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# ON THE TRACE INEQUALITIES RELATED TO LEFT-RIGHT MULTIPLICATION OPERATORS AND THEIR APPLICATIONS

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Abstract. Recently in [6] we obtained non-hermitian extensions of Heisenberg type and Schrödinger type uncertainty relations for generalized quasi-metric adjusted skew information or generalized quasi-metric adjusted correlation measure and applied to the inequalities related to fidelity and trace distance for different two generalized states which were given by Audenaert et al; and Powers-St$\phi$rmer [1, 2, 5]. In this paper we state the properties of left or right multiplication operators and obtain some related inequalities.

## 1. Introduction

Let $M_n(\mathbb{C})$(resp. $M_{n,sa}(\mathbb{C})$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices), endowed with the Hilbert-Schmidt scalar product $\langle X, Y\rangle_{HS} = Tr[X^*Y]$. Let $M_{n,+}(\mathbb{C})$ be the set of strictly positive elements of $M_n(\mathbb{C})$ and $M_{n,+,1}(\mathbb{C})$ be the set of density matrices. Let $L_A, R_A$ be left (right) multiplication operator for $A \in M_n(\mathbb{C})$ as follows.

$$L_A(X) = AX,\ R_A(X) = XA,\ (X \in M_n(\mathbb{C})).$$

**Proposition 1.1.** *$L_A, R_A$ are linear operators on $(M_n(\mathbb{C}), (\cdot,\cdot)_{HS})$ satisfying the following properties.*

(1) *For $A, B \in M_n(\mathbb{C})$, $L_A R_B = R_B L_A$,*
(2) *For $A, B \in M_n(\mathbb{C})$,*

$$L_{A+B} = L_A + L_B,\ R_{A+B} = R_A + R_B,\ L_{AB} = L_A L_B,\ R_{AB} = R_B R_A.$$

(3) *For $A \in M_n(\mathbb{C})$, $\lambda \in \mathbb{C}$, $L_{\lambda A} = \lambda L_A,\ R_{\lambda A} = \lambda R_A$.*
(4) *For $A \in M_n(\mathbb{C})$, $L_{A^*} = (L_A)^*,\ R_{A^*} = (R_A)^*$.*
(5) *For $A > 0$, $L_A > 0,\ R_A > 0$.*

**Proposition 1.2.** *Let $A, B \in M_{n,+}(\mathbb{C})$ have the following spectral decompositions*

$$A = \sum_{i=1}^n \alpha_i |\phi_i\rangle\langle\phi_i|,\ B = \sum_{j=1}^n \beta_j |\psi_j\rangle\langle\psi_j|,$$

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*where $\alpha_i > 0$ are positive eigenvalues of $A$, $|\phi_i\rangle$ are corresponding eigenvectors making orthnormal basis and $\beta_j > 0$ are positive eigenvalues of $B$, $|\psi_j\rangle$ are corrre-sponding eigenvectors making orthnormal basis. Then (1), (2) hold.*

(1) $L_A = L_{\sum_{i=1}^n \alpha_i|\phi_i\rangle\langle\phi_i|} = \sum_{i=1}^n \alpha_i L_{|\phi_i\rangle\langle\phi_i|} = \sum_{i=1}^n \sum_{j=1}^n \alpha_i L_{|\phi_i\rangle\langle\phi_i|} R_{|\psi_j\rangle\langle\psi_j|}$.
(2) $R_B = R_{\sum_{j=1}^n \beta_j|\psi_j\rangle\langle\psi_j|} = \sum_{j=1}^n \beta_j R_{|\psi_j\rangle\langle\psi_j|} = \sum_{i=1}^n \sum_{j=1}^n \beta_j L_{|\phi_i\rangle\langle\phi_i|} R_{|\psi_j\rangle\langle\psi_j|}$.

In this paper we state the properties of $L_A, R_A$ in section 2. In setion 3, we give some uncertainty relations for generalized quasi-metric adjusted skew informations. In section 4, we state the trace inequality representing the relationship between fidelity and trace distance. Also we give some interesting inequalities of generalized quasi-metric adjusted skew informations by using refined norm inequalities.

## 2. Properties of $L_A, R_A$

Now we give the theorem.

**Theorem 2.1.** *For $A \in M_n(\mathbb{C})$,*

$$\mathbb{T}\mathrm{r}(L_A) = n\mathrm{Tr}(A),\ \mathbb{T}\mathrm{r}(R_A) = n\mathrm{Tr}(A),$$

*where $\mathbb{T}\mathrm{r}$ represents the trace of operator on $(M_n(\mathbb{C}), (\cdot,\cdot)_{HS})$.*

*Proof.* Let $e_{ij}$ be matrix unit. That is $n \times n$ positive matrices with 1 for $(i,j)$ entry and 0 for other entries. Then since $\{e_{ij}\}$ are orthnormal basis of $M_n(\mathbb{C})$,

$$\begin{aligned}
\mathbb{T}\mathrm{r}(L_A) &= \sum_{i,j=1}^n \langle e_{ij}, Ae_{ij}\rangle_{HS} = \sum_{i,j=1}^n Tr(e_{ij}^* Ae_{ij}) = \sum_{i,j=1}^n Tr(Ae_{ij}e_{ij}^*) \\
&= \sum_{i,j=1}^n Tr(Ae_{ii}) = \sum_{j=1}^n Tr(\sum_{i=1}^n Ae_{ii}) = \sum_{j=1}^n Tr(A(\sum_{i=1}^n e_{ii})) \\
&= \sum_{j=1}^n Tr(A) = nTr(A).
\end{aligned}$$

It is similar to show about $\mathbb{T}\mathrm{r}(R_A)$. □

**Theorem 2.2.** *For $A, B \in M_{n,+}(\mathbb{C})$,*

(1) $|L_A - R_B| = \sum_{i=1}^n \sum_{j=1}^n |\alpha_i - \beta_j| L_{|\phi_i\rangle\langle\phi_i|} R_{|\psi_j\rangle\langle\psi_j|}$.
(2) $\mathbb{T}\mathrm{r}(|L_A - R_B|) = \sum_{i=1}^n \sum_{j=1}^n |\alpha_i - \beta_j|$.
(3) $\mathrm{Tr}(|L_A - R_B|I) = \sum_{i=1}^n \sum_{j=1}^n |\alpha_i - \beta_j||\langle\phi_i|\psi_j\rangle|^2$.

*Proof.* (1) It is clear from Proposition 1.2.
(2) We put $P_i = |\phi_i\rangle\langle\phi_i|,\ Q_j = |\psi_j\rangle\langle\psi_j|$.

$$\begin{aligned}\mathbb{T}\mathrm{r}(|L_A - R_B|) &= \sum_{i,j=1}^{n} |\alpha_i - \beta_j| \sum_{s,t=1}^{n} \langle e_{st}, L_{P_i} R_{Q_j} e_{st}\rangle_{HS} \\ &= \sum_{i,j=1}^{n} |\alpha_i - \beta_j| \sum_{s,t=1}^{n} \langle e_{st}, P_i e_{st} Q_j\rangle_{HS} \\ &= \sum_{i,j=1}^{n} |\alpha_i - \beta_j| \sum_{s,t=1}^{n} Tr(e_{st}^* P_i e_{st} Q_j) \\ &= \sum_{i,j=1}^{n} |\alpha_i - \beta_j| \sum_{s,t=1}^{n} Tr(e_{ts} P_i e_{st} Q_j).\end{aligned}$$

Since

$$Tr(e_{ts}|\phi_i\rangle\langle\phi_i|e_{st}|\psi_j\rangle\langle\psi_j|) = \langle\psi_j|e_{ts}|\phi_i\rangle\langle\phi_i|e_{st}|\psi_j\rangle,$$

$$\sum_{s,t=1}^{n} \langle\psi_j|e_{ts}|\phi_i\rangle\langle\phi_i|e_{st}|\psi_j\rangle = \sum_{s,t=1}^{n} |\langle\phi_i|e_{st}|\psi_j\rangle|^2.$$

Let $\phi_{is}$ be $s$ component of $|\phi_i\rangle$, $\psi_{tj}$ be $t$ component of $|\psi_j\rangle$. Since

$$|\langle\phi_i|e_{st}|\psi_j\rangle|^2 = |\overline{\phi}_{is}\psi_{tj}|,$$

$$\begin{aligned}\sum_{s,t=1}^{n} |\langle\phi_i|e_{st}|\psi_j\rangle|^2 &= \sum_{s,t=1}^{n} |\overline{\phi}_{is}\psi_{tj}|^2 = \sum_{s=1}^{n} |\overline{\phi}_{is}|^2 \sum_{t=1}^{n} |\psi_{tj}|^2 \\ &= \sum_{s=1}^{n} |\phi_{si}|^2 \sum_{t=1}^{n} |\psi_{tj}|^2 = 1.\end{aligned}$$

Thus

$$\mathbb{T}\mathrm{r}(|L_A - R_B|) = \sum_{i=1}^{n}\sum_{j=1}^{n} |\alpha_i - \beta_j|.$$

(3) By (1),

$$\begin{aligned}|L_A - R_B|I &= \sum_{i=1}^{n}\sum_{j=1}^{n} |\alpha_i - \beta_j| L_{|\phi_i\rangle\langle\phi_i|} R_{|\psi_j\rangle\langle\psi_j|} I \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} |\alpha_i - \beta_j| |\phi_i\rangle\langle\phi_i|\psi_j\rangle\langle\psi_j|.\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{Tr}(|L_A - R_B|I) &= \sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} |\alpha_i - \beta_j| \langle\psi_k|\phi_i\rangle\langle\phi_i|\psi_j\rangle\langle\psi_j|\psi_k\rangle \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} |\alpha_i - \beta_j| \langle\psi_j|\phi_i\rangle\langle\phi_i|\psi_j\rangle \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} |\alpha_i - \beta_j| |\langle\phi_i|\psi_j\rangle|^2.
\end{aligned}$$

□

**Theorem 2.3.** *We define $D(A,B) = \mathrm{Tr}(|L_A - R_B|I)$ for $A, B \in M_{n,+}(\mathbb{C})$. Then $D(A,B)$ is a metric on $M_{n,+}(\mathbb{C})$.*

(1) *$D(A,B) \geq 0$, $D(A,B) = 0$ is equivalent to $A = B$.*
(2) *$D(A,B) = D(B,A)$.*
(3) *$D(A,B) \leq D(A,C) + D(C,B)$.*

*Proof.* (1) $D(A,B) = \mathrm{Tr}(|L_A - R_B|I) = \sum_{i,j=1}^{n} |\alpha_i - \beta_j||\langle\phi_i|\psi_j\rangle|^2 \geq 0$.
Since $A = B \Longrightarrow D(A,B) = 0$ is clear, we prove the reverse. If $D(A,B) = 0$, then $|\alpha_i - \beta_j||\langle\phi_i|\psi_j\rangle| = 0$ for all $i,j$. Then for all $i,j$, $\alpha_i = \beta_j$ or $\langle\phi_i|\psi_j\rangle = 0$. For simplicity we prove the case of $n = 2$. For

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},\ 0 \leq \theta \leq 2\pi,$$

we don't lose the generization by putting

$$|\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ |\phi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and

$$|\psi_i\rangle = U|\phi_i\rangle$$

In this case $A, B$ are represented by

$$A = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix},$$

$$B = \beta_1 \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} + \beta_2 \begin{pmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix}.$$

When $\theta \neq 0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi, 2\pi$, $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = \lambda$. Then we obtain $A = B = \lambda I$. When $\cos\theta = 0$, $|\sin\theta| = 1$. Then $\alpha_1 = \beta_2, \alpha_2 = \beta_1$. And

$$A = B = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}.$$

When $\sin\theta = 0$, $|\cos\theta| = 1$. Then $\alpha_1 = \beta_1, \alpha_2 = \beta_2$. And

$$A = B = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}.$$

Thus we give $A = B$.
(2) We put $A = \sum_i \alpha_i P_i = \sum_i \alpha_i |\phi_i\rangle\langle\phi_i|,\ B = \sum_j \beta_j Q_j = \sum_j \beta_j |\psi_j\rangle\langle\psi_j|$. Then

$$\mathrm{Tr}(|L_A - R_B|I) = \sum_{i,j} |\alpha_i - \beta_j||\langle\phi_i|\psi_j\rangle|^2,$$

$$\mathrm{Tr}(|L_B - R_A|I) = \sum_{j,i} |\beta_j - \alpha_i||\langle\psi_j|\phi_i\rangle|^2 = \sum_{i,j} |\alpha_i - \beta_j||\langle\phi_i|\psi_j\rangle|^2.$$

Then $\mathrm{Tr}(|L_A - R_B|I) = \mathrm{Tr}(|L_B - R_A|I)$. We give $D(A,B) = D(B,A)$.
(3) We show the triangle inequality. We put

$$L_A = \sum_i \alpha_i L_{P_i},\ L_B = \sum_i \beta_i L_{Q_i},\ L_C = \sum_i \gamma_i L_{R_i},$$

$$R_A = \sum_i \alpha_i R_{P_i},\ R_B = \sum_i \beta_i R_{Q_i},\ R_C = \sum_i \gamma_i R_{R_i},$$

where $P_i = |\phi_i\rangle\langle\phi_i|,\ Q_i = |\psi_i\rangle\langle\psi_i|,\ R_i = |\xi_i\rangle\langle\xi_i|$. Then

$$|L_A - R_C| = \sum_{i,\ell} |\alpha_i - \gamma_\ell| L_{P_i} R_{Q_\ell} = \sum_{i,j,k,\ell} |\alpha_i - \gamma_\ell| L_{P_i} L_{S_k} R_{S_j} R_{Q_\ell},$$

$$|L_A - R_B| = \sum_{i,j} |\alpha_i - \beta_j| L_{P_i} R_{S_j} = \sum_{i,j,k,\ell} |\alpha_i - \beta_j| L_{P_i} L_{S_k} R_{S_j} R_{Q_\ell},$$

$$|L_B - R_C| = \sum_{k,\ell} |\beta_k - \gamma_\ell| L_{S_k} R_{Q_\ell} = \sum_{i,j,k,\ell} |\beta_k - \gamma_\ell| L_{P_i} L_{S_k} L_{S_j} R_{Q_\ell}.$$

Now we have

$$\begin{aligned}
& |L_A - R_C| \\
= & \sum_{i,j,k,\ell} |\alpha_i - \gamma_\ell| L_{P_i} L_{S_k} R_{S_j} R_{Q_\ell} \\
\leq & \sum_{i,j,k,\ell} (|\alpha_i - \beta_j| + |\beta_j - \beta_k| + |\beta_k - \gamma_\ell|) L_{P_i} L_{S_k} R_{S_j} R_{Q_\ell} \\
= & \sum_{i,j,k,\ell} |\alpha_i - \beta_j| L_{P_i} L_{S_k} R_{S_j} R_{Q_\ell} + \sum_{i,j,k\ell} |\beta_j - \beta_k| L_{P_i} L_{S_k} R_{S_j} R_{Q_\ell} \\
& \quad + \sum_{i,j,k,\ell} |\beta_k - \gamma_\ell| L_{P_i} L_{S_k} R_{S_j} R_{Q_\ell} \\
= & \sum_{i,j} |\alpha_i - \beta_j| L_{P_i} R_{S_j} + \sum_{j,k} |\beta_j - \beta_k| L_{S_k} R_{S_j} + \sum_{k,\ell} |\beta_k - \gamma_\ell| L_{S_k} R_{Q_\ell} \\
= & |L_A - R_B| + |L_B - R_B| + |L_B - R_C|.
\end{aligned}$$

Since $|L_B - R_B| = \sum_{i,j} |\beta_i - \beta_j| L_{P_i} R_{P_j}$,

$$|L_B - R_B| I = \sum_{i,j} |\beta_i - \beta_j| P_i P_j = \sum_i |\beta_i - \beta_i| P_i = 0.$$

Then

$$Tr(|L_A - R_C|I) \leq Tr(|L_A - R_B|I) + Tr(|L_B - R_C|I).$$

We give $D(A,C) \leq D(A,B) + D(B,C)$. $\square$

**Remark 2.1.** *We state the two remarks.*

(1) *For* $A, B \in M_{n,+}(\mathbb{C})$, *there are no relationship between* $\mathrm{Tr}(|L_A - R_B|I)$ *and* $\mathrm{Tr}(|A-B|)$. *Because when*
$$A = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}, \; B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix},$$
$\mathrm{Tr}(|L_A - R_B|I) = 3$, $\mathrm{Tr}(|A-B|) = \sqrt{10}$. *On the other hand when*
$$A = \begin{pmatrix} \frac{13}{2} & \frac{7}{2} \\ \frac{7}{2} & \frac{13}{2} \end{pmatrix}, \; B = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix},$$
$\mathrm{Tr}(|L_A - R_B|I) = 8$, $\mathrm{Tr}(|A-B|) = \sqrt{58}$.

(2) *For* $A, B \in M_{2,+,1}(\mathbb{C})$, *we can prove* $\mathrm{Tr}(|L_A - R_B|I) \le \mathrm{Tr}(|A-B|)$ *but we expect the same result in the case of* $n \ge 3$.

## 3. Generalized quasi-metric adjusted skew information and Correlation measure

A function $f : (0, +\infty) \to \mathbb{R}$ is said operator monotone if, for any $n \in \mathbb{N}$, and $A, B \in M_{n,+}(\mathbb{C})$ such that $0 \le A \le B$, the inequality $0 \le f(A) \le f(B)$ holds. An operator monotone function is said symmetric if $f(x) = xf(x^{-1})$ and normalized if $f(1) = 1$.

**Definition 3.1.** *Let* $\mathfrak{F}_{op}$ *be the class of functions* $f : (0, +\infty) \to (0, +\infty)$ *satisfying*

(1) $f(1) = 1$,
(2) $tf(t^{-1}) = f(t)$,
(3) $f$ *is operator monotone.*

**Example 3.1.** *Examples of elements of* $\mathfrak{F}_{op}$ *are given by the following list, for any* $x > 0$,
$$f_{RLD}(x) = \frac{2x}{x+y}, \; f_{SLD}(x) = \frac{x+1}{2}, \; f_{BKM}(x) = \frac{x-1}{\log x},$$
$$f_{WY}(x) = \left(\frac{\sqrt{x}+1}{2}\right)^2, \; f_{WYD}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^{\alpha}-1)(x^{1-\alpha}-1)}, \alpha \in (0,1).$$

For $f \in \mathfrak{F}_{op}$ define $f(0) = \lim_{x \to 0} f(x)$. We introduce the sets of regular and non-regular functions
$$\mathfrak{F}_{op}^r = \{f \in \mathfrak{F}_{op} | f(0) \neq 0\}, \; \mathfrak{F}_{op}^n = \{f \in \mathfrak{F}_{op} | f(0) = 0\}$$
and notice that trivially $\mathfrak{F}_{op} = \mathfrak{F}_{op}^r \cup \mathfrak{F}_{op}^n$. In Kubo-Ando theory of matrix means one associates a mean to each operator monotone function $f \in \mathfrak{F}_{op}$ by the formula
$$m_f(A, B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},$$
where $A, B \in M_{n,+}(\mathbb{C})$. By using the notion of matrix means we define the generalized monotone metrics for $X, Y \in M_n(\mathbb{C})$ by the following formula
$$\langle X, Y \rangle_f = Tr[X^* m_f(L_A, R_B)^{-1} Y],$$
where $L_A(X) = AX, R_B(X) = XB$.

**Definition 3.2.** *Let $g, f \in \mathfrak{F}_{op}^r$ satisfy*

$$g(x) \geq k\frac{(x-1)^2}{f(x)}$$

*for some $k > 0$. We define*

$$\Delta_g^f(x) = g(x) - k\frac{(x-1)^2}{f(x)} \in \mathfrak{F}_{op}. \tag{3.1}$$

**Definition 3.3.** *Notation as in Definition 3.2. For $X, Y \in M_n(\mathbb{C})$ and $A, B \in M_{n,+}(\mathbb{C})$, we define the following quantities:*

(1) $\Gamma_{A,B}^{(g,f)}(X,Y) = k\langle (L_A - R_B)X, (L_A - R_B)Y\rangle_f$
$= kTr[X^*(L_A - R_B)m_f(L_A, R_B)^{-1}(L_A - R_B)Y]$
$= Tr[X^* m_g(L_A, R_B)Y] - Tr[X^* m_{\Delta_g^f}(L_A, R_B)Y]$,

(2) $I_{A,B}^{(g,f)}(X) = \Gamma_{A,B}^{(g,f)}(X,X)$,

(3) $\Psi_{A,B}^{(g,f)}(X,Y) = Tr[X^* m_g(L_A, R_B)Y] + Tr[X^* m_{\Delta_g^f}(L_A, R_B)Y]$,

(4) $J_{A,B}^{(g,f)}(X) = \Psi_{A,B}^{(g,f)}(X,X)$,

(5) $U_\rho^{(g,f)}(X) = \sqrt{I_{A,B}^{(g,f)}(X) J_{A,B}^{(g,f)}(X)}$.

*The quantities $I_{A,B}^{(g,f)}(X)$ and $\Gamma_{A,B}^{(g,f)}(X,Y)$ are said generalized quasi-metric adjusted skew information and generalized quasi-metric adjusted correlation measure, respectively.*

We state an interesting uncertainty relation which is proved in [7]. We omit the proof.

**Theorem 3.1** ([7])**.** *For $f \in \mathfrak{F}_{op}^r$, it holds*

$$I_{A,B}^{(g,f)}(X) \cdot I_{A,B}^{(g,f)}(Y) \geq |\Gamma_{A,B}^{(g,f)}(X,Y)|^2 \geq \frac{1}{16}\left(I_{A,B}^{(g,f)}(X+Y) - I_{A,B}^{(g,f)}(X-Y)\right)^2,$$

*where $X, Y \in M_n(\mathbb{C})$ and $A, B \in M_{n,+}(\mathbb{C})$.*

By setting $g = f_{SLD},\ f = f_{WY},\ k = \frac{1}{4},\ A = B = \rho \in M_{n,+,1}(\mathbb{C})$, we have the following corollary.

**Corollary 3.1** ([4], Theorem 3.3)**.** *Let $X, Y \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$ be a quantum state. Then*

$$|I_\rho|(X) \cdot |I_\rho|(Y) \geq \frac{1}{16}(|I_\rho|(X+Y) - |I_\rho|(X-Y))^2,$$

*where $|I_\rho|(X) = -\frac{1}{2}Tr[[\rho^{1/2}, X^*][\rho^{1/2}, X]]$ and $[X,Y] = XY - YX$.*

We note the equation

$$|L_A - R_B| = \sum_{i=1}^n \sum_{j=1}^n |\lambda_i - \mu_j| L_{|\phi_i\rangle\langle\phi_i|} R_{|\psi_j\rangle\langle\psi_j|},$$

where $A = \sum_{i=1}^n \lambda_i |\phi_i\rangle\langle\phi_i|$, $B = \sum_{j=1}^n \mu_j |\psi_j\rangle\langle\psi_j|$ are the spectral decompositions. Next we state an interesting uncertainty relation which is proved in [7]. We omit the proof.

**Theorem 3.2** ([7]). *For $f \in \mathfrak{F}_{op}^r$, if*

$$g(x) + \Delta_g^f(x) \geq \ell f(x) \tag{3.2}$$

*for some $\ell > 0$, then the followings hold for $X, Y \in M_n(\mathbb{C})$ and $A, B \in M_{n,+}(\mathbb{C})$*

(1) $U_{A,B}^{(g,f)}(X) \cdot U_{A,B}^{(g,f)}(Y) \geq k\ell |Tr[X^* |L_A - R_B| Y]|^2$.

(2) $U_{A,B}^{(g,f)}(X) \cdot U_{A,B}^{(g,f)}(Y) \geq \frac{f(0)^2 \ell}{k} |\Gamma_{A,B}^{(g,f)}(X, Y)|^2$.

By setting $A = B = \rho \in M_{n,+,1}(\mathbb{C})$ we have the following corollary.

**Corollary 3.2** ([3], Theorem 3.5). *If $f, g \in \mathfrak{F}_{op}$ satisfy (3.2), then*

$$U_\rho^{(g,f)}(X) \cdot U_\rho^{(g,f)}(Y) \geq \frac{f(0)^2 \ell}{k} |Corr_\rho^{(g,f)}(X, Y)|^2,$$

*where $X, Y \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$. Here $U_\rho^{(g,f)}(X)$ and $Corr_\rho^{(g,f)}(X, Y)$ are defined in* [3].

## 4. Other trace inequalities

We assume that

$$g(x) = \frac{x+1}{2}, \ f(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}, \ k = \frac{f(0)}{2}, \ \ell = 2.$$

Then, since (3.1), (3.2) are satisfied for $g, f, k$ and $\ell$, we have the following trace inequality by putting $X = I$.

$$\begin{aligned} & \alpha(1-\alpha)(Tr[|L_A - R_B| I])^2 \\ \leq & \left(\frac{1}{2} Tr[A + B]\right)^2 - \left(\frac{1}{2} Tr[A^\alpha B^{1-\alpha} + A^{1-\alpha} B^\alpha]\right)^2. \end{aligned}$$

This is a generalization of trace inequality given in [2]. And also we give the following new inequality by combining the Chernoff type inequality with the above theorem. We omit the proof.

**Theorem 4.1** ([8]). *We have the following:*

$$\begin{aligned} & \frac{1}{2} Tr[A + B - |L_A - R_B| I] \leq \inf_{0 \leq \alpha \leq 1} Tr[A^{1-\alpha} B^\alpha] \\ \leq & Tr[A^{1/2} B^{1/2}] \leq \frac{1}{2} Tr[A^\alpha B^{1-\alpha} + A^{1-\alpha} B^\alpha] \\ \leq & \sqrt{\left(\frac{1}{2} Tr[A + B]\right)^2 - \alpha(1-\alpha)(Tr[|L_A - R_B| I)^2}. \end{aligned}$$

**Theorem 4.2.** *Let $A, B \in M_{n,+}(\mathbb{C})$ have the following spectral decompositions*

$$A = \sum_{i=1}^{n} \alpha_i |\phi_i\rangle\langle\phi_i|, \ B = \sum_{j=1}^{n} \beta_j |\psi_j\rangle\langle\psi_j|.$$

*Then we give an inequality.*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} |\alpha_i - \alpha_j| + \sum_{i=1}^{n}\sum_{j=1}^{n} |\beta_i - \beta_j| \le 2\sum_{i=1}^{n}\sum_{j=1}^{n} |\alpha_i - \beta_j|.$$

*That is*

$$\sum_{i<j} |\alpha_i - \alpha_j| + \sum_{i<j} |\beta_i - \beta_j| \le \sum_{i=1}^{n}\sum_{j=1}^{n} |\alpha_i - \beta_j|.$$

*We also represent as follows.*

$$\mathbb{T}\mathrm{r}(|L_A - R_A|) + \mathbb{T}\mathrm{r}(|L_B - R_B|) \le 2\mathbb{T}\mathrm{r}(|L_A - R_B|).$$

We need the following lemma in order to prove.

**Lemma 4.1.** *For any $x \in \mathbb{R}$, let $A_x$ be the numbers of the line segments combining $\alpha_i$ and $\alpha_j$ through $x$, $B_x$ be the numbers of the line segments combining $\beta_i$ and $\beta_j$ through $x$ and $D_x$ be the numbers of the line segments combing $\alpha_i$ and $\beta_j$ through $x$, respectively. We put $S_x = A_x + B_x$, $S = \int_{-\infty}^{+\infty} S_x dx$ and $D = \int_{-\infty}^{+\infty} D_x dx$. Then we have $S \le D$.*

*Proof.* We may prove $S_x \le D_x$ for any $x \in \mathbb{R}$. Let $B_x$ be the numbers of $\alpha_i$ which are located in left place of $x$ and $R_x$ be the numbers of $\beta_i$ which are located in left place of $x$. Since

$$S_x = B_x(n - B_x) + R_x(n - R_x), \ D_x = B_x(n - R_x) + R_x(n - B_x),$$

we have

$$D_x - S_x = (B_x - R_x)(n - R_x) + (R_x - B_x)(n - B_x) = (R_x - B_x)^2 \ge 0.$$

Thus $S \le D$. □

**Theorem 4.3.** *Let $\{X_i\}_{i-1}^N, \{Y_j\}_{j=1}^N \subset M_n(\mathbb{C})$. Then (1), (2) hold.*

(1) $\sum_{i=1}^N \sum_{j=1}^N \|X_i - X_j\| + \sum_{i=1}^N \sum_{j=1}^N \|Y_i - Y_j\| \le 2\sum_{i=1}^N \sum_{j=1}^N \|X_i - Y_j\|$

(2) $\sum_{i=1}^N \sum_{j=1}^N \|X_i \pm X_j\|^2 + \sum_{i=1}^N \sum_{j=1}^N \|Y_i \pm Y_j\|^2$
$= 2\left(\sum_{i=1}^N \sum_{j=1}^N \|X_i \pm Y_j\|^2 \pm \|\sum_{i=1}^N X_i - \sum_{j=1}^N Y_j\|^2\right)$

*Proof.* (1) The result is an extension of Theorem 4.2. We omit the proof.
(2) Since $\|X\|^2 = \langle X, X\rangle_{HS}$, it is easy to get the result. □

**Corollary 4.1.** *Let $\{X_i\}_{i-1}^N, \{Y_j\}_{j=1}^N \subset M_n(\mathbb{C})$, $A, B \in M_{n,+}(\mathbb{C})$ and $f, g \in \mathfrak{F}_{op}^r$. Then (1), (2) hold.*

(1) $\sum_{i=1}^N \sum_{j=1}^N \sqrt{I_{A,B}^{(g,f)}(X_i - X_j)} + \sum_{i=1}^N \sum_{j=1}^N \sqrt{I_{A,B}^{(g,f)}(Y_i - Y_j)}$
$\le 2\sum_{i=1}^N \sum_{j=1}^N \sqrt{I_{A,B}^{(g,f)}(X_i - Y_j)}$

$$(2)\ \sum_{i=1}^N \sum_{j=1}^N I_{A,B}^{(g,f)}(X_i \pm X_j) + \sum_{i=1}^N \sum_{j=1}^N I_{A,B}^{(g,f)}(Y_i \pm Y_j)$$
$$= 2\left(\sum_{i=1}^N \sum_{j=1}^N I_{A,B}^{(g,f)}(X_i \pm Y_j) \pm I_{A,B}^{(g,f)}\left(\sum_{i=1}^N X_i - \sum_{j=1}^N Y_j\right)\right)$$

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