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STRONG CONVERGENCE THEOREMS BY HYBRID METHODS FOR NONOMMUTATIVE DEMIGENERIC GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, using the hybrid method defined by Nakajo and Takahashi [15], we first obtain a strong convergence theorem for finding a common fixed point of two noncommutative demigeneric generalized hybrid mappings and a zero point of a maximal monotone operator in a Hilbert space. Next, using the shrinking projection method defined by Takahashi, Takeuchi and Kubota [22], we prove another strong convergence theorem for finding a common fixed point of the mappings and a zero point of the monotone operator in a Hilbert space. Using these results, we get well-known and new strong convergence theorems by the hybrid method and the shrinking projection method in a Hilbert space.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. Let T be a mapping of C into H. We denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{z \in C : Tz = z\}$. A mapping $T : C \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. It is well-known that if C is a bounded, closed and convex subset of H and $T : C \to C$ is nonexpansive, then $F(T)$ is nonempty. Furthermore, from Baillon [3] we know the first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space. Let C be a nonempty, closed and convex subset of H and let $T : C \to C$ be a nonexpansive mapping such that $F(T)$ is nonempty. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$.

In 2010, Kocourek, Takahashi and Yao [10] defined a broad class of nonlinear mappings in a Hilbert space: Let C be a nonempty subset of H. A mapping $T : C \to H$ is called *generalized hybrid* [10] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{1.1}$$

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for all $x, y \in C$. Such a mapping T is called (α, β)-*generalized hybrid*. We also know the following: For $\lambda \in \mathbb{R}$, a mapping $U : C \to H$ is called λ-*hybrid* [1] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Ux, y - Uy\rangle \tag{1.2}$$

for all $x, y \in C$. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is *nonspreading* [12, 13] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also *hybrid* [19] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [7]. We also know that the class of λ-hybrid mappings is contained in the class of generalized hybrid mappings; see [6]. Kohsaka [11] proved the following theorem.

Theorem 1.1 ([11]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let S and T be commutative λ and μ-hybrid mappings of C into itself such that the set $F(S) \cap F(T)$ of common fixed points of S and T is nonempty. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to a point of $F(S) \cap F(T)$.

On the other hand, in 2003, Nakajo and Takahashi [15] introduced the following hybrid method for finding a fixed point of a nonexpansive mapping in a Hilbert space: Let C be a nonempty, closed and convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\alpha_n \in [0, 1]$ and $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$.

Takahashi, Takeuchi and Kubota [22] also introduced the shrinking projection method for finding a fixed point of a nonexpansive mapping in a Hilbert space: Let C be a nonempty, closed and convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. For $x_1 \in C$ and $C_1 = C$, define $\{x_n\} \subset C$ be a sequence generated by

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\alpha_n \in [0, 1]$ and $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1}.

In this paper, using the hybrid method defined by Nakajo and Takahashi [15], we first obtain a strong convergence theorem for finding a common fixed point of two noncommutative demigeneric generalized hybrid mappings and a zero point of a maximal monotone operator in a Hilbert space. Next, using the shrinking projection method defined by Takahashi, Takeuchi and Kubota [22], we prove another strong convergence theorem for finding a common fixed point of the mappings and a zero point of the monotone operator in a Hilbert space. Using these results, we get well-known and new strong convergence theorems by the hybrid method and the shrinking projection method in a Hilbert space.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $\|\cdot\|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. In a Hilbert space, it is known that

$$\|x + y - z\|^2 - \|x - z\|^2 \geq 2\langle y, x - z\rangle \tag{2.1}$$

for all $x, y, z \in H$ and

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha \|x\|^2 + (1 - \alpha) \|y\|^2 - \alpha(1 - \alpha) \|x - y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$; see [18]. Furthermore, in a Hilbert space, we have that

$$2 \langle x - y, z - w\rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2 \tag{2.3}$$

for all $x, y, z, w \in H$. Let H be a Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if

$$\|Tx - u\| \leq \|x - u\|, \quad \forall x \in C,\ u \in F(T).$$

If C is closed and convex and $T : C \to H$ with $F(T) \neq \emptyset$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [8]. For a nonempty, closed and convex subset D of H, the nearest point projection of H onto D is denoted by P_D, that is, $\|x - P_Dx\| \leq \|x - y\|$ for all $x \in H$ and $y \in D$. Such a mapping P_D is called the metric projection of H onto D. We know that the metric projection P_D is firmly nonexpansive; $\|P_Dx - P_Dy\|^2 \leq \langle P_Dx - P_Dy, x - y\rangle$ for all $x, y \in H$. Furthermore, $\langle x - P_Dx, y - P_Dx\rangle \leq 0$ holds for all $x \in H$ and $y \in D$; see [17, 18]. Using this inequality and (2.3), we have that

$$\|P_Dx - y\|^2 + \|P_Dx - x\|^2 \leq \|x - y\|^2, \quad \forall x \in H,\ y \in D. \tag{2.4}$$

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be a monotone operator on H if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in \mathrm{dom}(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1}\colon H \to \mathrm{dom}(B)$, which is called the resolvent of B for $r > 0$. We denote by $A_r = \frac{1}{r}(I - J_r)$ the Yosida approximation of B for $r > 0$. We know from [18] that

$$A_rx \in BJ_rx, \quad \forall x \in H,\ r > 0. \tag{2.5}$$

Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is known that $B^{-1}0 = F(J_r)$ for all $r > 0$ and the resolvent J_r is firmly nonexpansive, i.e.,

$$\|J_r x - J_r y\|^2 \leq \langle x - y, J_r x - J_r y\rangle, \quad \forall x, y \in H. \tag{2.6}$$

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Then a mapping T from C into H is said to be *demigeneric generalized hybrid* [21] if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta > 0$ and

$$\begin{aligned} \alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 \leq 0 \end{aligned} \tag{2.7}$$

for all $x, y \in C$. Such a mapping T is called $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-*demigeneric generalized hybrid.* This concept of demigeneric generalized hybrid mappings generalizes the concept of generic generalized hybrid mappings [23], i.e., a mapping T from C into H is said to be *generic generalized hybrid* [23] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0 \tag{2.8}$$

for all $x, y \in C$.

We also know the following definition from Kawasaki and Takahashi [9]. Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let θ be a real number with $\theta \neq 0$. A mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called θ-generalized demimetric [9] if

$$\theta\langle x - q, J(x - Ux)\rangle \geq \|x - Ux\|^2 \tag{2.9}$$

for all $x \in C$ and $q \in F(U)$, where J is the duality mapping on E. This concept of generalized demimetric mapings covers demimetric mappings defined by Takahashi [20]: Let η be a real number with $\eta \in (-\infty, 1)$. A mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called η-demimetric [20] if

$$2\langle x - q, J(x - Ux)\rangle \geq (1 - \eta)\|x - Ux\|^2$$

for all $x \in C$ and $q \in F(U)$. When a Banach space E is a Hilbert space, the concept of generalized demimetric mapings is as follows:

$$\theta\langle x - q, x - Ux\rangle \geq \|x - Ux\|^2$$

for all $x \in C$ and $q \in F(U)$.

Examples. We know examples of demigeneric generalized hybrid mappings and generalized demimetric mappings.

(1) Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let k be a real number with $0 \leq k < 1$. A mapping $T : C \to H$ is called a k-strict pseudo-contraction [4] if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2$$

for all $x, y \in C$. If T is a k-strict pseudo-contraction and $F(T) \neq \emptyset$, then T is $(1 - k, k, k, -(1 + k), -k, -k)$-demigeneric generalized hybrid and $\frac{2}{1-k}$-generalized demimetric; see [21].

(2) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called generalized hybrid [10] if it satisfies (1.1), i.e., there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1-\alpha)\|x - Uy\|^2 \le \beta\|Ux - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. If U is (α, β)-generalized hybrid and $F(U) \neq \emptyset$, then U is $(\alpha, 1-\alpha, -\beta, -1+\beta, 0, 0)$-demigeneric generalized hybrid and 2-generalized demimetric; see [21].

(3) Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $T : C \to H$ is a Lipschitzian mapping, i.e., there exists $L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\|$$

for all $x, y \in C$. If $T : C \to H$ is a L-Lipschitzian mapping and $F(\frac{T}{L}) \neq \emptyset$, then $S = (L+1)I - T$ is $(1, -L-1, -L-1, 2L+1, L+1, L+1)$-demigeneric generalized hybrid and $(-2L)$-generalized demimetric; see [21].

(4) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $A : C \to H$ is called inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \ge \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Such a mapping U is called α-inverse strongly monotone. Let A be an α-inverse strongly monotone mapping from C into H with $A^{-1}0 \neq \emptyset$ and define $T = I + A$. Then T is $(2\alpha, -(2\alpha+1), -(2\alpha+1), 2(\alpha+1), 2\alpha+1, 2\alpha+1)$ -demigeneric generalized hybrid and $(-\frac{1}{\alpha})$-generalized demimetric; see [21].

We know the following results from [21] and [9].

Lemma 2.1 ([21])**.** *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-demigeneric generalized hybrid mapping from C into H such that $F(T) \neq \emptyset$ and $\alpha + \beta \neq 0$. Then T is $\frac{2(\alpha+\beta)}{\alpha+\beta+\zeta}$-generalized demimetric.*

Lemma 2.2 ([9])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let θ be a real number with $\theta \neq 0$. Let T be a θ-generalized demimetric mapping of C into E. Then $F(T)$ is closed and convex.*

For a sequence $\{C_n\}$ of nonempty closed convex subsets of a Hilbert space H, define s-$\text{Li}_n C_n$ and w-$\text{Ls}_n C_n$ as follows: $x \in$s-$\text{Li}_n C_n$ if and only if there exists $\{x_n\} \subset H$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in$w-$\text{Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset H$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies

$$C_0 = \text{s-Li}_n C_n = \text{w-Ls}_n C_n, \tag{2.10}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [14] and we write $C_0 =$M-$\lim_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\cap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [14].

Tsukada [24] proved the following theorem.

Theorem 2.3 ([24]). *Let H be a Hilbert space. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of H. If $C_0 =$M-$\lim_{n\to\infty} C_n$ exists and nonempty, then for each $x \in H$, $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$, where P_{C_n} and P_{C_0} are the mertic projections of H onto C_n and C_0, respectively.*

3. Strong convergence theorems by hybrid methods

In this section, using the hybrid method by Nakajo and Takahashi [15], we first prove a strong convergence theorem for noncommutative two demigeneric generalized hybrid mappings in a Hilbert space. Before proving the result, we need the following lemma [21, 9]

Lemma 3.1 ([21, 9]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $U : C \to H$ be demigeneric generalized hybrid. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Theorem 3.2. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let A be a maximal monotone operator on H and let J_t be the resolvent of A for $t > 0$. Let S and T be $(\alpha_1^*, \beta_1^*, \gamma_1^*, \delta_1^*, \varepsilon_1^*, \zeta_1^*)$ and $(\alpha_2^*, \beta_2^*, \gamma_2^*, \delta_2^*, \varepsilon_2^*, \zeta_2^*)$-demigeneric generalized hybrid mappings of C into H, respectively, such that $\alpha_1^* + \beta_1^* \neq 0$ and $\alpha_2^* + \beta_2^* \neq 0$. Put*

$$\lambda_1 = \frac{\alpha_1^* + \beta_1^* + \zeta_1^*}{\alpha_1^* + \beta_1^*} \quad \text{and} \quad \lambda_2 = \frac{\alpha_2^* + \beta_2^* + \zeta_2^*}{\alpha_2^* + \beta_2^*}.$$

Suppose that $\Omega := A^{-1}0 \cap F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and

$$\begin{cases} y_n = J_{t_n}\big(\alpha_n x_n + (1-\alpha_n)z_n\big), \\ z_n = \gamma_n((1-\lambda_1)I + \lambda_1 S)x_n + (1-\gamma_n)((1-\lambda_2)I + \lambda_2 T)x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n\cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$, $t_o, a, b, c, d \in \mathbb{R}$, $\{t_n\} \subset (0,\infty)$ and $\{\gamma_n\}, \{\alpha_n\} \subset [0,1]$ satisfy the following:

$$0 < t_0 \leq t_n,\ 0 < a \leq \gamma_n \leq b < 1 \quad \text{and} \quad 0 < c \leq \alpha_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = P_\Omega x_1$, where P_Ω is the metric projection of H onto Ω.

Proof. We first prove that $(1-\lambda_1)I + \lambda_1 S$ is quasi-nonexpansive. In fact, we have by Lemma 2.1 that S is $\frac{2(\alpha_1^*+\beta_1^*)}{\alpha_1^*+\beta_1^*+\zeta_1^*}$-generalized demimetric. Then we have that for any $x \in C$ and $p \in F(S)$,

$$\frac{2(\alpha_1^* + \beta_1^*)}{\alpha_1^* + \beta_1^* + \zeta_1^*}\langle x - Sx, x - p\rangle \geq \|x - Sx\|^2,$$

From $\lambda_1 = \frac{\alpha_1^*+\beta_1^*+\zeta_1^*}{\alpha_1^*+\beta_1^*}$, we get that $\frac{2}{\lambda_1}\langle x - Sx, x - p\rangle \geq \|x - Sx\|^2$. Thus we have that

$$\lambda_1^2 \frac{2}{\lambda_1}\langle x - Sx, x - p\rangle \geq \lambda_1^2\|x - Sx\|^2$$

and hence

$$2\langle x - ((1-\lambda_1)I + \lambda_1 S)x, x - p\rangle \geq \|x - ((1-\lambda_1)I + \lambda_1 S)x\|^2.$$

Using (2.3), we have that

$$\begin{aligned} &\|x - p\|^2 + \|((1-\lambda_1)I + \lambda_1 S)x - x\|^2 - \|((1-\lambda_1)I + \lambda_1 S)x - p\|^2 \\ &\quad \geq \|x - ((1-\lambda_1)I + \lambda_1 S)x\|^2 \end{aligned}$$

and hence

$$\|x - p\|^2 - \|((1-\lambda_1)I + \lambda_1 S)x - p\|^2 \geq 0.$$

This implies that $(1-\lambda_1)I + \lambda_1 S$ is quasi-nonexpansive. Similarly, $(1-\lambda_2)I + \lambda_2 T$ is quasi-nonexpansive. Since S and T are generalized demimetric, we also have from Lemma 2.2 that $F(S) \cap F(T)$ is closed and convex. Furthermore, since A is a maximal monotone operator, $A^{-1}0$ is closed and convex. So, $\Omega = A^{-1}0 \cap F(S) \cap F(T)$ is closed and convex. Thus, there exists the mertic projection of H onto Ω. Since

$$\begin{aligned} &\|y_n - z\|^2 \leq \|x_n - z\|^2 \\ \Longleftrightarrow &\|y_n\|^2 - \|x_n\|^2 - 2\langle y_n - x_n, z\rangle \leq 0, \end{aligned}$$

we have that C_n, Q_n and $C_n \cap Q_n$ are closed and convex for all $n \in \mathbb{N}$. We next show that $C_n \cap Q_n$ is nonempty. Put $S_1 = (1-\lambda_1)I + \lambda_1 S$ and $T_1 = (1-\lambda_2)I + \lambda_2 T$ and let $z \in \Omega$. Since J_{t_n} is nonexpansive and S_1 and T_1 are quasi-nonexpansive, we have that

$$\begin{aligned} \|z_n - z\|^2 &= \|\gamma_n S_1 x_n + (1-\gamma_n)T_1 x_n - z\|^2 \\ &\leq \|\gamma_n(S_1 x_n - z) + (1-\gamma_n)(T_1 x_n - z)\|^2 \\ &= \gamma_n\|S_1 x_n - z\|^2 + (1-\gamma_n)\|T_1 x_n - z\|^2 - \gamma_n(1-\gamma_n)\|S_1 x_n - T_1 x_n\|^2 \\ &\leq \gamma_n\|x_n - z\|^2 + (1-\gamma_n)\|x_n - z\|^2 - \gamma_n(1-\gamma_n)\|S_1 x_n - T_1 x_n\|^2 \\ &= \|x_n - z\|^2 - \gamma_n(1-\gamma_n)\|S_1 x_n - T_1 x_n\|^2 \\ &\leq \|x_n - z\|^2 \end{aligned}$$

and hence

$$\begin{aligned} \|y_n - z\|^2 &= \|J_{t_n}\big(\alpha_n x_n + (1-\alpha_n)z_n\big) - z\|^2 \\ &\leq \|\alpha_n x_n + (1-\alpha_n)z_n - z\|^2 \\ &\leq \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\ &\leq \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 \end{aligned}$$

$$= \|x_n - z\|^2.$$

Thus we have $z \in C_n$ and hence $\Omega \subset C_n$ for all $n \in \mathbb{N}$. Next, we show by induction that $\Omega \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. From $\Omega \subset Q_1$, it follows that $\Omega \subset C_1 \cap Q_1$. Suppose that $\Omega \subset C_k \cap Q_k$ for some $k \in \mathbb{N}$. From $x_{k+1} = P_{C_k \cap Q_k} x_1$, we have

$$\langle x_{k+1} - z, x_1 - x_{k+1} \rangle \geq 0, \quad \forall z \in C_k \cap Q_k.$$

Since $\Omega \subset C_k \cap Q_k$, we also have

$$\langle x_{k+1} - z, x_1 - x_{k+1} \rangle \geq 0, \quad \forall z \in \Omega.$$

This implies $\Omega \subset Q_{k+1}$. So, we have $\Omega \subset C_{k+1} \cap Q_{k+1}$. By induction, we have $\Omega \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. This means that $\{x_n\}$ is well-defined.

Since $x_n = P_{Q_n} x_1$ and $x_{n+1} = P_{C_n \cap Q_n} x_1 \in Q_n$, we have from (2.3) that

$$\begin{aligned} 0 &\leq 2\langle x_1 - x_n, x_n - x_{n+1} \rangle \\ &= \|x_1 - x_{n+1}\|^2 - \|x_1 - x_n\|^2 - \|x_n - x_{n+1}\|^2 \\ &\leq \|x_1 - x_{n+1}\|^2 - \|x_1 - x_n\|^2. \end{aligned} \tag{3.1}$$

Thus we get that

$$\|x_1 - x_n\|^2 \leq \|x_1 - x_{n+1}\|^2. \tag{3.2}$$

Furthermore, since $x_n = P_{Q_n} x_1$ and $z \in \Omega \subset Q_n$, we have

$$\|x_1 - x_n\| \leq \|x_1 - z\|. \tag{3.3}$$

We have from (3.2) and (3.3) that $\lim_{n\to\infty} \|x_1 - x_n\|^2$ exists. This implies that $\{x_n\}$ is bounded. Hence, $\{z_n\}$ and $\{y_n\}$ are also bounded. From (3.1), we have

$$\|x_n - x_{n+1}\|^2 \leq \|x_1 - x_{n+1}\|^2 - \|x_1 - x_n\|^2$$

and hence

$$\|x_n - x_{n+1}\| \to 0. \tag{3.4}$$

From $x_{n+1} \in C_n$, we have that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. From (3.4), we have $\|y_n - x_{n+1}\| \to 0$. So, we have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{3.5}$$

Furthermore, we have that, for $w \in \Omega$,

$$\begin{aligned} \|y_n - w\|^2 &\leq \|\alpha_n x_n + (1-\alpha_n) z_n - w\|^2 \\ &= \|\alpha_n (x_n - w) + (1-\alpha_n)(z_n - w)\|^2 \\ &= \alpha_n \|x_n - w\|^2 + (1-\alpha_n)\|z_n - w\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|z_n - x_n\|^2 \\ &\leq \alpha_n \|x_n - w\|^2 + (1-\alpha_n)\|x_n - w\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|z_n - x_n\|^2 \\ &= \|x_n - w\|^2 - \alpha_n(1-\alpha_n)\|z_n - x_n\|^2. \end{aligned}$$

Then, we obtain that

$$\alpha_n(1-\alpha_n)\|z_n - x_n\|^2 \leq \|x_n - w\|^2 - \|y_n - w\|^2$$

$$
\begin{aligned}
&= (\|x_n - w\| - \|y_n - w\|)(\|x_n - w\| + \|y_n - w\|) \\
&\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|).
\end{aligned}
$$

From the assumption of $\{\alpha_n\}$ and $\|y_n - x_n\| \to 0$, we have that

$$
\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.6}
$$

Putting $w_n = \alpha_n x_n + (1-\alpha_n)z_n$, we also have that

$$
\|x_n - w_n\| = \|x_n - \alpha_n x_n - (1-\alpha_n)z_n\| = (1-\alpha_n)\|x_n - z_n\|
$$

and hence

$$
\|w_n - x_n\| \to 0. \tag{3.7}
$$

We have from (2.2) that, for any $z \in F(S) \cap F(T)$,

$$
\begin{aligned}
\|x_n - z\|^2 &= \|x_n - z_n + z_n - z\|^2 \\
&\leq \|z_n - z\|^2 + 2\langle x_n - z_n, x_n - z\rangle \\
&= \|\gamma_n S_1 x_n + (1-\gamma_n)T_1 x_n - z\|^2 + 2\langle x_n - z_n, x_n - z\rangle \\
&= \gamma_n \|S_1 x_n - z\|^2 + (1-\gamma_n)\|T_1 x_n - z\|^2 \\
&\quad - \gamma_n(1-\gamma_n)\|S_1 x_n - T_1 x_n\|^2 + 2\langle x_n - z_n, x_n - z\rangle \\
&\leq \gamma_n \|x_n - z\|^2 + (1-\gamma_n)\|x_n - z\|^2 \\
&\quad - \gamma_n(1-\gamma_n)\|S_1 x_n - T_1 x_n\|^2 + 2\langle x_n - z_n, x_n - z\rangle \\
&= \|x_n - z\|^2 - \gamma_n(1-\gamma_n)\|S_1 x_n - T_1 x_n\|^2 + 2\langle x_n - z_n, x_n - z\rangle
\end{aligned}
$$

and hence

$$
\gamma_n(1-\gamma_n)\|S_1 x_n - T_1 x_n\|^2 \leq 2\langle x_n - z_n, x_n - z\rangle.
$$

Since $x_n - z_n \to 0$ and $\{x_n\}$ is bounded, we have that $S_1 x_n - T_1 x_n \to 0$. Then we have that

$$
\begin{aligned}
\|x_n - S_1 x_n\| &= \|x_n - T_n x_n + T_n x_n - S_1 x_n\| \\
&\leq \|x_n - T_n x_n\| + \|T_n x_n - S_1 x_n\| \\
&= \|x_n - T_n x_n\| + (1-\gamma_n)\|T_1 x_n - S_1 x_n\| \\
&\to 0.
\end{aligned}
$$

Since $\|x_n - S_1 x_n\| = \|x_n - ((1-\lambda_1)I + \lambda_1 S)x_n\| = |\lambda_1|\|x_n - Sx_n\|$, we have that $\|x_n - Sx_n\| \to 0$. Similarly, we have that $\|x_n - Tx_n\| \to 0$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup z^*$. From Lemma 3.1, we have $z^* \in F(S) \cap F(T)$. On the other hand, we have that for any $w \in \Omega$.

$$
\begin{aligned}
2\|y_n - w\|^2 &= 2\|J_{t_n} w_n - J_{t_n} w\|^2 \\
&\leq 2\langle w_n - w, y_n - w\rangle \\
&= \|w_n - w\|^2 + \|y_n - w\|^2 - \|w_n - y_n\|^2
\end{aligned}
$$

and hence

$$
\|w_n - y_n\|^2 \leq \|w_n - w\|^2 - \|y_n - w\|^2.
$$

Using this and $w_n - x_n = (1-\alpha_n)(z_n - x_n)$, we have that

$$\begin{aligned}\|w_n - y_n\|^2 &\le \|w_n - x_n + x_n - w\|^2 - \|y_n - w\|^2 \\ &= \|w_n - x_n\|^2 + 2\langle w_n - x_n, x_n - w\rangle + \|x_n - w\|^2 - \|y_n - w\|^2 \\ &= (1-\alpha_n)^2\|z_n - x_n\|^2 + 2(1-\alpha_n)\langle z_n - x_n, x_n - w\rangle \\ &\quad + \|x_n - w\|^2 - \|y_n - w\|^2 \\ &\le (1-\alpha_n)\|z_n - x_n\|^2 + 2(1-\alpha_n)\langle z_n - x_n, x_n - w\rangle \\ &\quad + \|x_n - w\|^2 - \|y_n - w\|^2.\end{aligned}$$

Since $z_n - x_n \to 0$ and $\|x_n - y_n\| \to 0$, we have that

$$w_n - y_n \to 0. \tag{3.8}$$

Furthermore, since A is a monotone operator and $\frac{w_{n_i} - y_{n_i}}{t_{n_i}} \in Ay_{n_i}$, we have that for any $(u, v) \in A$,

$$\langle u - y_{n_i}, v - \frac{w_{n_i} - y_{n_i}}{t_{n_i}}\rangle \ge 0.$$

Since $t_{n_i} \ge t_0 > 0$, $x_{n_i} \rightharpoonup z^*$ and $x_{n_i} - y_{n_i} \to 0$, we have $\langle u - z^*, v\rangle \ge 0$. Since A is a maximal monotone operator, we have $0 \in Az^*$ and hence $z^* \in A^{-1}0$. Thus we have $z^* \in \Omega$.

Put $z_0 = P_\Omega x_1$. Since $z_0 = P_\Omega x_1 \in C_n \cap Q_n$ and $x_{n+1} = P_{C_n \cap Q_n} x_1$, we have that

$$\|x_1 - x_{n+1}\| \le \|x_1 - z_0\|. \tag{3.9}$$

Since $\|\cdot\|$ is weakly lower semicontinuous, from $x_{n_i} \rightharpoonup z^*$ we have that

$$\|x_1 - z^*\| \le \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \le \|x_1 - z_0\|.$$

From the definition of z_0, we have $z^* = z_0$. So, we obtain $x_n \rightharpoonup z_0$. We finally show that $x_n \to z_0$. We have

$$\|z_0 - x_n\|^2 = \|z_0 - x_1\|^2 + \|x_1 - x_n\|^2 + 2\langle z_0 - x_1, x_1 - x_n\rangle, \quad \forall n \in \mathbb{N}.$$

So, we have from (3.9) that

$$\begin{aligned}\limsup_{n\to\infty} \|z_0 - x_n\|^2 &= \limsup_{n\to\infty}(\|z_0 - x_1\|^2 + \|x_1 - x_n\|^2 + 2\langle z_0 - x_1, x_1 - x_n\rangle) \\ &\le \limsup_{n\to\infty}(\|z_0 - x_1\|^2 + \|x_1 - z_0\|^2 + 2\langle z_0 - x_1, x_1 - x_n\rangle) \\ &= \|z_0 - x_1\|^2 + \|x_1 - z_0\|^2 + 2\langle z_0 - x_1, x_1 - z_0\rangle \\ &= 0.\end{aligned}$$

Thus we obtain $\lim_{n\to\infty} \|z_0 - x_n\| = 0$. Hence, $\{x_n\}$ converges strongly to z_0. This completes the proof. □

Next, we prove a strong convergence theorem by the shrinking projection method [22] for noncommutative two generalized hybrid mappings in a Hilbert space.

Theorem 3.3. *Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let A be a maximal monotone operator on H and let J_t be the resolvent of A for $t > 0$. Let S and T be $(\alpha_1^*, \beta_1^*, \gamma_1^*, \delta_1^*, \varepsilon_1^*, \zeta_1^*)$ and*

$(\alpha_2^*, \beta_2^*, \gamma_2^*, \delta_2^*, \varepsilon_2^*, \zeta_2^*)$-demigeneric generalized hybrid mappings of C into H, espectively, such that $\alpha_1^* + \beta_1^* \neq 0$ and $\alpha_2^* + \beta_2^* \neq 0$. Put

$$\lambda_1 = \frac{\alpha_1^* + \beta_1^* + \zeta_1^*}{\alpha_1^* + \beta_1^*} \quad \textit{and} \quad \lambda_2 = \frac{\alpha_2^* + \beta_2^* + \zeta_2^*}{\alpha_2^* + \beta_2^*}.$$

Suppose that $\Omega := A^{-1}0 \cap F(S) \cap F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 \in C$ and $C_1 = C$, define a sequence $\{x_n\}$ in C as follows:

$$\begin{cases} y_n = J_{t_n}\big(\alpha_n x_n + (1-\alpha_n) z_n\big), \\ z_n = \gamma_n((1-\lambda_1)I + \lambda_1 S)x_n + (1-\gamma_n)((1-\lambda_2)I + \lambda_2 T)x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1}, $t_o, a, b, c, d \in \mathbb{R}$, $\{t_n\} \subset (0.\infty)$ and $\{\gamma_n\}, \{\alpha_n\} \subset [0,1]$ satisfy the following:

$$0 < t_0 \leq t_n, \ 0 < a \leq \gamma_n \leq b < 1 \quad \textit{and} \quad 0 < c \leq \alpha_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then, $\{x_n\}$ converges strongly to $z_0 = P_\Omega u$, where P_Ω is the metric projection of H onto Ω.

Proof. Put $S_1 = (1-\lambda_1)I + \lambda_1 S$ and $T_1 = (1-\lambda_2)I + \lambda_2 T$. As in the proof of Theorem 3.2, S_1 and T_1 are quasi-nonexpansive. Since S and T are genealized demimetric, we also have that $F(S) \cap F(T)$ is closed and convex. Furthermore, since A is a maximal monotone operator, $A^{-1}0$ is closed and convex. So, $\Omega = A^{-1}0 \cap F(S) \cap F(T)$ is closed and convex. Thus, there exists the mertic projection of H onto Ω.

Next, we shall show that C_n is closed and convex, and $\Omega \subset C_n$ for all $n \in \mathbb{N}$. It is obvious from assumption that $C_1 = C$ is closed and convex, and $\Omega \subset C_1$. Suppose that C_k is closed and convex, and $\Omega \subset C_k$ for some $k \in \mathbb{N}$. We know that for $z \in C_k$,

$$\begin{aligned} &\|y_k - z\|^2 \leq \|x_k - z\|^2 \\ \Longleftrightarrow &\|y_k\|^2 - \|x_k\|^2 - 2\langle y_k - x_k, z\rangle \leq 0. \end{aligned}$$

So, C_{k+1} is closed and convex. By induction, C_n are closed and convex for all $n \in \mathbb{N}$. Since J_{t_k} is nonexpansive and S and T are quasi-nonexpansive, we have that, for any $z \in \Omega$,

$$\begin{aligned} \|z_k - z\|^2 &= \|\gamma_n S_1 x_k + (1-\gamma_k) T_1 x_k - z\|^2 \\ &= \|\gamma_k (S_1 x_k - z) + (1-\gamma_k)(T_1 x_k - z)\|^2 \\ &= \gamma_k \|S_1 x_k - z\|^2 + (1-\gamma_k)\|T_1 x_k - z\|^2 - \gamma_k(1-\gamma_k)\|S_1 x_k - T_1 x_k\|^2 \\ &\leq \gamma_k \|x_k - z\|^2 + (1-\gamma_k)\|x_k - z\|^2 - \gamma_k(1-\gamma_k)\|S_1 x_k - T_1 x_k\|^2 \\ &= \|x_k - z\|^2 - \gamma_k(1-\gamma_k)\|S_1 x_k - T_1 x_k\|^2 \\ &\leq \|x_k - z\|^2 \end{aligned} \tag{3.10}$$

and hence

$$\begin{aligned}\|y_k - z\|^2 &= \|J_{t_k}\big(\alpha_k x_k + (1-\alpha_k)z_k\big) - z\|^2\\ &\le \|\alpha_k x_k + (1-\alpha_k)z_k - z\|^2\\ &\le \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\|z_k - z\|^2\\ &\le \alpha_k\|x_k - z\|^2 + (1-\alpha_k)\|x_k - z\|^2\\ &= \|x_k - z\|^2.\end{aligned}$$

Hence, we have $z \in C_{k+1}$. By induction, we have that $\Omega \subset C_n$ for all $n \in \mathbb{N}$. Since C_n is nonempty, closed and convex, there exists the metric projection P_{C_n} of H onto C_n. Thus, $\{x_n\}$ is well-defined.

Define $z_0 = P_\Omega u$. Putting $v_n = P_{C_n}u$, we have that

$$\|u - v_n\| \le \|u - y\|$$

for all $y \in C_n$. Since $z_0 \in \Omega \subset C_n$, we have that

$$\|u - v_n\| \le \|u - z_0\|. \tag{3.11}$$

This means that $\{v_n\}$ is bounded. From $v_n = P_{C_n}u$ and $v_{n+1} \in C_{n+1} \subset C_n$, we have that

$$\|u - v_n\| \le \|u - v_{n+1}\|.$$

Thus $\{\|u - v_n\|\}$ is bounded and nondecreasing. Then there exists the limit of $\{\|u - v_n\|\}$. Put $\lim_{n\to\infty}\|v_n - u\| = c$. For any $m, n \in \mathbb{N}$ with $m \ge n$, we have $C_m \subset C_n$. From $v_m = P_{C_m}u \in C_m \subset C_n$ and (2.4), we have that

$$\|v_m - P_{C_n}u\|^2 + \|P_{C_n}u - u\|^2 \le \|u - v_m\|^2.$$

This implies that

$$\|v_m - v_n\|^2 \le \|u - v_m\|^2 - \|v_n - u\|^2 \le c^2 - \|v_n - u\|^2. \tag{3.12}$$

Since $c^2 - \|v_n - u\|^2 \to 0$ as $n \to \infty$, we have that $\{v_n\}$ is a Cauchy sequence. By the completeness of C, there exists a point $v_0 \in C$ such that $v_n \to v_0$.

Using Theorem 2.3, we can also prove that $v_n \to v_0$. In fact, since $\{C_n\}$ is a nonincreasing sequence of nonempty, closed and convex subsets of H with respect to inclusion, it follows that

$$\emptyset \ne \Omega \subset \text{M-}\lim_{n\to\infty} C_n = \bigcap_{n=1}^{\infty} C_n. \tag{3.13}$$

Put $C_0 = \bigcap_{n=1}^{\infty} C_n$. Then, by Theorem 2.3 we have that $\{v_n\} = \{P_{C_n}u\}$ converges strongly to $v_0 = P_{C_0}u$, i.e., $v_n = P_{C_n}u \to v_0$.

Since the metric projection P_{C_n} is nonexpansive, it follows that

$$\begin{aligned}\|x_n - v_0\| &\le \|x_n - v_n\| + \|v_n - v_0\|\\ &= \|P_{C_n}u_n - P_{C_n}u\| + \|v_n - v_0\|\\ &\le \|u_n - u\| + \|v_n - v_0\|\end{aligned}$$

and hence

$$x_n \to v_0. \tag{3.14}$$

To complete the proof, it is sufficient to show that $z_0 = P_\Omega u = v_0$.

From (3.14), we have that

$$\|x_n - x_{n+1}\| \to 0. \tag{3.15}$$

From $x_{n+1} \in C_{n+1}$, we also have that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. So, we get that $\|y_n - x_{n+1}\| \to 0$. Using this, we have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{3.16}$$

Furthermore, we have from (3.10) that, for $w \in \Omega$,

$$\begin{aligned}
\|y_n - w\|^2 &= \|J_{t_n}\big(\alpha_n x_n + (1-\alpha_n)z_n\big) - w\|^2 \\
&\leq \|\alpha_n x_n + (1-\alpha_n)z_n - w\|^2 \\
&= \|\alpha_n(x_n - w) + (1-\alpha_n)(z_n - w)\|^2 \\
&= \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|z_n - w\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\|z_n - x_n\|^2 \\
&\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|x_n - w\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\|z_n - x_n\|^2 \\
&= \|x_n - w\|^2 - \alpha_n(1-\alpha_n)\|z_n - x_n\|^2.
\end{aligned}$$

Then, we obtain that

$$\begin{aligned}
\alpha_n(1-\alpha_n)\|z_n - x_n\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 \\
&= (\|x_n - w\| - \|y_n - w\|)(\|x_n - w\| + \|y_n - w\|) \\
&\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|).
\end{aligned}$$

From the assumption of $\{\alpha_n\}$ and $\|y_n - x_n\| \to 0$, we have that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.17}$$

Using this, as in the proof of Theorem 3.2, we have that $Sx_n - x_n \to 0$ and $Tx_n - x_n \to 0$. From (3.14), we also get that $x_n \rightharpoonup v_0$. From Lemma 3.1, we have $v_0 \in F(S) \cap F(T)$. On the other hand, putting $w_n = \alpha_n x_n + (1-\alpha_n)z_n$, we have that for any $w \in \Omega$.

$$\begin{aligned}
2\|y_n - w\|^2 &= 2\|J_{t_n}w_n - J_{t_n}w\|^2 \\
&\leq 2\langle w_n - w, y_n - w\rangle \\
&= \|w_n - w\|^2 + \|y_n - w\|^2 - \|w_n - y_n\|^2
\end{aligned}$$

and hence

$$\|w_n - y_n\|^2 \leq \|w_n - w\|^2 - \|y_n - w\|^2.$$

Using this and $w_n - x_n = (1-\alpha_n)(z_n - x_n)$, we have that

$$\begin{aligned}
\|w_n - y_n\|^2 &\leq \|w_n - x_n + x_n - w\|^2 - \|y_n - w\|^2 \\
&= \|w_n - x_n\|^2 + 2\langle w_n - x_n, x_n - w\rangle + \|x_n - w\|^2 - \|y_n - w\|^2 \\
&= (1-\alpha_n)^2\|z_n - x_n\|^2 + 2(1-\alpha_n)\langle z_n - x_n, x_n - w\rangle \\
&\quad + \|x_n - w\|^2 - \|y_n - w\|^2 \\
&\leq (1-\alpha_n)\|z_n - x_n\|^2 + 2(1-\alpha_n)\langle z_n - x_n, x_n - w\rangle
\end{aligned}$$

$$+\|x_n - w\|^2 - \|y_n - w\|^2.$$

Since $z_n - x_n \to 0$ and $x_n - y_n \to 0$, we have that

$$w_n - y_n \to 0. \tag{3.18}$$

Furthermore, since A is a monotone operator and $\frac{w_n - y_n}{t_n} \in Ay_n$, we have that for any $(u,v) \in A$,

$$\langle u - y_n, v - \frac{w_n - y_n}{t_n} \rangle \geq 0.$$

Since $t_n \geq t_0 > 0$, $x_n \rightharpoonup v_0$ and $x_n - y_n \to 0$, we have $\langle u - v_0, v\rangle \geq 0$. Since A is a maximal monotone operator, we have $0 \in Av_0$ and hence $v_0 \in A^{-1}0$. Thus we have $v_0 \in \Omega$.

Since $z_0 = P_\Omega u \in C_{n+1}$ and $x_{n+1} = P_{C_{n+1}} u_{n+1}$, we have that

$$\|u_{n+1} - x_{n+1}\| \leq \|u_{n+1} - z_0\|. \tag{3.19}$$

Thus we have that

$$\|u - v_0\| \leq \|u - z_0\|$$

and hence $z_0 = v_0$. Therefore, $\{x_n\}$ converges strongly to z_0. This completes the proof. □

4. Applications

In this section, using Theorems 3.2 and 3.3, we get well-known and new strong convergence theorems by the hybrid method and the shrinking projection method in a Hilbert space. Let H be a Hilbert space and let $Ax = 0$ for all $x \in H$. Then A is a maximal monotone operator. If J_t is the resovent of A for $t > 0$, then $J_t = I$ for all $t > 0$. Using this fact and Theorems 3.2, we have the following results.

Theorem 4.1. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let S and T be k_1 and k_2-strict pseudo-contractions of C into H, respectively, such that $F(S) \cap F(T) \neq \emptyset$. Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C as folows:*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) z_n, \\ z_n = \gamma_n (k_1 I + (1-k_1)S)x_n + (1-\gamma_n)(k_2 I + (1-k_2)T)x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$. $a, b, c, d \in \mathbb{R}$ and $\{\alpha_n\}, \{\gamma_n\} \subset [0,1]$ satisfy

$$0 < a \leq \gamma_n \leq b < 1 \quad \text{and} \quad 0 < c \leq \alpha_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)} x_1$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

Proof. Since S is a k_1-strict pseudo-contraction, it is demigeneric generalized hybrid and $\frac{2}{1-k_1}$-generalized demimetric. Then we have $\lambda_1 = 1 - k_1$ in Theorem 3.2. Similarly, since T is a k_2-strict pseudo-contraction, we have $\lambda_2 = 1 - k_2$. Thus, from Theorem 3.2, we have the desired result. □

Theorem 4.2. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let S and T be generalized hybrid mappings of C into H such that $F(S) \cap F(T) \neq \emptyset$. Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C as folows:*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\big(\gamma_n S x_n + (1-\gamma_n))T x_n\big), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$, $\{\alpha_n\}, \{\gamma_n\} \subset [0,1]$ and $a, b, c, d \in \mathbb{R}$ satisfy

$$0 < a \leq \gamma_n \leq b < 1 \quad \text{and} \quad 0 < c \leq \alpha_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)} x_1$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

Proof. Since S is a generalized hybrid mapping, it is demigeneric generalized hybrid and 2-generalized demimetric. Then we have $\lambda_1 = 1$ in Theorem 3.2. Similarly, since T is a generalized hybrid mapping, we have $\lambda_2 = 1$. Thus, from Theorem 3.2, we have the desired result. □

Theorem 4.3. *Let H be a Hilbert space. Let G and B be maximal monotone operators on H. Let J_s and Q_t be the resolvents of G for $s > 0$ and B for $t > 0$, respectively. Suppose that $G^{-1}0 \cap B^{-1}0 \neq \emptyset$. Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C as folows: $x_1 \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\Big(\gamma_n J_s x_n + (1-\gamma_n) Q_t x_n\Big), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$, $a, b, c, d \in \mathbb{R}$ and $\{\alpha_n\}, \{\gamma_n\} \subset [0,1]$ satisfy

$$0 < a \leq \gamma_n \leq b < 1 \quad \text{and} \quad 0 < c \leq \alpha_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = P_{G^{-1}0 \cap B^{-1}0} x_1$, where $P_{G^{-1}0\cap B^{-1}0}$ is the metric projection of H onto $G^{-1}0 \cap B^{-1}0$.

Proof. Since J_s is the resolvent of G for $s > 0$, J_s is nonexpansive. Similarly, since Q_t is the resolvent of B for $t > 0$, it is nonexpansive. Therefore, we have the desired result from Theorem 3.2. □

Using Theorem 3.3, we obtain the following strong convergence theorems in a Hilbert space. The proofs are as those of Theorems 4.1, 4.2 and 4.3.

Theorem 4.4. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let S and T be k_1 and k_2-strict pseudo-contractions of C into*

H, respectively, such that $F(S) \cap F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 \in C$ and $C_1 = C$, define a sequence $\{x_n\}$ in C as follows:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) z_n, \\ z_n = \gamma_n (k_1 I + (1-k_1)S) x_n + (1-\gamma_n)(k_2 I + (1-k_2)T) x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1}, $a, b, c, d \in \mathbb{R}$, and $\{\gamma_n\}, \{\alpha_n\} \subset [0,1]$ satisfy the following:

$$0 < a \le \gamma_n \le b < 1 \quad \text{and} \quad 0 < c \le \alpha_n \le d < 1, \quad \forall n \in \mathbb{N}.$$

Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)} u$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

Theorem 4.5. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let S and T be generalized hybrid mappings of C into H such that $F(S) \cap F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 \in C$ and $C_1 = C$, define a sequence $\{x_n\}$ in C as follows:*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\big(\gamma_n S x_n + (1-\gamma_n)) T x_n\big), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1}, $a, b, c, d \in \mathbb{R}$, and $\{\gamma_n\}, \{\alpha_n\} \subset [0,1]$ satisfy the following:

$$0 < a \le \gamma_n \le b < 1 \quad \text{and} \quad 0 < c \le \alpha_n \le d < 1, \quad \forall n \in \mathbb{N}.$$

Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)} u$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

Theorem 4.6. *Let H be a Hilbert space. Let G and B be maximal monotone operators on H. Let J_s and Q_t be the resolvents of G for $s > 0$ and B for $t > 0$, respectively. Suppose that $G^{-1}0 \cap B^{-1}0 \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 \in C$ and $C_1 = C$, define a sequence $\{x_n\}$ in C as follows:*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\big(\gamma_n J_s x_n + (1-\gamma_n) Q_t x_n\big), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1} and $a, b, c, d \in \mathbb{R}$ and $\{\alpha_n\}, \{\gamma_n\} \subset [0,1]$ satisfy

$$0 < a \le \gamma_n \le b < 1 \quad \text{and} \quad 0 < c \le \alpha_n \le d < 1, \quad \forall n \in \mathbb{N}.$$

Then, $\{x_n\}$ converges strongly to $z_0 = P_{G^{-1}0\cap B^{-1}0} u$, where $P_{G^{-1}0\cap B^{-1}0}$ is the metric projection of H onto $G^{-1}0 \cap B^{-1}0$.

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem (with respect to C) is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{4.1}$$

The set of such solutions $\hat{x}$ is denoted by $EP(f)$, i.e.,

$$EP(f) = \{\hat{x} \in C : f(\hat{x}, y) \geq 0, \ \forall y \in C\}.$$

For solving the equilibrium problem, let us assume that the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C$,

$$\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y);$$

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

The following lemmas were given in Combettes and Hirstoaga [5] and Takahashi, Takahashi and Toyoda [16]; see also [2].

Lemma 4.7 ([5])**.** *Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Assume that $f : C \times C \to \mathbb{R}$ satisfies (A1) – (A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \ \forall y \in C \right\}.$$

Then, the following hold:

(1) *T_r is single-valued;*
(2) *T_r is a firmly nonexpansive mapping, i.e., for all $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

(3) *$F(T_r) = EP(f)$;*
(4) *$EP(f)$ is closed and convex.*

We call such T_r the resolvent of f for $r > 0$.

Lemma 4.8 ([16])**.** *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $f : C \times C \to \mathbb{R}$ satisfy (A1) – (A4). Let A_f be a set-valued mapping of H into itself defined by*

$$A_f x = \begin{cases} \{z \in H : f(x, y) \geq \langle y - x, z \rangle, \ \forall y \in C\}, & \forall x \in C, \\ \emptyset, \quad \forall x \notin C. \end{cases}$$

Then, $EP(f) = A_f^{-1}0$ and A_f is a maximal monotone operator with $D(A_f) \subset C$. Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r of f coincides with the resolvent of A_f, i.e., $T_r x = (I + rA_f)^{-1}x$.

Using Lemmas 4.7 and 4.8 and Theorem 3.2, we have the following theorem.

Theorem 4.9. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions (A1)-(A4) and let T_{t_n} be the resolvent of A_f for $t_n > 0$. Let S and T be nonexpansive and nonspreading mappings of C into H, respectively. Suppose that $\Omega := EP(f) \cap F(S) \cap F(T) \neq \emptyset$. Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C as folows:*

$$\begin{cases} x_{n+1} = T_{t_n}\big(\alpha_n x_n + (1-\alpha_n)z_n\big), \\ z_n = \gamma_n S x_n + (1-\gamma_n)T x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $t_0, a, b, c, d \in \mathbb{R}$, $\{t_n\} \subset (0.\infty)$ and $\{\gamma_n\}, \{\alpha_n\} \subset [0,1]$ satisfy the following:

$$0 < t_0 \leq t_n, \; 0 < a \leq \gamma_n \leq b < 1 \;\; \text{and} \;\; 0 < c \leq \alpha_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = P_\Omega x_1$, where P_Ω is the metric projection of H onto Ω.

Proof. We have from Lemma 4.8 that A_f is a maximal monotone operator with $D(A_f) \subset C$ and $EP(f) = A_f^{-1}0$. Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r of f coincides with the resolvent of A_f, that is, $T_r x = (I + rA_f)^{-1}x$. Furthermore, nonexpansive mappings and nonspreading mappings are contained in the class of generalized hybrid mappings. So, put $\lambda_1 = \lambda_2 = 1$ in Theorem 3.2. Thus, we obtain the desired result from Theorem 3.2. □

Similarly, using Theorem 3.3, we have the following theorem.

Theorem 4.10. *Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions (A1)-(A4) and let T_{t_n} be the resolvent of A_f for $t_n > 0$. Let S and T be nonexpansive and nonspreading mappings of C into H, respectively. Suppose that $\Omega := EP(f) \cap F(S) \cap F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 \in C$ and $C_1 = C$, define a sequence $\{x_n\}$ in C as follows:*

$$\begin{cases} y_n = T_{t_n}\big(\alpha_n x_n + (1-\alpha_n)z_n\big), \\ z_n = \gamma_n S x_n + (1-\gamma_n)T x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1}, $t_o, a, b, c, d \in \mathbb{R}$, $\{t_n\} \subset (0.\infty)$ and $\{\gamma_n\}, \{\alpha_n\} \subset [0,1]$ satisfy the following:

$$0 < t_0 \leq t_n, \; 0 < a \leq \gamma_n \leq b < 1 \;\; \text{and} \;\; 0 < c \leq \alpha_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then, $\{x_n\}$ converges strongly to $z_0 = P_\Omega u$, where P_Ω is the metric projection of H onto Ω.

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