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ANOTHER SIMPLE PROOF OF THE LEBESGUE-RADON-NIKODYM THEOREM

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Abstract. We give a simple proof of the Lebesgue-Radon-Nikodym theorem by using the Riesz representation theorem.

1. Introduction

The Radon-Nikodym theorem is one of the important theorems in the measure theory. Using the Riesz representation theorem, von Neumann gave its elegant proof [9, Lemma 3.2.3]. However, his proof is a little bit complicated for beginners, and there are various proofs of the theorem; see [1, 2, 4, 5, 6, 8], [3, Theorem 6.10] and others. All the proofs have some devices and interesting points, and some of them depend on the Hahn decomposition theorem or the Riesz representation theorem.

In this note, we give one more alternative simple proof of the Lebesgue-Radon-Nikodym theorem. Our proof depends on the Riesz representation theorem and the fact that the total variation of a signed measure is a measure. Throughout this note, let X be a nonempty set and let $\mathscr{F}$ be a σ-algebra on X. We say $\nu\colon \mathscr{F} \to [-\infty, \infty]$ is a signed measure, if $\nu(\emptyset) = 0$, it takes at most one of the values $\pm\infty$, and

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n) \quad \text{for each disjoint sequence } \{A_n\} \subset \mathscr{F}. \tag{1.1}$$

We note that the value of the series $\sum_{n=1}^{\infty} \nu(A_n)$ in (1.1) does not depend on the reordering of the series. Unless otherwise noted, we assume that singed measures are defined on $\mathscr{F}$. We call a signed measure ν a measure if $\nu(A) \geq 0$ for each $A \in \mathscr{F}$. We define the total variation $|\nu|$ of a signed measure ν by

$$|\nu|(A) = \sup\left\{ \sum_{j=1}^{\infty} |\nu(A_j)| \colon \{A_j\} \subset \mathscr{F} \text{ is a partition of } A \right\} \quad \text{for each } A \in \mathscr{F}.$$

We say a signed measure ν is finite if $\nu(A) \in \mathbb{R}$ for each $A \in \mathscr{F}$, and it is σ-finite if there is a partition $\{X_n\} \subset \mathscr{F}$ of X such that $|\nu|(X_n) < \infty$ for each $n \in \mathbb{N}$. We say a signed measure ν is absolutely continuous with respect to a measure μ if $\nu(A) = 0$ for each $A \in \mathscr{F}$ with $\mu(A) = 0$. For a measure μ, we understand that the space $L^p(X, \mathscr{F}, \mu)$ for $1 \leq p < \infty$ is the set of all equivalent classes of

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measurable functions from X into $\mathbb{R}$ that satisfies $\|u\| \equiv (\int_X |u|^p \, d\mu)^{1/p} < \infty$ for each $u \in L^p(X, \mathscr{F}, \mu)$.

The following is the Lebesgue-Radon-Nikodym theorem. For $E \in \mathscr{F}$, we denote by $\mathscr{F}|_E$ the σ-algebra $\{F \cap E \colon F \in \mathscr{F}\}$ on E, and for a measure μ and $E \in \mathscr{F}$, we denote by $\mu|_E$ the restriction of μ on $\mathscr{F}|_E$. For $f \colon X \to \mathbb{R}$, we set $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$.

Theorem 1.1. *Let ν be a σ-finite signed measure and let μ be a σ-finite measure. Then there exist $E \in \mathscr{F}$ and a measurable function $f \colon E \to \mathbb{R}$ which satisfy the following* (i)–(iii)*.*

(i) $\mu(X \setminus E) = 0$.

(ii) *It holds that*

   (a) $f^- \in L^1(E, \mathscr{F}|_E, \mu|_E)$ *in the case that ν does not take the value $-\infty$,*

   (b) $f^+ \in L^1(E, \mathscr{F}|_E, \mu|_E)$ *in the case that ν does not take the value ∞,*

   (c) *in particular, $f \in L^1(E, \mathscr{F}|_E, \mu|_E)$ in the case that ν is finite.*

(iii) *It holds that*

$$\nu(A) = \nu(A \setminus E) + \int_{A \cap E} f \, d\mu \quad \text{for each } A \in \mathscr{F}.$$

In particular, if ν is absolutely continuous with respect to μ, then

$$\nu(A) = \int_A f \, d\mu \quad \text{for each } A \in \mathscr{F}.$$

Remark 1.2. When ν is absolutely continuous with respect to μ, the theorem above is called the Radon-Nikodym theorem.

2. Preliminaries

It is well known that for a signed measure $\nu \colon \mathscr{F} \to [-\infty, \infty]$, the total variation $|\nu| \colon \mathscr{F} \to [0, \infty]$ is a measure. For the proof, see [3, Theorems 6.2]. Although the proof in it is given for a complex valued measure, it works for our case.

In the rest of this section, we assume that a signed measure ν is finite. In this case, $|\nu|$ is a finite measure, i.e., $|\nu|(X) < \infty$; for the proof, see [3, Theorem 6.4]. We set $\nu^+, \nu^- \colon \mathscr{F} \to \mathbb{R}$ by

$$\nu^+(A) = \frac{1}{2}(|\nu|(A) + \nu(A)) \quad \text{and} \quad \nu^-(A) = \frac{1}{2}(|\nu|(A) - \nu(A))$$

for each $A \in \mathscr{F}$. We note that ν^+, ν^- are finite measures, $\nu(A) = \nu^+(A) - \nu^-(A)$ and $|\nu|(A) = \nu^+(A) + \nu^-(A)$ for each $A \in \mathscr{F}$, and the decomposition $\nu = \nu^+ - \nu^-$ is called the Jordan decomposition of ν.

We recall the definition of the integration by ν. For each $f \in L^1(X, \mathscr{F}, |\nu|)$, the integration $\int_X f \, d\nu$ is defined by

$$\int_X f \, d\nu = \int_X f \, d\nu^+ - \int_X f \, d\nu^-.$$

For the readers' convenience, we note that it holds

$$\int_X \sum_{j=1}^n a_j 1_{A_j} \, d\nu = \sum_{j=1}^n a_j \nu(A_j)$$

for each $a_1, \dots, a_n \in \mathbb{R}$ and $A_1, \dots, A_n \in \mathscr{F}$. We can easily see that the correspondence $f \mapsto \int_X f\,d\nu \colon L^1(X, \mathscr{F}, |\nu|) \to \mathbb{R}$ is linear and it holds that

$$\left| \int_X f\,d\nu \right| \le \int_X |f|\,d|\nu| \quad \text{for each } f \in L^1(X, \mathscr{F}, |\nu|).$$

We also recall the Riesz representation theorem for a Hilbert space; for the proof, see [7, Theorem C in Section 55].

Theorem 2.1. *Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, and let $F \colon H \to \mathbb{R}$ be a bounded linear functional. Then there exists a unique $v \in H$ such that $F(u) = \langle u, v \rangle$ for each $u \in H$.*

The following is the Hahn decomposition theorem for a finite signed measure. For the sake of completeness, we give the proof by using Theorem 2.1. Although we treat the finite case only, the Hahn decomposition theorem holds even if a signed measure is not σ-finite.

Theorem 2.2. *Let ν be a finite signed measure. Then there exist $P, N \in \mathscr{F}$ such that $X = P \cup N$, $P \cap N = \emptyset$, and*

$$\begin{cases} \nu(A) \ge 0 & \text{for each } A \in \mathscr{F} \text{ with } A \subset P, \\ \nu(B) \le 0 & \text{for each } B \in \mathscr{F} \text{ with } B \subset N. \end{cases}$$

Proof. We define $F \colon L^2(X, \mathscr{F}, |\nu|) \to \mathbb{R}$ by $F(h) = \int_X h\,d\nu$ for $h \in L^2(X, \mathscr{F}, |\nu|)$. By the Hölder inequality, we have

$$|F(h)| \le \int_X |h|\,d|\nu| \le (|\nu|(X))^{\frac{1}{2}} \left(\int_X |h|^2\,d|\nu| \right)^{\frac{1}{2}} \quad \text{for each } h \in L^2(X, \mathscr{F}, |\nu|).$$

Thus F is a bounded linear functional on $L^2(X, \mathscr{F}, |\nu|)$. By Theorem 2.1, there is a unique $g \in L^2(X, \mathscr{F}, |\nu|)$ such that

$$\int_X h\,d\nu = \int_X hg\,d|\nu| \quad \text{for each } h \in L^2(X, \mathscr{F}, |\nu|).$$

In particular, we have

$$\nu(A) = \int_A g\,d|\nu| \quad \text{for each } A \in \mathscr{F}.$$

Putting $P = \{x \in X \colon g(x) > 0\}$ and $N = \{x \in X \colon g(x) \le 0\}$, we can easily see that our assertion holds. □

Remark 2.3. In the proof above, we can also put $P = \{x \in X \colon g(x) \ge 0\}$ and $N = \{x \in X \colon g(x) < 0\}$. Thus the decomposition is not unique in general.

Remark 2.4. By the function g in the proof above, the Jordan decomposition $\nu = \nu^+ - \nu^-$ of ν is given by $\nu^+(A) = \int_A g^+\,d|\nu|$ and $\nu^-(A) = \int_A g^-\,d|\nu|$ for $A \in \mathscr{F}$.

3. Proof of Theorem 1.1

First, we study the finite case of Theorem 1.1.

Theorem 3.1. *Let ν be a finite signed measure and let μ be a finite measure. Then there exist $E \in \mathscr{F}$ and $f \in L^1(E, \mathscr{F}|_E, \mu|_E)$ such that $\mu(X \setminus E) = 0$ and*

$$\nu(A) = \nu(A \setminus E) + \int_{A\cap E} f\, d\mu \quad \textit{for each } A \in \mathscr{F}.$$

In particular, if ν is absolutely continuous with respect to μ, then

$$\nu(A) = \int_A f\, d\mu \quad \textit{for each } A \in \mathscr{F}.$$

Proof. For each $n \in \mathbb{N}$, let $P_n, N_n \in \mathscr{F}$ be a Hahn decomposition for the finite signed measure $n\mu - |\nu|$, i.e., $X = P_n \cup N_n$, $P_n \cap N_n = \emptyset$, and

$$n\mu(A) - |\nu|(A) \geq 0 \quad \text{for each } A \in \mathscr{F} \text{ with } A \subset P_n, \tag{3.1}$$

$$n\mu(B) - |\nu|(B) \leq 0 \quad \text{for each } B \in \mathscr{F} \text{ with } B \subset N_n. \tag{3.2}$$

We set $E_1 = P_1$, $E_n = P_n \setminus P_{n-1}$ for $n = 2, 3, \ldots$, and $E = \bigcup_{n=1}^{\infty} E_n$. We will show $\mu(X \setminus E) = 0$. If not, i.e., $\mu(X \setminus E) > 0$, then (3.2) yields $|\nu|(X \setminus E) = \infty$, which is a contradiction. Thus we have shown $\mu(X \setminus E) = 0$. Let $n \in \mathbb{N}$. From (3.1), we have

$$\left| \int_{E_n} h\, d\nu \right| \leq \int_{E_n} |h|\, d|\nu| \leq n \int_{E_n} |h|\, d\mu \leq n\mu(E_n)^{\frac{1}{2}} \left(\int_{E_n} |h|^2\, d\mu \right)^{\frac{1}{2}}$$

for each $h \in L^2(E_n, \mathscr{F}|_{E_n}, \mu|_{E_n})$. By Theorem 2.1, there is $f_n \in L^2(E_n, \mathscr{F}|_{E_n}, \mu|_{E_n})$ such that

$$\int_{E_n} h\, d\nu = \int_{E_n} h f_n\, d\mu \quad \text{for each } h \in L^2(E_n, \mathscr{F}|_{E_n}, \mu|_{E_n}).$$

In particular, we have

$$\nu(A) = \int_A f_n\, d\mu \quad \text{for each } A \in \mathscr{F} \text{ with } A \subset E_n.$$

Putting $f_n(x) = 0$ for $x \in E \setminus E_n$, we can consider that f_n is defined on E. We set$f = \sum_{n=1}^{\infty} f_n$, $E_+ = \{x \in E : f(x) \geq 0\}$ and $E_- = \{x \in E : f(x) < 0\}$. By the monotone convergence theorem, we have

$$\nu(A \cap E \cap E_+) = \sum_{n=1}^{\infty} \nu(A \cap E_n \cap E_+) = \sum_{n=1}^{\infty} \int_{A\cap E_n} f^+\, d\mu = \int_{A\cap E} f^+\, d\mu,$$

$$\nu(A \cap E \cap E_-) = \sum_{n=1}^{\infty} \nu(A \cap E_n \cap E_-) = \sum_{n=1}^{\infty} \int_{A\cap E_n} (-f^-)\, d\mu = -\int_{A\cap E} f^-\, d\mu$$

for each $A \in \mathscr{F}$. From these identities and the finiteness of ν, we have $f \in L^1(E, \mathscr{F}|_E, \mu|_E)$ and

$$\nu(A) = \nu(A \setminus E) + \nu(A \cap E \cap E_+) + \nu(A \cap E \cap E_-) = \nu(A \setminus E) + \int_{A\cap E} f\, d\mu$$

for each $A \in \mathscr{F}$. □

Now, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Since ν and μ are σ-finite, there is a partition $\{X_n\} \subset \mathscr{F}$ of X such that $|\nu|(X_n) < \infty$ and $\mu(X_n) < \infty$ for each $n \in \mathbb{N}$. By Theorem 3.1, for each $n \in \mathbb{N}$, there exist $E_n \in \mathscr{F}$ and $f_n \in L^1(E_n, \mathscr{F}|_{E_n}, \mu|_{E_n})$ such that $E_n \subset X_n$, $\mu(X_n \setminus E_n) = 0$ and

$$\nu(A) = \int_{A\cap E_n} f_n \, d\mu \quad \text{for each } A \in \mathscr{F} \text{ with } A \subset E_n.$$

We set $E = \bigcup_{n=1}^{\infty} E_n$, $f_n = 0$ on $E \setminus E_n$ for each $n \in \mathbb{N}$, and $f = \sum_{n=1}^{\infty} f_n$. Then we have

$$\mu(X \setminus E) = \sum_{n=1}^{\infty} \mu(X_n \setminus E_n) = 0.$$

Setting $E_+ = \{x \in E\colon f(x) \geq 0\}$ and $E_- = \{x \in E\colon f(x) < 0\}$, by the monotone convergence theorem, we have

$$\nu(A \cap E \cap E_+) = \sum_{n=1}^{\infty} \nu(A \cap E_n \cap E_+) = \sum_{n=1}^{\infty} \int_{A\cap E_n} f^+ \, d\mu = \int_{A\cap E} f^+ \, d\mu,$$

$$\nu(A \cap E \cap E_-) = \sum_{n=1}^{\infty} \nu(A \cap E_n \cap E_-) = \sum_{n=1}^{\infty} \int_{A\cap E_n} (-f^-) \, d\mu = -\int_{A\cap E} f^- \, d\mu$$

for each $A \in \mathscr{F}$. From these identities, we can easily see that (ii) holds, and we have

$$\nu(A) = \nu(A \setminus E) + \nu(A \cap E \cap E_+) + \nu(A \cap E \cap E_-) = \nu(A \setminus E) + \int_{A\cap E} f \, d\mu$$

for each $A \in \mathscr{F}$. ☐

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