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CONVERGENCE ANALYSIS OF MODIFIED PICARD-S HYBRID WITH ERRORS FOR TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

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Abstract. In this paper, we construct a new type iterative scheme is so call Picard-S hybrid with errors to prove Δ-convergence and strong convergence theorems under suitable conditions for total asymptotically nonexpansive mappings in CAT(0) spaces. Our results in the paper improve and extend many results appeared in the literature. Furthermore, we also illustrate numerical examples of proposed iterative scheme to compare speed of convergence among the existing iterative schemes.

1. Introduction

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a mapping c from a closed interval $[0, r] \subset \mathbb{R}$ to X such that

$$c(0) = x, c(r) = y, \quad d(c(t), c(s)) = |t - s|$$

for all $s, t \in [0, r]$. In particular, c is an isometry and $d(x, y) = r$. The image of c is call a *geodesic* (*or metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. We denote the point $w \in [x, y]$ such that $d(x, w) = \alpha d(x, y)$ by $w = (1 - \alpha)x \oplus \alpha y$, where $\alpha \in [0, 1]$.

The space (X, d) is called a *geodesic space* if any two points of X are joined by a geodesic and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $D \subseteq X$ is said to be *convex* if D includes geodesic segment joining every two points of itself. A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consist of three points (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A *comparison triangle* for geodesic triangle (or $\triangle(x_1, x_2, x_3)$) in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) = \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that

$$d_{\mathbb{R}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$$

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for $i, j \in \{1, 2, 3\}$. A geodesic metric space is called a *CAT(0) space* (see, [2, 5, 6, 13, 14, 15, 17, 20]) if all geodesic triangle satisfy the following comparison axiom:

Let $\triangle$ be a geodesic triangle in X and $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the *CAT(0) inequality* if, for all $x, y \in \triangle$ and all comparison points,

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}).$$

If x, y_1, y_2 are points of a CAT(0) space and y_0 is the midpoint of the segment $[y_1, y_2]$, which is denoted by $\frac{y_1 \oplus y_2}{2}$, then the CAT(0) inequality implies

$$d^2\Big(x, \frac{y_1 \oplus y_2}{2}\Big) \leq \frac{1}{2} d^2(x, y_1) + \frac{1}{2} d^2(x, y_2) - \frac{1}{4} d^2(y_1, y_2). \tag{1.1}$$

The inequality (1.1) is called the (CN) *inequality* (for more details, see Bruhat and Titz [4]). In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

It is well known that all complete, simply connected Riemannian manifold having non-positive section curvature is a CAT(0) space. For other examples, Euclidean buildings [3], Pre-Hilbert spaces, R-trees [2], the complex Hilbert ball with a Hyperbolic metric [9] is a CAT(0) space. Further, complete CAT(0) spaces are called *Hadamard spaces.*

In the sequel, we give some fundamental for nonlinear mappings in CAT(0) spaces. Let C be a nonempty closed subset of a CAT(0) space X. Let $T : C \to C$ be a self-mapping. Recall that a mapping T is said to be:

- (M_1) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ $\forall x, y \in C$;
- (M_2) asymptotically nonexpansive if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that
  $d(T^n x, T^n y) \leq k_n d(x, y)$ $\forall x, y \in C$ and $\forall n \geq 1$;
- (M_3) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that
  $d(T^n x, T^n y) \leq L d(x, y)$ $\forall x, y \in C$ and $\forall n \geq 1$.

In 2006, Alber *et al.* [1] first introduced the concept of total asymptotically nonexpansive mappings. Recall that mapping $T : C \to C$ is said to be $(\{\mu_n\}, \{\nu_n\}, \psi)$ total asymptotically nonexpansive (or briefly, total asymptotically nonexpansive) if there exist nonnegative real sequences $\{\mu_n\}$ and $\{\nu_n\}$ with $\mu_n \to 0, \nu_n \to 0$ as $n \to \infty$ and a continuous strictly increasing function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x, y) + \mu_n \psi(d(x, y)) + \nu_n$$

for all $x, y \in C$ and $n \geq 1$.
From definition, it is easy to see that, each nonexpansive mapping is asymptotically nonexpansive with $\{k_n = 1\}$, $\forall n \geq 1$, and each asymptotically nonexpansive mapping is total asymptotically nonexpansive mapping with $\mu_n = k_n - 1$, $\nu_n = 0$, $\forall n \geq 1$ and $\psi(t) = t$, $t \geq 0$. Also, for each asymptotically nonexpansive mapping is a uniformly L-Lipschitzian mapping with $L = \sup_{n\in N}\{k_n\}$.

A point $x \in C$ is called a fixed point of T if $x = T(x)$. We denote by $F(T)$ the set of fixed points of T. A sequence $\{x_n\}$ in C is called an approximation fixed point sequence of T if

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

On the other hand, in 2015 Thakur et al. [21] introduced the modified Picard-Mann hybrid iterative scheme for approximating a fixed point of a total asymptotically nonexpansive mapping T in complete CAT(0) spaces, the sequence $\{x_n\}$ is defined by

$$\begin{cases} x_1 \in C, \\ w_n = (1-\alpha_n)x_n \oplus \alpha_n T^n x_n, \\ x_{n+1} = T^n w_n \end{cases} \tag{1.2}$$

for all $n \geq 1$, where $\{\alpha_n\}$ is a real appropriate sequence in $[0,1]$. They proved some convergence theorems under mild conditions and they also examined speed convergence between the modified Picard-Mann hybrid iterative scheme (1.2) and the modified Mann iterative scheme.

Recently, Pansuwan and Sintunavarat [18] introduced the Picard-Ishikawa hybrid iterative scheme, the sequence $\{x_n\}$ is given by

$$\begin{cases} x_1 \in C, \\ w_n = (1-\alpha_n)x_n \oplus \alpha_n T^n x_n, \\ y_n = (1-\beta_n)w_n \oplus \beta_n T^n w_n \\ x_{n+1} = T^n y_n \end{cases} \tag{1.3}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real appropriate sequences in [0,1]. They proved strong and Δ-convergence theorems of iteration process (1.3) for total asymptotically nonexpansive mappings in complete CAT(0) spaces and they also gave numerical examples and convergence behavior to support the results.

Inspired and motivated by the above works, in this paper, we introduce a new type iterative scheme called "modified Picard-S hybrid with errors" which is defined the following manner:

$$\begin{cases} x_1 \in C, \\ w_n = \alpha_n^{(1)} x_n \oplus \alpha_n^{(2)} T^n x_n \oplus \alpha_n^{(3)} e_n^{(1)}, \\ y_n = \beta_n^{(1)} T^n x_n \oplus \beta_n^{(2)} T^n w_n \oplus \beta_n^{(3)} e_n^{(2)} \\ x_{n+1} = \gamma_n^{(1)} T^n y_n \oplus \gamma_n^{(2)} e_n^{(3)} \end{cases} \tag{1.4}$$

for all $n \geq 1$, where $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \{\alpha_n^{(3)}\}, \{\beta_n^{(1)}\}, \{\beta_n^{(2)}\}, \{\beta_n^{(1)}\}, \{\gamma_n^{(1)}\}, \{\gamma_n^{(2)}\}$ are real appropriate sequences in [0,1] and $\{e_n^{(1)}\}, \{e_n^{(2)}\}, \{e_n^{(3)}\}$ are bounded sequences in C. Under suitable conditions, we establish Δ-convergence theorem and strong theorem of iterative scheme (1.4) for total asymptotically nonexpansive mappings in framework of complete CAT(0) spaces. Moreover, we also give numerical examples to illustrate the speed of the convergence of proposed iterative scheme to compare with others iterative schemes.

2. Preliminaries

In this section, we give some elementary properties about CAT(0) spaces as follows:

Lemma 2.1 ([7]). *Let X be a CAT(0) space. Then, for any $x, y, z \in X$ and $\lambda \in [0,1]$,*

$$d\left((1-\lambda)\, x \oplus \lambda y, z\right) \leq (1-\lambda)\, d\,(x,z) + \lambda d\,(y,z)\,.$$

Lemma 2.2 ([7]). *Let X be a CAT(0) space. Then, for any $x, y, z \in X$ and $\lambda \in [0,1]$,*

$$d^2\left((1-\lambda)\, x \oplus \lambda y, z\right) \leq (1-\lambda)\, d^2\,(x,z) + \lambda d^2\,(y,z) - \lambda\,(1-\lambda)\, d^2\,(x,y)\,.$$

In 2008, Kirk and Panyanak [10] specialized Lim's concept of the Δ-convergence in a general metric space [16] to CAT(0) spaces and proved some weak convergence theorems in Banach spaces by using the Δ-convergence. Now, we introduce some basic properties and the concept of the Δ-convergence.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space (X,d). For any $x \in X$, we put

$$r(x,\{x_n\}) = \limsup_{n\to\infty} \rho(x,\{x_n\}).$$

(1) The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\};$$

(2) The *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}.$$

It is well known that, in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point (see [8]).

(3) A sequence $\{x_n\} \subset X$ is said to be Δ-*convergent* to a point $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

In 2014, Karapinar *et al.* [11] proved the existence theorem of fixed points for uniformly continuous and total asymptotically nonexpansive mappings in CAT(0) spaces which is also useful in our main results.

Lemma 2.3 ([11]). *Let C be a nonempty bounded closed convex subset of a complete CAT(0) space (X,d) and $T : C \to C$ be a uniformly continuous total asymptotically nonexpansive mapping. Then T has a fixed point and the fixed point set of $F(T)$ are closed and convex.*

Lemma 2.4 ([22]). *Let $\{a_n\}$, $\{\lambda_n\}$ and $\{c_n\}$ be the sequences of nonnegative numbers such that*

$$a_{n+1} \leq (1+\lambda_n)a_n + c_n,$$

for all $n \geq 1$. If $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. Whenever, if there exists a subsequence $\{a_{n_i}\} \subseteq \{a_n\}$ such that $a_{n_i} \to 0$ as $i \to \infty$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. *Suppose that (X,d) is a complete CAT(0) space . Then the following statement hold:*

- (S_1) *every bounded sequence in X always has a Δ-convergent subsequence (see* [12]*);*
- (S_2) *if $\{x_n\}$ is a bounded sequence in a closed convex subset C of X, then the asymptotic center of $\{x_n\}$ is in C (see* [8]*);*
- (S_3) *if $\{x_n\}$ is a bounded sequence X with $A(\{x_n\})=\{p\},\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\})=\{u\}$ and a sequence $\{d(x_n,u)\}$ converges, then $p=u$ (see* [7]*).*

Lemma 2.6 ([11])**.** *Let C be a closed convex subset of a complete CAT(0) space (X,d) and $T:C\to C$ be a uniformly continuous and total asymptotically nonexpansive mapping. If $\{x_n\}$ is a bounded sequence in C such that $\lim_{n\to\infty} d(x_n,Tx_n)=0$ and $\Delta-\lim_{n\to\infty} x_n=w$, then $Tw=w$.*

3. Main results

Theorem 3.1. *Let C be a bounded closed convex subset of a complete CAT(0) space (X,d) and $T:C\to C$ be a uniformly L-Lipschitzian and $(\{\mu_n\},\{\nu_n\},\psi)$-total asymptotically nonexpansive mapping. Suppose that the following conditions are satisfied:*

- (C_1) $\sum_{n=1}^{\infty}\mu_n<\infty$ *and* $\sum_{n=1}^{\infty}\nu_n<\infty$*;*
- (C_2) $\sum_{n=1}^{\infty} d(e_n^{(1)},p)<\infty$, $\sum_{n=1}^{\infty} d(e_n^{(2)},p)<\infty$ *and* $\sum_{n=1}^{\infty} d(e_n^{(3)},p)<\infty$ *for all* $p\in F(T)$*;*
- (C_3) *there exist constants b,d with $0<b\le \alpha_n^{(1)},\alpha_n^{(2)},\alpha_n^{(3)}\le d<1$ such that $\alpha_n^{(1)}+\alpha_n^{(2)}+\alpha_n^{(3)}=1$ for each $n\in\mathbb{N}$;*
- (C_4) *there exist constants a,c with $0<a\le \beta_n^{(1)},\beta_n^{(2)},\beta_n^{(3)}\le c<1$ such that $\beta_n^{(1)}+\beta_n^{(2)}+\beta_n^{(3)}=1$ for each $n\in\mathbb{N}$;*
- (C_5) *there exist constants f,h with $0<f\le\gamma_n^{(1)},\gamma_n^{(2)}\le h<1$ such that $\gamma_n^{(1)}+\gamma_n^{(2)}=1$ for each $n\in\mathbb{N}$;*
- (C_6) $\lim_{n\to\infty}\alpha_n^{(3)}=lim_{n\to\infty}\beta_n^{(3)}=lim_{n\to\infty}\gamma_n^{(2)}=0$*;*
- (C_7) *there exists a constant M^* such that $\psi(r)\le M^*r$ for each $r\ge 0$.*

Then the sequence $\{x_n\}$ defined by (1.4) Δ-converges to a fixed point of T.

Proof. By using Lemma 2.3, we get $F(T)\neq\emptyset$. Next, we will divide the proof into three steps.

Step1 First, we will prove that $\lim_{n\to\infty} d(x_n,p)$ exists for all $p\in F(T)$, where the sequence $\{x_n\}$ is generated by (1.4). Let $p\in F(T)$. By Lemma 2.1 and (1.4) we have

$$\begin{aligned}
d(w_n,p) &= d(\alpha_n^{(1)}x_n\oplus\alpha_n^{(2)}T^nx_n\oplus\alpha_n^{(3)}e_n^{(1)},p)\\
&\le \alpha_n^{(1)}d(x_n,p)+\alpha_n^{(2)}d(T^nx_n,p)+\alpha_n^{(3)}d(e_n^{(1)},p)\\
&\le \alpha_n^{(1)}d(x_n,p)+\alpha_n^{(2)}\left[d(x_n,p)+\mu_n\psi(d(x_n,p))+\nu_n\right]+\alpha_n^{(3)}d(e_n^{(1)},p)\\
&= (\alpha_n^{(1)}+\alpha_n^{(2)})d(x_n,p)+\alpha_n^{(2)}\left[\mu_n\psi(d(x_n,p))+\nu_n\right]+\alpha_n^{(3)}d(e_n^{(1)},p)\\
&= (1-\alpha_n^{(3)})d(x_n,p)+\alpha_n^{(2)}\left[\mu_n\psi(d(x_n,p))+\nu_n\right]+\alpha_n^{(3)}d(e_n^{(1)},p)
\end{aligned}$$

$$\begin{aligned} &\leq d(x_n,p)+[\mu_n\psi(d(x_n,p))+\nu_n]+d(e_n^{(1)},p) \\ &\leq (1+\mu_n M^*)d(x_n,p)+\nu_n+d(e_n^{(1)},p) \qquad (3.1)\end{aligned}$$

for each $n \in \mathbb{N}$. Also, we have

$$\begin{aligned} d(y_n,p) &= d(\beta_n^{(1)}T^n x_n \oplus \beta_n^{(2)}T^n w_n \oplus \beta_n^{(3)}e_n^{(2)},p) \\ &\leq \beta_n^{(1)}d(T^n x_n,p)+\beta_n^{(2)}d(T^n w_n,p)+\beta_n^{(3)}d(e_n^{(2)},p) \\ &\leq \beta_n^{(1)}[d(x_n,p)+\mu_n\psi(d(x_n,p))+\nu_n] \\ &\quad +\beta_n^{(2)}[d(w_n,p)+\mu_n\psi(d(w,p))+\nu_n]+\beta_n^{(3)}d(e_n^{(2)},p) \\ &\leq \beta_n^{(1)}[(1+\mu_n M^*)d(x_n,p) \\ &\quad +\nu_n]+\beta_n^{(2)}[(1+\mu_n M^*)d(w_n,p)+\nu_n]+\beta_n^{(3)}d(e_n^{(2)},p) \\ &\leq \beta_n^{(1)}[(1+\mu_n M^*)d(x_n,p)+\nu_n] \\ &\quad +\beta_n^{(2)}[(1+\mu_n M^*)[(1+\mu_n M^*)d(x_n,p)+\nu_n+d(e_n^{(1)},p)]+\nu_n] \\ &\quad +\beta_n^{(3)}d(e_n^{(2)},p) \\ &= \beta_n^{(1)}[(1+\mu_n M^*)d(x_n,p)+\nu_n] \\ &\quad +\beta_n^{(2)}[(1+\mu_n M^*)^2 d(x_n,p)+(1+\mu_n M^*)\nu_n \\ &\quad +(1+\mu_n M^*)d(e_n^{(1)},p)+\nu_n]+\beta_n^{(3)}d(e_n^{(2)},p) \\ &\leq (1+\mu_n M^*)^2 d(x_n,p)+(2+\mu_n M^*)\nu_n \\ &\quad +(1+\mu_n M^*)d(e_n^{(1)},p)+d(e_n^{(2)},p), \qquad (3.2)\end{aligned}$$

for each $n \in \mathbb{N}$. From (1.4), (3.1) and (3.2), for each $n \in \mathbb{N}$, we get

$$\begin{aligned} d(x_{n+1},p) &= d(\gamma_n^{(1)}T^n y_n \oplus \gamma_n^{(2)}e_n^{(3)},p) \\ &\leq \gamma_n^{(1)}d(T^n y_n,p)+\gamma_n^{(2)}d(e_n^{(3)},p) \\ &\leq \gamma_n^{(1)}[d(y_n,p)+\mu_n\psi(d(y_n,p))+\nu_n]+\gamma_n^{(2)}d(e_n^{(3)},p) \\ &\leq \gamma_n^{(1)}[(1+\mu_n M^*)d(y_n,p)+\nu_n]+\gamma_n^{(2)}d(e_n^{(3)},p) \\ &\leq \gamma_n^{(1)}[(1+\mu_n M^*)[(1+\mu_n M^*)^2 d(x_n,p) \\ &\qquad +(2+\mu_n M^*)\nu_n+(1+\mu_n M^*)d(e_n^{(1)},p)+d(e_n^{(2)},p)]+\nu_n] \\ &\qquad\quad +\gamma_n^{(2)}d(e_n^{(3)},p) \\ &= (1+\mu_n M^*)^3 d(x_n,p)+(2+\mu_n M^*)(1+\mu_n M^*)\nu_n \\ &\quad +(1+\mu_n M^*)^2 d(e_n^{(1)},p)+(1+\mu_n M^*)d(e_n^{(2)},p) \\ &\quad +\nu_n+d(e_n^{(3)},p) \\ &= (1+\mu_n M^*)^3 d(x_n,p)+(3+3\mu_n M^*+(\mu_n M^*)^2)\nu_n \\ &\quad +(1+2\mu_n M^*+(\mu_n M^*)^2)d(e_n^{(1)},p) \\ &\quad +(1+\mu_n M^*)d(e_n^{(2)},p)+d(e_n^{(3)},p) \\ &= (1+\xi_n)d(x_n,p)+\delta_n, \qquad (3.3)\end{aligned}$$

where $\xi_n := [3M^*+3\mu_n(M^*)^2+\mu_n^2(M^*)^3]\mu_n$ and

$$\delta_n := (3+3\mu_n M^*+(\mu_n M^*)^2)\nu_n+(1+2\mu_n M^*+(\mu_n M^*)^2)d(e_n^{(1)},p)$$

$$+ (1+\mu_n M^*)d(e_n^{(2)}, p) + d(e_n^{(3)}, p).$$

By assumption (C_1) and (C_2), we have

$$\sum_{n=1}^{\infty} \xi_n < \infty \quad and \quad \sum_{n=1}^{\infty} \delta_n < \infty \tag{3.4}$$

By assertion (3.3), (3.4) and Lemma 2.4, we obtain $\lim_{n\to\infty} d(x_n, p)$ exists.

Step 2 Next, we will show that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Without loss of generality, we can assume that

$$r := \lim_{n\to\infty} d(x_n, p) \geq 0. \tag{3.5}$$

From (3.1), we have

$$\lim_{n\to\infty} d(w_n, p) \leq r. \tag{3.6}$$

And also (3.2), we have

$$\lim_{n\to\infty} d(y_n, p) \leq r. \tag{3.7}$$

By Lemma 2.2 and (1.4) we have

$$\begin{aligned}
d^2(w_n, p) &= d^2(\alpha_n^{(1)} x_n \oplus \alpha_n^{(2)} T^n x_n \oplus \alpha_n^{(3)} e_n^{(1)}, p) \\
&\leq \alpha_n^{(1)} d^2(x_n, p) + \alpha_n^{(2)} d^2(T^n x_n, p) \\
&\quad + \alpha_n^{(3)} d^2(e_n^{(1)}, p) - \alpha_n^{(1)} \alpha_n^{(2)} d^2(x_n, T^n x_n) \\
&\leq \alpha_n^{(1)} d^2(x_n, p) + \alpha_n^{(2)} [(1+\mu_n M^*) d(x_n, p) + \nu_n]^2 \\
&\quad + \alpha_n^{(3)} d^2(e_n^{(1)}, p) - \alpha_n^{(1)} \alpha_n^{(2)} d^2(x_n, T^n x_n) \\
&= \alpha_n^{(1)} d^2(x_n, p) + \alpha_n^{(2)} [(1+\mu_n M^*)^2 d^2(x_n, p) \\
&\quad + 2(1+\mu_n M^*) d^2(x_n, p)\nu_n + \nu_n^2] + \alpha_n^{(3)} d^2(e_n^{(1)}, p) \\
&\quad - \alpha_n^{(1)} \alpha_n^{(2)} d^2(x_n, T^n x_n) \\
&= (1-\alpha_n^{(3)}) d^2(x_n, p) + \alpha_n^{(2)} [(2+\mu_n M^*) M^* d^2(x_n, p)\mu_n \\
&\quad + (2(1+\mu_n M^*) d(x_n, p) + \nu_n)\nu_n] \\
&\quad + \alpha_n^{(3)} d^2(e_n^{(1)}, p) - \alpha_n^{(1)} \alpha_n^{(2)} d^2(x_n, T^n x_n) \\
&\leq d^2(x_n, p) + ((2+\mu_n M^*) M^* d^2(x_n, p))\mu_n \\
&\quad + (2(1+\mu_n M^*) d(x_n, p) + \nu_n)\nu_n \\
&\quad + \alpha_n^{(3)} d^2(e_n^{(1)}, p) - \alpha_n^{(1)} \alpha_n^{(2)} d^2(x_n, T^n x_n), \qquad (3.8)
\end{aligned}$$

where $Q_n = (2+\mu_n M^*) M^* d^2(x_n, p)$ and $R_n = 2(1+\mu_n M^*) d(x_n, p) + \nu_n$, which yield

$$d^2(w_n, p) \leq d^2(x_n, p) + Q_n \mu_n + R_n \nu_n + \alpha_n^{(3)} d^2(e_n^{(1)}, p). \tag{3.9}$$

It follows from (3.8) we have

$$\alpha_n^{(1)} \alpha_n^{(2)} d^2(x_n, T^n x_n) \leq d^2(x_n, p) - d^2(w_n, p) + Q_n \mu_n + R_n \nu_n + \alpha_n^{(3)} d^2(e_n^{(1)}, p).$$

By assumptions (C_1), (C_2), (C_3) and assertions (3.5), (3.6), (3.8) we get

$$d^2(x_n, T^n x_n) \leq \frac{1}{bd} \left[d^2(x_n, p) - d^2(w_n, p) + Q_n \mu_n + R_n \nu_n + \alpha_n^{(3)} d^2(e_n^{(1)}, p) \right]$$

$$\to \quad 0 \qquad as \qquad n\to\infty. \tag{3.10}$$

Since

$$\begin{aligned}
d^2(y_n,p) &= d^2(\beta_n^{(1)}T^nx_n \oplus \beta_n^{(2)}T^nw_n \oplus \beta_n^{(3)}e_n^{(2)},p)\\
&\le \beta_n^{(1)}d^2(T^nx_n,p)+\beta_n^{(2)}d^2(T^nw_n,p)\\
&\quad +\beta_n^{(3)}d^2(e_n^{(2)},p)-\beta_n^{(1)}\beta_n^{(2)}d^2(T^nx_n,T^nw_n)\\
&\le \beta_n^{(1)}[(1+\mu_nM^*)d(x_n,p)+\nu_n]^2\\
&\quad +\beta_n^{(2)}[(1+\mu_nM^*)d(w_n,p)+\nu_n]^2+\beta_n^{(3)}d(e_n^{(2)},p)\\
&\quad -\beta_n^{(1)}\beta_n^{(2)}d^2(T^nx_n,T^nw_n)\\
&= \beta_n^{(1)}(1+\mu_nM^*)^2d^2(x_n,p)+\beta_n^{(1)}[2(1+\mu_nM^*)d(x_n,p)+\nu_n]\nu_n\\
&\quad +\beta_n^{(2)}(1+\mu_nM^*)^2d^2(w_n,p)\\
&\quad +\beta_n^{(2)}[2(1+\mu_nM^*)d(w_n,p)+\nu_n]\nu_n\\
&\quad +\beta_n^{(3)}d(e_n^{(2)},p)-\beta_n^{(1)}\beta_n^{(2)}d^2(T^nx_n,T^nw_n)\\
&\le d^2(x_n,p)+((2+\mu_nM^*)M^*d^2(w_n,p)+2Q_n)\mu_n\\
&\quad +(2(1+\mu_nM^*)d(w_n,p)+\nu_n+R_n)\nu_n\\
&\quad +\alpha_n^{(3)}d^2(e_n^{(1)},p)\\
&\quad +\beta_n^{(3)}d(e_n^{(2)},p)-\beta_n^{(1)}\beta_n^{(2)}d^2(T^nx_n,T^nw_n),
\end{aligned} \tag{3.11}$$

where $P_n=(2+\mu_nM^*)M^*d^2(w_n,p)+2Q_n$ and $S_n=2(1+\mu_nM^*)d(w_n,p)+\nu_n+R_n$, which implies that

$$\begin{aligned}
d^2(y_n,p)\le d^2(x_n,p)+P_n\mu_n+S_n\nu_n+\alpha_n^{(3)}d^2(e_n^{(1)},p)\\
+\beta_n^{(3)}d(e_n^{(2)},p)-\beta_n^{(1)}\beta_n^{(2)}d^2(T^nx_n,T^nw_n).
\end{aligned}$$

By assumptions (C_1), (C_2), (C_3), (C_4) and assertions (3.5), (3.7), (3.11) we get

$$\begin{aligned}
&d^2(T^nx_n,T^nw_n)\\
&\le \frac{1}{ef}\left[d^2(x_n,p)-d^2(y_n,p)+P_n\mu_n+S_n\nu_n+\alpha_n^{(3)}d^2(e_n^{(1)},p)+\beta_n^{(3)}d^2(e_n^{(2)},p)\right]\\
&\to 0 \qquad as \qquad n\to\infty.
\end{aligned} \tag{3.12}$$

By assumption (C_6) and assertions (3.10), (3.12) we get

$$\begin{aligned}
d(y_n,x_n) &= d(\beta_n^{(1)}T^nx_n\oplus\beta_n^{(2)}T^nw_n\oplus\beta_n^{(3)}e_n^{(2)},x_n)\\
&\le \beta_n^{(1)}d(T^nx_n,x_n)+\beta_n^{(2)}d(T^nw_n,x_n)+\beta_n^{(3)}d(e_n^{(2)},x_n)\\
&\le d(T^nx_n,x_n)+[d(T^nw_n,T^nx_n)+d(T^nx_n,x_n)]+\beta_n^{(3)}d(e_n^{(2)},x_n)\\
&\to \quad 0 \qquad as \qquad n\to\infty.
\end{aligned} \tag{3.13}$$

By assumption (C_3), (C_6) and assertion (3.10) we have

$$\begin{aligned}
d(w_n,x_n) &= d(\alpha_n^{(1)}T^nx_n\oplus\alpha_n^{(2)}T^nw_n\oplus\alpha_n^{(3)}e_n^{(2)},x_n)\\
&\le d(T^nx_n,x_n)+\alpha_n^{(3)}d(e_n^{(1)},x_n)\\
&\to \quad 0 \qquad as \qquad n\to\infty.
\end{aligned} \tag{3.14}$$

It follows from (3.13) and (3.14) which implies that

$$\begin{aligned} d(w_n, y_n) &\leq d(w_n, x_n) + d(x_n, y_n) \\ &\to 0 \quad as \quad n \to \infty. \end{aligned} \tag{3.15}$$

By assumptions (C_1), (C_2) and assertions (3.13) and (3.15) we have

$$\begin{aligned} d(y_n, T^n y_n) &\leq \beta_n^{(1)} d(T^n x_n, T^n y_n) + \beta_n^{(2)} d(T^n w_n, T^n y_n) + \beta_n^{(3)} d(e_n^{(2)}, T^n y_n) \\ &\leq (1 + \mu_n M^*)[d(x_n, y_n) + d(w_n, y_n)] + 2\nu_n + \beta_n^{(3)} d(e_n^{(2)}, T^n y_n) \\ &\to 0 \quad as \quad n \to \infty. \end{aligned} \tag{3.16}$$

By assumptions (C_2), (C_5) and (C_6) we have

$$\begin{aligned} d(x_{n+1}, T^n y_n) &= d(\gamma_n^{(1)} T^n y_n \oplus \gamma_n^{(2)} e_n^{(3)}, T^n y_n) \\ &\leq \gamma_n^{(2)} d(e_n^{(3)}, T^n y_n) \\ &\to 0 \quad as \quad n \to \infty. \end{aligned} \tag{3.17}$$

From assertions (3.13), (3.16) and (3.17) which implies that

$$\begin{aligned} d(x_{n+1}, x_n) &\leq d(x_{n+1}, T^n y_n) + d(T^n y_n, y_n) + d(y_n, x_n) \\ &\to 0 \quad as \quad n \to \infty. \end{aligned} \tag{3.18}$$

Since T is uniformly L-Lipschitzian,

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) + d(T^{n+1} x_{n+1}, T^{n+1} x_n) \\ &\quad + d(T^{n+1} x_n, Tx_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) + Ld(x_{n+1}, x_n) \\ &\quad + Ld(T^n x_n, x_n) \\ &= (1 + L) d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) + Ld(T^n x_n, x_n) \\ &\to 0 \quad as \quad n \to \infty. \end{aligned}$$

Step 3 Next, we claim that the sequence $\{x_n\}$ Δ-converges to a fixed point of T. Indeed, we will show that

$$w_\Delta(x_n) := \bigcup_{\{v_n\} \subseteq \{x_n\}} A(\{v_n\}) \subseteq F(T)$$

and $w_\Delta(x_n)$ consists of exactly one point. Let $v \in w_\Delta(x_n)$. By the definition of $w_\Delta(x_n)$, there exists a subsequence $\{v_n\}$ of $\{x_n\}$ such that $A(\{v_n\}) = \{v\}$. From Lemma 2.5(S_1), there is a subsequence $\{u_n\}$ of $\{v_n\}$ such that $\Delta - \lim_{n \to \infty} u_n = u$ and $u \in C$. By Lemma 2.6, we have $u \in F(T)$. Since $\{d(v_n, u)\}$ converges, by Lemma 2.5(S_2), we get $v = u$. Thus $w_\Delta(x_n) \subseteq F(T)$.
Finally, we prove that $w_\Delta(x_n)$ consists of exactly one point. Let $\{v_n\}$ be a subsequence of $\{x_n\}$ by the uniqueness asymptotic center such that $A(\{v_n\}) = v$ and let $A(\{x_n\}) = \{x\}$. Since $v = u \in F(T)$ and $\{d(x_n, u)\}$ converges, by using Lemma 2.5(S_3), we see that $x = u \in F(T)$, Therefore $w_\Delta(x_n) = \{x\}$. This completes the proof. □

By using a similar technique as in the proof of Theorem 3.2 in the paper of Thakur et al. [21], we obtain strong convergence theorem without the proof immediately.

Theorem 3.2. *Let* $X, T, C, (C_1), (C_2), (C_3), (C_4), (C_5), (C_6), \{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \{\alpha_n^{(3)}\}, \{\beta_n^{(1)}\}, \{\beta_n^{(2)}\}, \{\beta_n^{(3)}\}, \{\gamma_n^{(1)}\}, \{\gamma_n^{(2)}\}, \{e_n^{(1)}\}, \{e_n^{(2)}\}, \{e_n^{(3)}\}$ *satisfy the assumption of Theorem 3.1. Then the sequence* $\{x_n\}$ *which is defined by (1.4) converges strongly to a fixed point of* T *if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0,$$

where $d(x, F(T)) = \inf\{d(x,p) : p \in F(T)\}$.

The concept of special self mapping is called Condition(I) introduced by Senter and Dotson [19] as follows.

Definition 3.3 ([19])**.** Let (X, d) be a CAT(0) space and C a nonempty subset. A self mapping T with $F(T) \neq \emptyset$ is said to satisfy condition (I) if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(l) > 0$ for all $l > 0$ such that

$$d(x, Tx) \geq f(d(x, F(T)))$$

for all $x \in C$.

By using Condition (I) with the similar technique as in the proof of Theorem 3.3 in Thakur et al. [21], we obtain the following result.

Theorem 3.4. *Let* $X, T, C, (C_1), (C_2), (C_3), (C_4), (C_5), (C_6), \{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \{\alpha_n^{(3)}\}, \{\beta_n^{(1)}\}, \{\beta_n^{(2)}\}, \{\beta_n^{(3)}\}, \{\gamma_n^{(1)}\}, \{\gamma_n^{(2)}\}, \{e_n^{(1)}\}, \{e_n^{(2)}\}, \{e_n^{(3)}\}$ *satisfy the assumption of Theorem 3.1 and let self mapping of* T *satisfy Condition (I). Then the sequence* $\{x_n\}$ *which is defined by (1.4) converges strongly to a fixed point of* T.

Definition 3.5. Let (X, d) be a CAT(0) space and C a nonempty subset. Recall that a self mapping T is said to be $semi-compact$ if C is closed and for all bounded sequence $\{x_n\} \subset C$ with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\} \to p \in C$.

Using a similar technique as in the proof of Theorem 22 in Karapinar et al. [11], we obtain the following result.

Theorem 3.6. *Let* $X, T, C, (C_1), (C_2), (C_3), (C_4), (C_5), (C_6), \{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \{\alpha_n^{(3)}\}, \{\beta_n^{(1)}\}, \{\beta_n^{(2)}\}, \{\beta_n^{(3)}\}, \{\gamma_n^{(1)}\}, \{\gamma_n^{(2)}\}, \{e_n^{(1)}\}, \{e_n^{(2)}\}, \{e_n^{(3)}\}$ *satisfy the assumption of Theorem 3.1 and let* T *be* $semi-compact$*. Then the sequence* $\{x_n\}$ *which is defined by (1.4) converges strongly to a fixed point of* T.

4. Numerical examples

In this section, we will illustrate numerical examples to compare speed convergence of the modified Picard-S hybrid with error iterative scheme (1.4) with the modified Picard-Mann hybrid iterative scheme (1.2) and the modified Picard-Ishikawa hybrid iterative scheme (1.3) by using the mapping same [14, Example 4.1] and difference a set C and the initial point.

Example 4.1. *Let $X = \mathbb{R}$ be a Euclidean metric space and $C = [1, 50]$. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by*

$$Tx = \sqrt[3]{x^2 + 18}.$$

It is obvious that T is a continuous uniformly L-Lipschitzian and a total asymptotically nonexpansive mapping (see more [14, Example 4.1, pp 10.]*) with $F(T) = \{3\}$.*

Let $x_1 = 50$. By using MATLAB compute the iterates of (1.2),(1.3) and (1.4) with different control parameters as follows:

Case 1) $\alpha_n = \alpha_n^{(1)} = \frac{5n}{15n+7}, \alpha_n^{(2)} = \frac{280n^2+49n}{420n^2+256n+28}, \alpha_n^{(3)} = \frac{187n+28}{420n^2+256n+28}, \beta_n = \beta_n^{(1)} = \frac{7n}{10n+3}, \beta_n^{(2)} = \frac{3n}{10n+3},$
$\beta_n^{(3)} = \frac{3}{10n+3}, \gamma_n^{(1)} = \frac{2n}{2n+1}, \gamma_n^{(2)} = \frac{1}{2n+1}, e_n^{(1)} = e_n^{(2)} = e_n^{(3)} = \frac{3n^2-1}{n^2}.$

Case 2) $\alpha_n^{(1)} = \frac{4n}{5n+2}, \alpha_n^{(2)} = \frac{10n^2+2n}{50n^2+25n+2}, \alpha_n^{(3)} = \frac{19n+2}{50n^2+25n+2}, \beta_n^{(1)} = \frac{2n}{7n+11}, \beta_n^{(2)} = \frac{5n+2}{7n+11}, \beta_n^{(3)} = \frac{9}{7n+11},$
$\gamma_n^{(1)} = \frac{3n}{3n+2}, \gamma_n^{(2)} = \frac{2}{3n+2}, e_n^{(1)} = e_n^{(2)} = e_n^{(3)} = \frac{3n^2-1}{n^2}.$
Then we obtain numerical results in Table 1, Figures 1 and 2.

TABLE 1. The values of the sequence $\{x_n\}$ with different control parameters.

Iterate	Case 1			Case 2		
	Iterative scheme			Iterative scheme		
	Picard-Mann	Picard-Ishikawa	Picard-S with error	Picard-Mann	Picard-Ishikawa	Picard-S with error
x_1	50.00000000	50.00000000	50.00000000	50.00000000	50.00000000	50.00000000
x_2	12.07191130	8.80033965	3.565161735	9.54930216	9.31484614	3.34214043
x_3	3.46195853	3.10805800	3.000384129	3.14850148	3.130444174	3.00017264
x_4	3.00376045	3.00031523	3.000000014	3.00049474	3.000393847	3.00000001
x_5	3.00000644	3.00000019	3.00000000	3.00000033	3.000000237	3.00000000
x_6	3.00000000	3.00000000	3.00000000	3.00000000	3.00000000	3.00000000
x_7	3.00000000	3.00000000	3.00000000	3.00000000	3.00000000	3.00000000
x_8	3.00000000	3.00000000	3.00000000	3.00000000	3.00000000	3.00000000
x_9	3.00000000	3.00000000	3.00000000	3.00000000	3.00000000	3.00000000
x_10	3.00000000	3.00000000	3.00000000	3.00000000	3.00000000	3.00000000

Figures 1 and 2 illustrate the speed of convergence with different control parameters for approximating the fixed point set of $F(T) = \{3\}$.

Example 4.2. *Let $X = \mathbb{R}$ be a Euclidean metric space and $C = [1, 20]$. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by*

$$Tx = \ln x + 1.$$

It is obvious that T is a continuous uniformly L-Lipschitzian and a total asymptotically nonexpansive mapping (see more [14, Example 4.2, pp 11-12.]*) with $F(T) = \{1\}$.*

Let $x_1 = 20$. By using MATLAB compute the iterates of (1.2),(1.3) and (1.4) with different control parameters as follows:

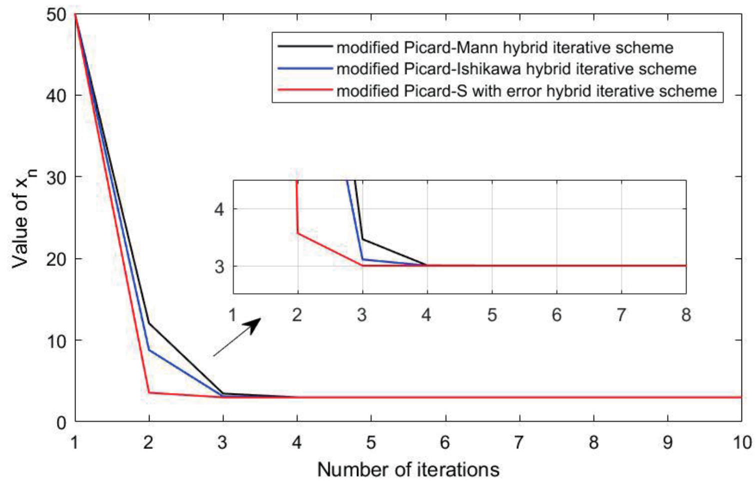

FIGURE 1. Comparison speed of convergence with Case 1 of Table 1

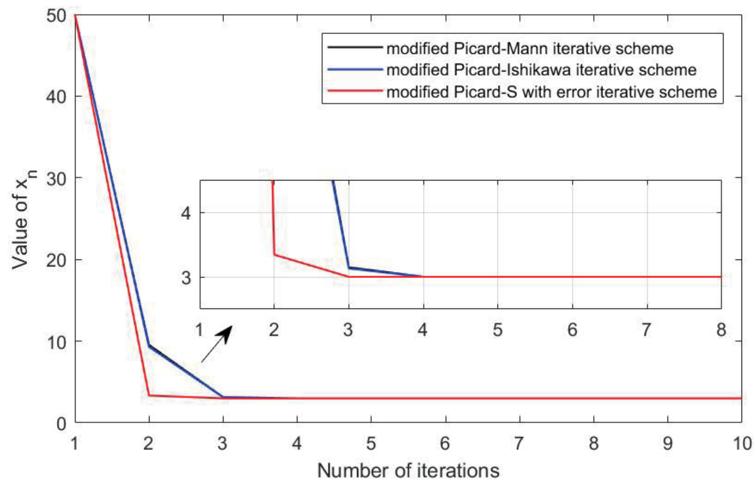

FIGURE 2. Comparison speed of convergence with Case 2 of Table 1

Case 1) $\alpha_n = \alpha_n^{(1)} = \frac{5n}{15n+7}, \alpha_n^{(2)} = \frac{280n^2+49n}{420n^2+256n+28}, \alpha_n^{(3)} = \frac{187n+28}{420n^2+256n+28}$, $\beta_n = \beta_n^{(1)} = \frac{7n}{10n+3}, \beta_n^{(2)} = \frac{3n}{10n+3}$,
$\beta_n^{(3)} = \frac{3}{10n+3}, \gamma_n^{(1)} = \frac{2n}{2n+1}, \gamma_n^{(2)} = \frac{1}{2n+1}, e_n^{(1)} = e_n^{(2)} = e_n^{(3)} = \frac{n^2-1}{n^2}$.

Case 2) $\alpha_n^{(1)} = \frac{4n}{5n+2}, \alpha_n^{(2)} = \frac{10n^2+2n}{50n^2+25n+2}, \alpha_n^{(3)} = \frac{19n+2}{50n^2+25n+2}$, $\beta_n^{(1)} = \frac{2n}{7n+11}, \beta_n^{(2)} = \frac{5n+2}{7n+11}, \beta_n^{(3)} = \frac{9}{7n+11}$,
$\gamma_n^{(1)} = \frac{3n}{3n+2}, \gamma_n^{(2)} = \frac{2}{3n+2}, e_n^{(1)} = e_n^{(2)} = e_n^{(3)} = \frac{n^2-1}{n^2}$.
Then we get numerical results in Table 2, and we also illustrate the convergence speed is shown in Figure 3 and 4 respectively.

Table 2. The values of the sequence $\{x_n\}$ with different control conditions.

Iterate	Case 1			Case 2		
	Iterative scheme			Iterative scheme		
	Picard-Mann	Picard-Ishikawa	Picard-S with error	Picard-Mann	Picard-Ishikawa	Picard-S with error
x_1	20.00000000	20.00000000	20.00000000	20.00000000	20.00000000	20.00000000
x_{10}	1.03702638	1.02454010	1.00003167	1.02714689	1.02568315	1.00003340
x_{20}	1.00826911	1.00544124	1.00000001	1.00608066	1.00570201	1.00000002
x_{30}	1.00355046	1.00233278	1.00000001	1.00261609	1.00244453	1.00000001
x_{40}	1.00196428	1.00128981	1.00000000	1.00144904	1.00135143	1.00000001
x_{50}	1.00124498	1.00081721	1.00000000	1.00091911	1.00085616	1.00000000
x_{60}	1.00085907	1.00056380	1.00000000	1.00063455	1.00059059	1.00000000
x_{70}	1.00062832	1.00041231	1.00000000	1.00046428	1.00043186	1.00000000
x_{80}	1.00046755	1.00030678	1.00000000	1.00035438	1.00032948	1.00000000
x_{90}	1.00037785	1.00024790	1.00000000	1.00027935	1.00025963	1.00000000
x_{100}	1.00030543	1.00020038	1.00000000	1.00022585	1.00020984	1.00000000
⋮	⋮	⋮	⋮	⋮	⋮	⋮

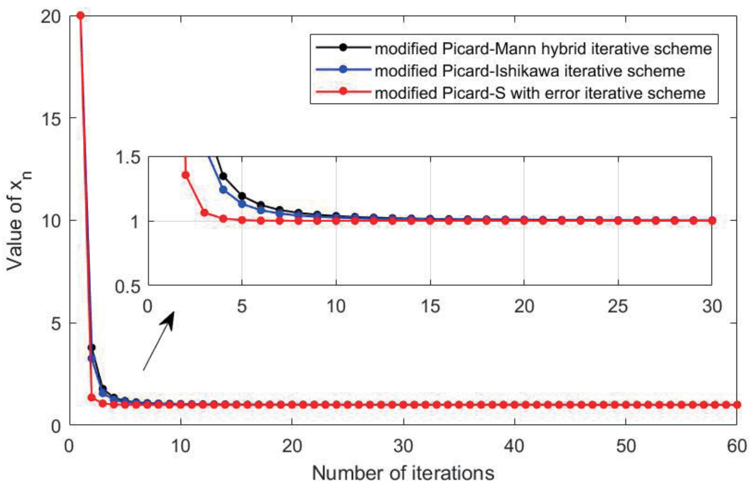

Figure 3. Comparison speed of convergence with Case 1 of Table 2

From the numerical results of Example 4.1 and 4.2. We see that the convergence speed of the modified Picard-S hybrid with error iterative scheme (1.4) is faster than that of the modified Picard-Mann hybrid iterative scheme (1.2) and that of the modified Picard-Ishikawa hybrid iterative scheme (1.3).

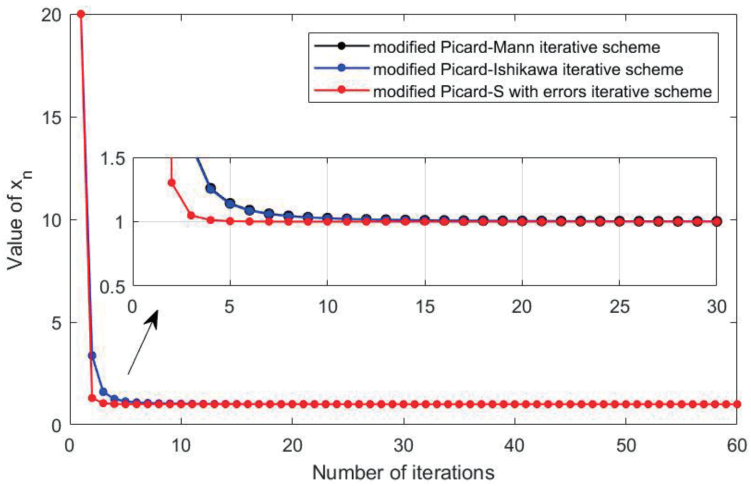

FIGURE 4. Comparison speed of convergence with Case 2 of Table 2

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