minimax programming problems, researchers have studied several second-order duality results for some minimax programming problems, and derived the duality theorems under the variety of generalized convexity, please see [1–6,10,11,15]. As on complex variables, Huang [7] constructed the second-order duality model for non-differentiable minimax programming problem in 2017, and established the duality theorems under second-order generalized Θ-bonvexity. Huang et. al. [8,9] also established the second-order parametric dual model, second-order Mond-Weir type and Wolfe type dual models for problem (P) with derived their duality theorems under second-order generalized Θ-bonvexity.

The purpose of this paper is to construct a second-order mixed type(MD) dual model of (P), such that dual problem (MD) includes problems (MWD) and (WD) as the special cases, and then to derive their duality theorems under second-order generalized Θ-bonvexity.

2. Notations and preliminaries

Throughout the paper, we introduce some notations, definitions and lemmas used as in [7,13]. Let $v \in \mathbb{C}^p$, we denote $\overline{v}$, v^T and v^H are conjugate, transpose and transpose conjugate of v. The set $S = \{\xi \in \mathbb{C}^p \mid Re(K\xi) \geq 0\}$ is a polyhedral cone where $K \in \mathbb{C}^{k\times p}$ is a $k \times p$ matrix, the dual cone S^* of S is defined by

$$S^* = \{\mu \in \mathbb{C}^p \mid \mathrm{Re}\langle \xi, \mu\rangle \geq 0 \text{ for } \xi \in S\}.$$

Given $\zeta = (z, \overline{z}) \in \mathbb{C}^{2n}$ and a twice differentiable analytic function $f : \mathbb{C}^{2n} \to \mathbb{C}$, the gradient expression $\nabla f(\zeta)$ is denoted by

$$\nabla f(\zeta) = \Big(\nabla_z f(\zeta), \nabla_{\overline{z}} f(\zeta)\Big) \in \mathbb{C}^{2n}$$

with

$$\nabla_z f(\zeta) = \Big(f_{z_1}(\zeta), \ldots, f_{z_n}(\zeta)\Big) \in \mathbb{C}^n, \;\; \nabla_{\overline{z}} f(\zeta) = \Big(f_{\overline{z_1}}(\zeta), \ldots, f_{\overline{z_n}}(\zeta)\Big) \in \mathbb{C}^n.$$

The second-order gradient expression $\nabla^2 f(\zeta)$ is denoted by

$$\nabla^2 f(\zeta) = \begin{pmatrix} \nabla_{zz} f(\zeta), & \nabla_{z\overline{z}} f(\zeta) \\ \nabla_{\overline{z}z} f(\zeta), & \nabla_{\overline{z}\overline{z}} f(\zeta) \end{pmatrix} \in \mathbb{C}^{2n\times 2n}$$

with

$$\nabla_{zz} f(\zeta) = \big(f_{z_i z_j}(\zeta)\big)_{n\times n}, \; i, \; j = 1, \ldots n, \quad \nabla_{\overline{z}z} f(\zeta) = \big(f_{\overline{z_i} z_j}(\zeta)\big)_{n\times n}, \; i, \; j = 1, \ldots n,$$
$$\nabla_{z\overline{z}} f(\zeta) = \big(f_{z_i \overline{z_j}}(\zeta)\big)_{n\times n}, \; i, \; j = 1, \ldots n, \quad \nabla_{\overline{z}\overline{z}} f(\zeta) = \big(f_{\overline{z_i}\overline{z_j}}(\zeta)\big)_{n\times n}, i, \; j = 1, \ldots n.$$

For convenience to express the constraint conditions of second-order duality models, we need the following two lemmas.

Lemma 2.1 ([13, Lemma 2]). *For $\eta \in Y \subset \mathbb{C}^{2m}$ and $\zeta = (z, \overline{z}) \in Q \subset \mathbb{C}^{2n}$, we denote the function*

$$\Phi(\zeta) = f(\zeta, \eta) + \langle h(\zeta), \mu \rangle.$$

Then $\Phi(\zeta)$ is differentiable at $\zeta_0 = (z_0, \overline{z_0})$, and

$$\begin{aligned} &\operatorname{Re}\left[\Phi'(\zeta_0)(\zeta - \zeta_0)\right] \\ &\quad = \operatorname{Re}\left[\left\langle\ z - z_0,\ \overline{\nabla_z f(\zeta_0, \eta)} + \nabla_{\overline{z}} f(\zeta_0, \eta) + \mu^T \overline{\nabla_z h(\zeta_0)} + \mu^H \nabla_{\overline{z}} h(\zeta_0)\ \right\rangle\right]. \end{aligned}$$

Lemma 2.2 ([7, Lemma 3.1]). *Given $\zeta = (z, \overline{z}), \zeta_0 = (z_0, \overline{z_0}) \in Q \subset \mathbb{C}^{2n}$ and let $(v, \overline{v}) = \zeta - \zeta_0$. Then the twice differentiable analytic mappings $f(\zeta)$ and $\langle h(\zeta), \mu \rangle$ have the second-order gradient representations at $\zeta_0 = (z_0, \overline{z_0})$ as follows.*

(a)

$$\begin{aligned} (\zeta - \zeta_0)^T \nabla^2 f(\zeta_0)(\zeta - \zeta_0) = &\left\langle\ v, v^H [\overline{\nabla_{zz} f(\zeta_0)}]\ \right\rangle + \left\langle\ v^H [\nabla_{\overline{z}\overline{z}} f(\zeta_0)], v\ \right\rangle \\ &+ \left\langle\ v, v^T [\overline{\nabla_{\overline{z}z} f(\zeta_0)}]\ \right\rangle + \left\langle\ v^T [\nabla_{z\overline{z}} f(\zeta_0)], v\ \right\rangle. \end{aligned}$$

The real part of the above identity is equal to

$$\operatorname{Re}\left(\left\langle\ z - z_0, v^H \left[\overline{\nabla_{zz} f(\zeta_0)} + \nabla_{\overline{z}\overline{z}} f(\zeta_0)\right] + v^T \left[\overline{\nabla_{\overline{z}z} f(\zeta_0)} + \nabla_{z\overline{z}} f(\zeta_0)\right]\ \right\rangle\right).$$

(b)

$$\begin{aligned} (\zeta - \zeta_0)^T \nabla^2 \langle h(\zeta_0), \mu \rangle (\zeta - \zeta_0) = &\left\langle\ v, v^H [\mu^T \overline{\nabla_{zz} h(\zeta_0)}]\ \right\rangle + \left\langle\ v^H [\mu^H \nabla_{\overline{z}\overline{z}} h(\zeta_0)], v\ \right\rangle \\ &+ \left\langle\ v, v^T [\mu^T \overline{\nabla_{\overline{z}z} h(\zeta_0)}]\ \right\rangle + \left\langle\ v^T [\mu^H \nabla_{z\overline{z}} h(\zeta_0)], v\ \right\rangle. \end{aligned}$$

The real part of the above identity is equal to

$$\operatorname{Re}\left(\left\langle\ z - z_0, v^H \left[\mu^T \overline{\nabla_{zz} h(\zeta_0)} + \mu^H \nabla_{\overline{z}\overline{z}} h(\zeta_0)\right] + v^T \left[\mu^T \overline{\nabla_{\overline{z}z} h(\zeta_0)} + \mu^H \nabla_{z\overline{z}} h(\zeta_0)\right]\ \right\rangle\right).$$

The definition of the second-order generalized Θ-bonvexity for the real part of complex function as follows.

Definition 2.3 ([7, Definition 4.1]). The real part of a twice differentiable analytic function $f(\cdot)$ from $\mathbb{C}^{2n}$ to $\mathbb{R}$ is called, respectively,

(i) (strictly)Θ-bonvex at $\zeta_0 \in Q \subset \mathbb{C}^{2n}$ if there exists a certain mapping $\Theta : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ such that for any $\zeta \in Q$,

$$\begin{aligned} &\operatorname{Re}\left\{f(\zeta) - f(\zeta_0) + \frac{1}{2}(\zeta - \zeta_0)^T \nabla^2 f(\zeta_0)(\zeta - \zeta_0)\right\} \\ &\qquad \geq \operatorname{Re}\left\{[\nabla f(\zeta_0) + (\zeta - \zeta_0)^T \nabla^2 f(\zeta_0)] \Theta(\zeta, \zeta_0)\right\}, \\ &\qquad (>) \end{aligned}$$

(ii) (strictly)Θ-pseudobonvex at $\zeta_0 \in Q \subset \mathbb{C}^{2n}$ if theRe exists a certain mapping $\Theta : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ such that for any $\zeta \in Q$,

$$\begin{aligned} &\operatorname{Re}\left\{[\nabla f(\zeta_0) + (\zeta - \zeta_0)^T \nabla^2 f(\zeta_0)] \Theta(\zeta, \zeta_0)\right\} \geq 0 \Rightarrow \\ &\qquad Re\left\{f(\zeta) - f(\zeta_0) + \frac{1}{2}(\zeta - \zeta_0)^T \nabla^2 f(\zeta_0)(\zeta - \zeta_0)\right\} \geq 0, (> 0) \end{aligned}$$

(iii) Θ-quasibonvex at $\zeta_0 \in Q$ if there exists a certain mapping $\Theta : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ such that for any $\zeta \in Q$,

$$\operatorname{Re}\left\{ f(\zeta) - f(\zeta_0) + \frac{1}{2}(\zeta - \zeta_0)^T \nabla^2 f(\zeta_0)(\zeta - \zeta_0) \right\} \leq 0 \Rightarrow$$
$$\operatorname{Re}\left\{ [\nabla f(\zeta_0) + (\zeta - \zeta_0)^T \nabla^2 f(\zeta_0)]\Theta(\zeta, \zeta_0) \right\} \leq 0.$$

The following lemma and remark could easily shown that the relations of the second-order generalized Θ-bonvexity.

Lemma 2.4 ([9, Lemma 2.4]).

(a) *If* $\operatorname{Re}(f)$ *is* Θ*-bonvex at* $\zeta_0 \in Q$*, then* $\operatorname{Re}(f)$ *are also* Θ*-pseudobonvex and* Θ*-quasibonvex at* $\zeta_0 \in Q$*.*

(b) *If* $\operatorname{Re}(f)$ *is strictly* Θ*-bonvex at* $\zeta_0 \in Q$*, then* $\operatorname{Re}(f)$ *strictly* Θ*-pseudobonvex at* $\zeta_0 \in Q$*.*

Remark 2.5 ([9, Remark]).

(a) If $\operatorname{Re}(f)$ is strictly Θ-bonvex at $\zeta_0 \in Q$, then $\operatorname{Re}(f)$ is Θ-bonvex at $\zeta_0 \in Q$.

(b) If $\operatorname{Re}(f)$ is strictly Θ-pseudobonvex at $\zeta_0 \in Q$, then $\operatorname{Re}(f)$ is Θ-pseudobonvex at $\zeta_0 \in Q$.

3. Second-order mixed type dual model

The second-order mixed type dual (MD) for problem (P) is considered the objective of fractional functional added a part of the constraint function of (P) with a part of multiplier $\mu \in S^*$ into the numerator of the fractional objective in (P). In order to construct the second-order mixed type dual problem, we will represent the constraint function in (P) as follows.

$$h(\zeta) = (h_1(\zeta), h_2(\zeta), \ldots, h_p(\zeta)) \in (-S) \subset \mathbb{C}^p,$$

and the multiplier $\mu = (\mu_1, \ldots, \mu_p) \in S^* \subset \mathbb{C}^p$. Now, we partition the index set $P = \{1, \ldots, p\}$ of the constraint function $h(\zeta)$ to be $P = P_0 \cup P_1 \cup \cdots \cup P_t$ such that

$$\operatorname{Re} \langle\, h_{P_r}(\zeta), \mu_{P_r} \,\rangle \leq 0 \text{ for } r = 0, 1, \ldots, t, \tag{3.1}$$

where $h_{P_r}(\zeta) \equiv \big(h_i(\zeta)\big)_{i\in P_r}$ and $\mu_{P_r} \equiv \big(\mu_i\big)_{i\in P_r}$.
Thus,

$$\operatorname{Re} \langle h(\zeta), \mu \,\rangle = \operatorname{Re} \langle\, h_{P_0}(\zeta), \mu_{P_0} \,\rangle + \sum_{r=1}^{t} \operatorname{Re} \langle\, h_{P_r}(\zeta), \mu_{P_r} \,\rangle \leq 0.$$

Here, $\langle h_{P_r}, \mu_{P_r}\rangle = \sum_{i\in P_r} \mu_i h_i(\zeta)$ for $r = 0,\ 1, \ldots,\ t$. By lemma 2.1 and lemma 2.2, we could know that

$$\operatorname{Re}\left\langle h'_{P_r}(\xi)(\zeta - \xi), \mu_{P_r} \right\rangle = \operatorname{Re}\left\langle\ z - u\ ,\ \mu_{P_r}^T \overline{\nabla_z h_{P_r}(\xi)} + \mu_{P_r}^H \nabla_{\overline{z}} h_{P_r}(\xi)\ \right\rangle,$$

$$\begin{aligned} \operatorname{Re}\left[(\zeta - \xi)^T \langle\, \nabla^2 h_{P_r}(\xi), \mu_{P_r}\rangle(\zeta - \xi)\right] &\\ = \operatorname{Re}\Big\langle\ z - u, v^H &\left[\mu_{P_r}^T \overline{\nabla_{zz} h_{P_r}(\xi)} + \mu_{P_r}^H \nabla_{\overline{z}\overline{z}} h_{P_r}(\xi)\right] \\ &+ v^T \left[\mu_{P_r}^T \overline{\nabla_{\overline{z}z} h_{P_r}(\xi)} + \mu_{P_r}^H \nabla_{z\overline{z}} h_{P_r}(\xi)\right]\ \Big\rangle\ , \end{aligned} \tag{3.2}$$

where $\zeta = (z, \overline{z})$, $\xi = (u, \overline{u})$ and $(v, \overline{v}) = \zeta - \xi$.

For convenience to build the parametric free dual models, we given some symbols as the following:

$$F^{(1)}(\xi,\eta_i) = \overline{\nabla_z f(\xi,\eta_i)} + \nabla_{\overline{z}} f(\xi,\eta_i);$$
$$F_1^{(2)}(\xi,\eta_i) = \overline{\nabla_{zz} f(\xi,\eta_i)} + \nabla_{\overline{z}\overline{z}} f(\xi,\eta_i); \quad F_2^{(2)}(\xi,\eta_i) = \overline{\nabla_{\overline{z}z} f(\xi,\eta_i)} + \nabla_{z\overline{z}} f(\xi,\eta_i);$$
$$G^{(1)}(\xi,\eta_i) = \overline{\nabla_z g(\xi,\eta_i)} + \nabla_{\overline{z}} g(\xi,\eta_i);$$
$$G_1^{(2)}(\xi,\eta_i) = \overline{\nabla_{zz} g(\xi,\eta_i)} + \nabla_{\overline{z}\overline{z}} g(\xi,\eta_i); \quad G_2^{(2)}(\xi,\eta_i) = \overline{\nabla_{\overline{z}z} g(\xi,\eta_i)} + \nabla_{z\overline{z}} g(\xi,\eta_i);$$

and for $r = 0,\ 1, \dots,\ t$,

$$H_{P_r}^{(1)}(\xi,\mu_{P_r}) = \mu_{P_r}^T \overline{\nabla_z h_{P_r}(\xi)} + \mu_{P_r}^H \nabla_{\overline{z}} h_{P_r}(\xi);$$
$$H_{1;P_r}^{(2)}(\xi,\mu_{P_r}) = \mu_{P_r}^T \overline{\nabla_{zz} h_{P_r}(\xi)} + \mu_{P_r}^H \nabla_{\overline{z}\overline{z}} h_{P_r}(\xi);$$
$$H_{2;P_r}^{(2)}(\xi,\mu_{P_r}) = \mu_{P_r}^T \overline{\nabla_{\overline{z}z} h_{P_r}(\xi)} + \mu_{P_r}^H \nabla_{z\overline{z}} h_{P_r}(\xi).$$

Thus, the formula (3.3) could be express by

$$\begin{aligned}
&\mathrm{Re}\left\langle h'_{P_r}(\xi)(\zeta-\xi), \mu_{P_r}\right\rangle = \mathrm{Re}\left\langle\ z-u\ ,\ H_{P_r}^{(1)}(\xi,\mu_{P_r})\ \right\rangle,\\
&\mathrm{Re}\left[(\zeta-\xi)^T\langle\ \nabla^2 h_{P_r}(\xi), \mu_{P_r}\rangle(\zeta-\xi)\right]\\
&\qquad = \mathrm{Re}\left\langle\ z-u, v^H H_{1;P_r}^{(2)}(\xi,\mu_{P_r}) + v^T H_{2;P_r}^{(2)}(\xi,\mu_{P_r})\ \right\rangle\ .
\end{aligned}$$

We could set up the second-order mixed type dual (MD) for problem (P) as the following:

$$\text{(MD)} \quad \max_{(k,\widetilde{\lambda},\widetilde{\eta})\in K(\xi)}\ \max_{(\xi,\mu,v)\in X(k,\widetilde{\lambda},\widetilde{\eta})} \frac{\sum_{i=1}^k \lambda_i Re\left[f(\xi,\eta_i) + \langle h_{P_0}(\xi), \mu_{P_0}\rangle\right]}{\sum_{i=1}^k \lambda_i Re\left[g(\xi,\eta_i)\right]}.$$

Here,

(1) $K(\xi)$ is the set of the component $(k, \widetilde{\lambda}, \widetilde{\eta})$ (where $\widetilde{\lambda} = (\lambda_1, \dots, \lambda_k)$ and $\widetilde{\eta} = (\eta_1, \dots, \eta_k)$) satisfied the necessary optimality conditions theorem of problem (P)(see Theorem 1.1).

(2) $X(k, \widetilde{\lambda}, \widetilde{\eta})$ is the set of all feasible solutions (ξ, μ, w_1, w_2) of (MD) satisfied the following expressions:

For $\xi = (u, \overline{u}) \in Q$ and $0 \neq \mu \in S^*$, such that

$$\begin{aligned}
&\left\{\sum_{i=1}^k \lambda_i \left(\left[F^{(1)}(\xi,\eta_i) + H_{P_0}^{(1)}(\xi,\mu_{P_0})\right] + \nu^H\left[F_1^{(2)}(\xi,\eta_i) + H_{1;P_0}^{(2)}(\xi,\mu_{P_0})\right]\right.\right.\\
&\qquad \left.\left. +\nu^T\left[F_2^{(2)}(\xi,\eta_i) + H_{2;P_0}^{(2)}(\xi,\mu_{P_0})\right]\right)\right\} \times \left(\sum_{i=1}^k \lambda_i \mathrm{Re}\ g(\xi,\eta_i)\right)\\
&\qquad\quad - \left(\sum_{i=1}^k \lambda_i \mathrm{Re}[f(\xi,\eta_i) + \langle h_{P_0}(\xi), \mu_{P_0}\rangle]\right)
\end{aligned}$$

(3.3)

$$\times\left\{\sum_{i=1}^{k}\lambda_i\big[G^{(1)}(\xi,\eta_i)+\nu^H G_1^{(2)}(\xi,\eta_i)+\nu^T G_2^{(2)}(\xi,\eta_i)\big]\right\}$$
$$+\sum_{r=1}^{t}\Big[H_{P_r}^{(1)}(\xi,\mu_{P_r})+\nu^H H_{1;P_r}^{(2)}(\xi,\mu_{P_r})+\nu^T H_{2;P_r}^{(2)}(\xi,\mu_{P_r})\Big]=0,$$

$$\left(\sum_{i=1}^{k}\lambda_i\mathrm{Re}\left[f(\xi,\eta_i)+\langle h_{P_0}(\xi),\mu_{P_0}\rangle\right]\right)\cdot$$
$$\left\{\sum_{i=1}^{k}\lambda_i\mathrm{Re}\Big\langle\ \nu\ ,\ \nu^H G_1^{(2)}(\xi,\eta_i)+\nu^T G_2^{(2)}(\xi,\eta_i)\Big\rangle\right\}$$

(3.4)

$$\geq\left\{\sum_{i=1}^{k}\lambda_i\mathrm{Re}\Big\langle\ \nu\ ,\ \nu^H\big[F_1^{(2)}(\xi,\eta_i)+H_{1;P_0}^{(2)}(\xi,\mu_{P_0})\big]\right.$$
$$\left.+\nu^T\big[F_2^{(2)}(\xi,\eta_i)+H_{2;P_0}^{(2)}(\xi,\mu_{P_0})\big]\Big\rangle\right\}\cdot\Big(\sum_{i=1}^{k}\lambda_i\mathrm{Re}\ g(\xi,\eta_i)\Big),$$

$$\mathrm{Re}\,\langle h_{P_r}(\xi),\mu_{P_r}\rangle\geq\frac{1}{2}\mathrm{Re}\Big\langle\ \nu\ ,\ \nu^H H_{1;P_r}^{(2)}(\xi,\mu_{P_r})+\nu^T H_{2;P_r}^{(2)}(\xi,\mu_{P_r})\ \Big\rangle,$$

(3.5)

$$\text{for } r=1,\ldots,t.$$

In 2018, Huang [9] established the second-order Mond-Weir type(MWD) and Wolfe type(WD) dual models for problem (P).

$$\text{(MWD)}\qquad \max_{(k,\widetilde{\lambda},\widetilde{\eta})\in K(\xi)}\quad \max_{(\xi,\mu,v)\in X_1(k,\widetilde{\lambda},\widetilde{\eta})}\quad \frac{\sum_{i=1}^{k}\lambda_i\ \mathrm{Re}\ f(\xi,\eta_i)}{\sum_{i=1}^{k}\lambda_i\ \mathrm{Re}\ g(\xi,\eta_i)},$$

where the constraint set $X_1(k,\widetilde{\lambda},\widetilde{\eta})$ satisfied suitable constraint conditions.

$$\text{(WD)}\qquad \max_{(k,\widetilde{\lambda},\widetilde{\eta})\in K(\xi)}\quad \max_{(\xi,\mu,v)\in X_2(k,\widetilde{\lambda},\widetilde{\eta})}\quad \frac{\sum_{i=1}^{k}\lambda_i\ \mathrm{Re}\ [f(\xi,\eta_i)+\langle h(\xi),\mu\rangle]}{\sum_{i=1}^{k}\lambda_i\ \mathrm{Re}\ g(\xi,\eta_i)},$$

where the constraint set $X_2(k,\widetilde{\lambda},\widetilde{\eta})$ satisfied suitable constraint conditions.

In problem (MD), if the index set P of the constraint function $h(\zeta)$ in (P) is separated only by P_0 and P_1, (it means that $P=P_0\cup P_1$ and $P_r=\emptyset$ for $r=2,\cdots,t$), then

$$\text{(MD)}\equiv\text{(MWD)},\quad \text{when } P_0=\emptyset \text{ and } P_1=P,$$

$$(\mathrm{MD}) \equiv (\mathrm{WD}), \quad \text{when } P_0 = P \text{ and } P_1 = \emptyset.$$

We obtain that the second-order dual problems (MWD) and (MD) are the special cases of the dual problem (MD).

4. The duality theorems

In order to state and prove the duality theorems, given function $\Phi(\bullet)$ by

$$\Phi(\bullet) = \left\{ \sum_{i=1}^{k} \lambda_i \mathrm{Re}\ [f(\bullet,\eta_i) + \langle h_{P_0}(\bullet), \mu_{P_0}\rangle] \right\} \times \left(\sum_{i=1}^{k} \lambda_i \mathrm{Re}\ g(\xi,\eta_i) \right)$$
$$- \left(\sum_{i=1}^{k} \lambda_i \mathrm{Re}\ [f(\xi,\eta_i) + \langle h_{P_0}(\xi), \mu_{P_0}\rangle] \right) \times \left\{ \sum_{i=1}^{k} \lambda_i \mathrm{Re}\ g(\bullet,\eta_i) \right\},$$

where $\bullet = (\cdot, \bar{\cdot}) \in \mathbb{C}^{2n}$. Under the second-order generalized Θ-bonvexities, we could be derive the duality theorems of problem (MD).

Theorem 4.1 (Weak Duality)**.** *Let $\zeta = (z, \overline{z})$ be (P)-feasible solution, $(k, \widetilde{\lambda}, \widetilde{\eta}, \xi, \mu, \nu)$ be (MD)-feasible solution. If any one of the following conditions holds:*

(i) $\Phi(\bullet)$ *is Θ-pseudobonvex and* Re $\langle h_{P_r}(\bullet), \mu_{P_r}\rangle$ *for $r = 1, \ldots, t$ are Θ-quasibonvex in Q,*

(ii) $\Phi(\bullet)$ *is is Θ-quasibonvex and* Re $\langle h_{P_r}(\bullet), \mu_{P_r}\rangle$ *for $r = 1, \ldots, t$ are strictly Θ-pseudobonvex in Q,*

(iii) $\Phi(\bullet)$ *and* Re $\langle h_{P_r}(\bullet), \mu_{P_r}\rangle$ *for $r = 1, \ldots, t$ are Θ-bonvex at $\xi \in Q$,*

then

$$\sup_{\eta\in Y} \frac{\mathrm{Re}\ f(\zeta,\eta)}{\mathrm{Re}\ g(\zeta,\eta)} \geq \frac{\sum_{i=1}^{k} \lambda_i \mathrm{Re}\,[f(\xi,\eta_i) + \langle h_{P_0}(\xi), \mu_{P_0}\rangle]}{\sum_{i=1}^{k} \lambda_i \mathrm{Re}\big[g(\xi,\eta_i)\big]}.$$

Proof. Suppose on the contrary that

$$\sup_{\eta\in Y} \frac{\mathrm{Re}\ f(\zeta,\eta)}{\mathrm{Re}\ g(\zeta,\eta)} < \frac{\sum_{i=1}^{k} \lambda_i \mathrm{Re}\big[f(\xi,\eta_i) + \langle h_{P_0}(\xi), \mu_{P_0}\rangle\big]}{\sum_{i=1}^{k} \lambda_i \mathrm{Re}\big[g(\xi,\eta_i)\big]}.$$

Then for all $\eta \in Y$,

$$(\mathrm{Re}\ f(\zeta,\eta)) \cdot \left\{ \sum_{i=1}^{k} \lambda_i\, \mathrm{Re}\ g(\xi,\eta_i) \right\}$$
$$< (\mathrm{Re}\ g(\zeta,\eta)) \cdot \left\{ \sum_{i=1}^{k} \lambda_i\, \mathrm{Re}\big[f(\xi,\eta_i) + \langle h_{P_0}(\xi), \mu_{P_0}\rangle\big] \right\}.$$

Since $\lambda_i > 0$ with $\sum_{i=1}^{k} \lambda_i = 1$ and given $\eta_i \in Y$ for $i = 1, \cdots, k$, we obtain

$$\left[\sum_{i=1}^{k} \lambda_i \text{Re } f(\zeta, \eta_i)\right] \cdot \left\{\sum_{i=1}^{k} \lambda_i \text{ Re } g(\xi, \eta_i)\right\} - \left[\sum_{i=1}^{k} \lambda_i \text{Re } g(\zeta, \eta)\right] \cdot \left\{\sum_{i=1}^{k} \lambda_i \text{ Re } \left[f(\xi, \eta_i) + \langle h_{P_0}(\xi), \mu_{P_0}\rangle\right]\right\} < 0.$$

Since Re $\langle h_{P_0}(\zeta), \mu_{P_0}\rangle \le 0$, $\left(\sum_{i=1}^{k} \lambda_i \text{Re } g(\xi, \eta_i)\right) > 0$ and $\Phi(\xi) = 0$. Thus,

$$\Phi(\zeta) < 0 = \Phi(\xi). \tag{4.1}$$

And

$$\text{Re } \left[(\zeta-\xi)^T \nabla^2 \Phi(\xi)(\zeta-\xi)\right] = \left\{\sum_{i=1}^{k} \lambda_i \text{Re } \left\langle \nu \ , \ \nu^H \left[F_1^{(2)}(\xi, \eta_i) + H_{1;P_0}^{(2)}(\xi, \mu_{P_0})\right] + \nu^T \left[F_2^{(2)}(\xi, \eta_i) + H_{2;P_0}^{(2)}(\xi, \mu_{P_0})\right]\right\rangle\right\} \times \left(\sum_{i=1}^{k} \lambda_i \text{Re } g(\xi, \eta_i)\right) - \left(\sum_{i=1}^{k} \lambda_i \text{Re}[f(\xi, \eta_i) + \langle h_{P_0}(\xi), \mu_{P_0}\rangle]\right) \times \left\{\sum_{i=1}^{k} \lambda_i \text{Re } \left\langle \nu \ , \ \nu^H G_1^{(2)}(\xi, \eta_i) + \nu^T G_2^{(2)}(\xi, \eta_i)\right\rangle\right\}.$$

By constraint condition (3.4) of (MD), we obtain

$$\text{Re } \left[(\zeta-\xi)^T \nabla^2 \Phi(\xi)(\zeta-\xi)\right] < 0. \tag{4.2}$$

On the other hand, by (3.1) and condition (3.5) of (MD), we obtain

(4.3)

$$\text{Re}\langle h_{P_r}(\zeta) - h_{P_r}(\xi), \mu_{P_r}\rangle + \frac{1}{2}\text{Re}\left\langle \nu \ , \ \nu^H H_{1;P_r}^{(2)}(\xi, \mu_{P_r}) + \nu^T H_{2;P_r}^{(2)}(\xi, \mu_{P_r}) \ \right\rangle \le 0, \quad r = 1, \ldots, t.$$

(1) If hypothesis (i) holds, $\Phi(\bullet)$ is Θ-pseudobonvex at ξ, by inequalities (4.1) and (4.2), then

$$\text{Re } \left\{[\nabla\Phi(\xi) + (\zeta-\xi)^T \nabla^2 \Phi(\xi)]\Theta(\zeta, \xi)\right\} < 0.$$

Hence,

$$\left\{\sum_{i=1}^{k} \lambda_i \Big(\left[F^{(1)}(\xi, \eta_i) + H_{P_0}^{(1)}(\xi, \mu_{P_0})\right] + \nu^H \left[F_1^{(2)}(\xi, \eta_i) + H_{1;P_0}^{(2)}(\xi, \mu_{P_0})\right] + \nu^T \left[F_2^{(2)}(\xi, \eta_i) + H_{2;P_0}^{(2)}(\xi, \mu_{P_0})\right]\Big)\right\} \times \left(\sum_{i=1}^{k} \lambda_i \text{Re } g(\xi, \eta_i)\right) - \left(\sum_{i=1}^{k} \lambda_i \text{Re}[f(\xi, \eta_i) + \langle h_{P_0}(\xi), \mu_{P_0}\rangle]\right) \tag{4.4}$$

$$\times \left\{ \sum_{i=1}^{k} \lambda_i \left[G^{(1)}(\xi, \eta_i) + \nu^H G_1^{(2)}(\xi, \eta_i) + \nu^T G_2^{(2)}(\xi, \eta_i) \right] \right\} < 0.$$

Since Re $\langle h_{P_r}(\bullet), \mu_{P_r} \rangle$ for $r = 1, \ldots, t$ aRe Θ-quasibonvex at ξ, then by (4.3) and Lemma 2.2(b), we have

$$\sum_{r=1}^{t} \left[H_{P_r}^{(1)}(\xi, \mu_{P_r}) + \nu^H H_{1;P_r}^{(2)}(\xi, \mu_{P_r}) + \nu^T H_{2;P_r}^{(2)}(\xi, \mu_{P_r}) \right] \leq 0. \tag{4.5}$$

From (4.4) and (4.5), it implies that

$$\begin{aligned} &\left\{ \sum_{i=1}^{k} \lambda_i \Big(\left[F^{(1)}(\xi, \eta_i) + H_{P_0}^{(1)}(\xi, \mu_{P_0}) \right] \right. \\ &\quad \left. + \nu^H \left[F_1^{(2)}(\xi, \eta_i) + H_{1;P_0}^{(2)}(\xi, \mu_{P_0}) \right] + \nu^T \left[F_2^{(2)}(\xi, \eta_i) + H_{2;P_0}^{(2)}(\xi, \mu_{P_0}) \right] \Big) \right\} \\ &\quad \times \left(\sum_{i=1}^{k} \lambda_i \mathrm{Re} g(\xi, \eta_i) \right) - \left(\sum_{i=1}^{k} \lambda_i \mathrm{Re}[f(\xi, \eta_i) + \langle h_{P_0}(\xi), \mu_{P_0} \rangle] \right) \\ &\quad \times \left\{ \sum_{i=1}^{k} \lambda_i \left[G^{(1)}(\xi, \eta_i) + \nu^H G_1^{(2)}(\xi, \eta_i) + \nu^T G_2^{(2)}(\xi, \eta_i) \right] \right\} \\ &\quad + \sum_{r=1}^{t} \left[H_{P_r}^{(1)}(\xi, \mu_{P_r}) + \nu^H H_{1;P_r}^{(2)}(\xi, \mu_{P_r}) + \nu^T H_{2;P_r}^{(2)}(\xi, \mu_{P_r}) \right] < 0. \end{aligned} \tag{4.6}$$

This contradicts the constraint condition (3.3) of (MD).

(2) If hypothesis (*ii*) holds, $\Phi(\bullet)$ is Θ-quasibonvex at ξ, then the inequality (4.4) is become to less than or equal to zero. Since Re $\langle h_{P_r}(\bullet), \mu_{P_r} \rangle$ for $r = 1, \ldots, t$ are strictly Θ-pseudobonvex in Q, then the inequality (4.5) is become to less than zero. The inequality (4.6) holds, it still contradicts the constraint condition (3.3), we done.

(3) If hypothesis (*iii*) holds, $\Phi(\bullet)$ and Re $\langle h_{P_r}(\bullet), \mu_{P_r} \rangle$ for $r = 1, \ldots, t$ are Θ-bonvex at $\xi \in Q$. From Lemma 2.4(a), $\Phi(\bullet)$ is Θ-pseudobonvex, Re $\langle h_{P_r}(\bullet), \mu_{P_r} \rangle$ for $r = 1, \ldots, t$ are Θ-quasibonvex in Q, it is become to the hypotheses of (*i*), we done.

This is the complete proof. □

Theorem 4.2. *(Strong Duality)*
Let $\zeta_0 = (z_0, \overline{z_0})$ be an optimal solution of problem (P). Then there are $(k, \widetilde{\lambda}, \widetilde{\eta}) \in K(\zeta_0)$ and $(\zeta_0, \mu, v) \in X(k, \widetilde{\lambda}, \widetilde{\eta})$ such that $(k, \widetilde{\lambda}, \widetilde{\eta}, \zeta_0, \mu, v)$ is a feasible solution of the dual problem (MD). If the hypotheses of weak duality theorem are fulfilled, then $(k, \widetilde{\lambda}, \widetilde{\eta}, \zeta_0, \mu, v)$ is an optimal solution of (MD), and problems (P) and (MD) have the same optimal values.

Proof. Let ζ_0 is an optimal solution of (P) with optimal value

$$\gamma^* = \phi(\zeta_0) = \frac{\sum_{i=1}^{k} \lambda_i \mathrm{Re} f(\zeta_0, \eta_i)}{\sum_{i=1}^{k} \lambda_i \mathrm{Re} g(\zeta_0, \eta_i)}.$$

From Theorem 1.1, we could obtain the non-zero $\mu \in S^* \subset \mathbb{C}^p$, positive integer k with $\eta_i \in Y(\zeta_0)$, multipliers $\lambda_i > 0$ for $i = 1, \ldots, k$ and $\sum_{i=1}^k \lambda_i = 1$ such that

$$\left\{ \sum_{i=1}^k \lambda_i \left[\overline{\nabla_z f(\zeta_0, \eta_i)} + \nabla_{\overline{z}} f(\zeta_0, \eta_i) \right] + \mu^T \ \overline{\nabla_z h(\zeta_0)} + \mu^H \nabla_{\overline{z}} h(\zeta_0) \right\} \times \left(\sum_{i=1}^k \lambda_i \ \mathrm{Re}\ g(\zeta_0, \eta_i) \right) - \left(\sum_{i=1}^k \lambda_i \ \mathrm{Re}\ f(\zeta_0, \eta_i) \right) \times \left\{ \sum_{i=1}^k \lambda_i \left[\overline{\nabla_z g(\zeta_0, \eta_i)} + \nabla_{\overline{z}} g(\zeta_0, \eta_i) \right] \right\} = 0.$$

Since $\mu = (\mu_1, \ldots, \mu_p) = (\mu_{P_0}, \mu_{P_1}, \ldots, \mu_{P_t})$ for the partition $P = P_0 \cup P_1 \cup \cdots \cup P_t$. If we replace μ_{P_r} by $\mu_{P_r} \times \left(\sum_{i=1}^k \lambda_i \ \mathrm{Re}\ g(\zeta_0, \eta_i) \right)$ for $r = 1, \ldots, t$, and take $\nu = z_0 - z_0 = 0$, then

$$\left\{ \sum_{i=1}^k \lambda_i \Big([F^{(1)}(\zeta_0, \eta_i) + H_{P_0}^{(1)}(\zeta_0, \mu_{P_0})] + \nu^H [F_1^{(2)}(\zeta_0, \eta_i) + H_{1;P_0}^{(2)}(\zeta_0, \mu_{P_0})] + \nu^T [F_2^{(2)}(\zeta_0, \eta_i) + H_{2;P_0}^{(2)}(\zeta_0, \mu_{P_0})] \Big) \right\} \times \left(\sum_{i=1}^k \lambda_i \mathrm{Re}\ g(\zeta_0, \eta_i) \right) - \left(\sum_{i=1}^k \lambda_i \mathrm{Re}[f(\zeta_0, \eta_i) + \langle h_{P_0}(\zeta_0), \mu_{P_0} \rangle] \right) \times \left\{ \sum_{i=1}^k \lambda_i [G^{(1)}(\zeta_0, \eta_i) + \nu^H G_1^{(2)}(\zeta_0, \eta_i) + \nu^T G_2^{(2)}(\zeta_0, \eta_i)] \right\} + \sum_{r=1}^t \left[H_{P_r}^{(1)}(\zeta_0, \mu_{P_r}) + \nu^H H_{1;P_r}^{(2)}(\zeta_0, \mu_{P_r}) + \nu^T H_{2;P_r}^{(2)}(\zeta_0, \mu_{P_r}) \right] = 0.$$

The component $(\zeta_0, \mu, \nu = 0) \in X_1(k, \widetilde{\lambda}, \widetilde{\eta})$ is satisfying conditions (3.3) $\sim$ (3.5) of problem (MD). It follows that $(k, \widetilde{\lambda}, \widetilde{\eta}, \zeta_0, \mu, \nu = 0)$ is a feasible solution of (MD). If the hypotheses of Theorem 4.1 are fulfilled, then $(k, \widetilde{\lambda}, \widetilde{\eta}, \zeta_0, \mu, \nu = 0)$ is an optimal solution of (MD), and the two problems (P) and (MD) have the same optimal values. □

Theorem 4.3. *(Strict Converse Duality)*
Let ζ and $(k, \widetilde{\lambda}, \widetilde{\eta}, \xi, \mu, v)$ be optimal solutions of (P) and (MD), and assume that the assumptions of strong duality theorem are fulfilled. In addition, if $\Phi(\bullet)$ is strictly Θ-pseudobonvex and $\mathrm{Re}\langle h_{P_r}(\bullet), \mu_{P_r} \rangle$, $r = 1, \ldots, t$ are Θ-quasibonvex, then $\zeta = \xi$ and the optimal values of (P) and (MD) are equal.

Proof. Assume that $\zeta \neq \xi$, and reach a contradiction.

By strong duality theorem (Theorem 4.2),

$$\sup_{\eta\in Y}\frac{\text{Re } f(\zeta,\eta)}{\text{Re } g(\zeta,\eta)}=\frac{\sum_{i=1}^{k}\lambda_i\text{Re}\big[f(\xi,\eta_i)+\langle h_{P_0}(\xi),\mu_{P_0}\rangle\big]}{\sum_{i=1}^{k}\lambda_i\text{Re}\big[g(\xi,\eta_i)\big]}.$$

Then for all $\eta\in Y$,

$$(\text{Re } f(\zeta,\eta))\cdot\left\{\sum_{i=1}^{k}\lambda_i\text{ Re } g(\xi,\eta_i)\right\}\leq(\text{Re } g(\zeta,\eta))\cdot\left\{\sum_{i=1}^{k}\lambda_i\text{ Re }\big[f(\xi,\eta_i)+\langle h_{P_0}(\xi),\mu_{P_0}\rangle\big]\right\}.$$

Since $\lambda_i>0$ with $\sum_{i=1}^{k}\lambda_1=1$ and given $\eta_i\in Y$ for $i=1,\dots,k$, we have that

$$\left(\sum_{i=1}^{k}\lambda_i\text{Re } f(\zeta,\eta_i)\right)\cdot\left\{\sum_{i=1}^{k}\lambda_i\text{ Re } g(\xi,\eta_i)\right\}-\left(\sum_{i=1}^{k}\lambda_i\text{Re } g(\zeta,\eta)\right)\cdot\left\{\sum_{i=1}^{k}\lambda_i\text{ Re }\big[f(\xi,\eta_i)+\langle h_{P_0}(\xi),\mu_{P_0}\rangle\big]\right\}\leq 0.$$

Since Re $\langle h_{P_0}(\zeta),\mu_{P_0}\rangle\leq 0$, $\left(\sum_{i=1}^{k}\lambda_i\text{Re } g(\xi,\eta_i)\right)>0$ and $\Phi(\xi)=0$. Thus,

$$\Phi(\zeta)\leq 0=\Phi(\xi).$$

If $\Phi(\bullet)$ is strictly Θ-pseudobonvex at ξ, and by similar process as the proof of Theorem 4.1, then inequality (4.4) still holds. Since Re $\langle h_{P_r}(\bullet),\mu_{P_r}\rangle$ for $r=1,\dots,t$ are Θ-quasibonvex, then inequality (4.5) still holds. By (4.4) and (4.5), the inequality (4.6) holds. It still contradicts the equality of (3.3). Therefore, the result of theorem is proved. □

References

[1] I. Ahmad, Z. Husain and S. Sharma, *Second order duality in nondifferentiable minimax programming involving type-I functions*, J. Comput. Appl. Math. **215** (2008), 91–102.

[2] I. Ahmad, *Second-order nondifferentiable minimax fractional programming with square root terms*, Filomat **27** (2013), 135–142.

[3] C. R. Bector, S. Chandra and I. Husain, *Second-order duality for a minimax programming problem*, Opsearch **28** (1991), 249–263.

[4] S. K. Gupta and D. Dangar, *On Second-order nondifferentiable minimax fractional programming*, J. Comput. Appl. Math. **225** (2014), 878–886.

[5] Q. Hu, G. Yang and J. Jian, *On second-order duality for minimax fractional programming*, Nonlinear Anal. Real World Appl. **12** (2011), 3509–3514.

[6] Q. Hu, Y. Chen and J. Jian, *Second-order duality for non-differentiable minimax programming*, Int. J. Compu. Math. **89(1)** (2012), 11–16.

[7] T. Y. Huang, *Second-order duality for a non-differentiable minimax programming in complex spaces*, Int. J. Compu. Math. **94(12)** (2017), 2508–2519.
[8] T. Y. Huang and H. C. Lai ,*Second-order parametric duality for a minimax fractional programming in complex spaces*, J. Nonlinear Convex Anal. **18(9)** (2017), 1685–1697.
[9] T. Y. Huang, *Second-order Mond-Weir type and Wolfe type dualities for minimax fractional programming in complex spaces*, submitted.
[10] Z. Husain, I. Ahmad and S. Sharma, *second order duality for minimax fractional programming*, Optim. Lett. **3** (2009) 277–286.
[11] Z. Husain, A. Jayswal and I. Ahmad, *Second-order duality for nondifferentiable minimax programming problems with generalized convexity*, J. Global Optim. **44** (2009), 593–608.
[12] H. C. Lai, J. C. Liu and S. Schaible, *Complex minimax fractional programming of analytic functions*, J. Optim. Theory Appl. **137** (2008), 171–184.
[13] H. C. Lai and T. Y. Huang, *Optimality conditions for a nondifferentiable minimax programming in complex spaces*, Nonlinear Anal. **71** (2009), 1205–1212.
[14] H. C. Lai and T. Y. Huang, *Mixed type duality for a nondiffrentiable minimax fractional complex programming*, Pac. J. Optim. **10(2)** (2014), 305–319.
[15] S. K. Mishra and N. G. Rueda, *Second order duality for nondifferentiable minimax programming involving generalized type-I functions*, J. Optim. Theory Appl., **130** (2006) 479–488.

Manuscript received 17 July 2018

T. Y. Huang
Department of Applied Mathematics, Feng-Chia University, Tai-Chung 40724, Taiwan.
E-mail address: huangty@fcu.edu.tw toneyau@yahoo.com.tw