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ON CIRCUMCENTERS OF FINITE SETS IN HILBERT SPACES

HEINZ H. BAUSCHKE*, HUI OUYANG, AND XIANFU WANG†

ABSTRACT. A well-known object in classical Euclidean geometry is the circumcenter of a triangle, i.e., the point that is equidistant from all vertices. The purpose of this paper is to provide a systematic study of the circumcenter of sets containing finitely many points in a Hilbert space. This is motivated by recent works of Behling, Bello Cruz, and Santos on accelerated versions of the Douglas–Rachford method. We present basic results and properties of the circumcenter. Several examples are provided to illustrate the tightness of various assumptions.

1. Introduction and standing assumption

Throughout this paper,

$$\mathcal{H} \text{ is a real Hilbert space}$$

with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. We denote by $\mathcal{P}(\mathcal{H})$ the set of all nonempty subsets of $\mathcal{H}$ containing *finitely many* elements. Assume that

$$S = \{x_1, x_2, \ldots, x_m\} \in \mathcal{P}(\mathcal{H}).$$

The goal of this paper is to provide a systematic study of the circumcenter of S, i.e., of the (unique if it exists) point in the affine hull of S that is equidistant from all points in S. The classical case in trigonometry or Euclidean geometry arises when $m = 3$ and $\mathcal{H} = \mathbb{R}^2$. Recent applications of the circumcenter focus on the present much more general case. Indeed, our work is motivated by recent works of Behling, Bello Cruz, and Santos (see [4] and [5]) on accelerating the Douglas–Rachford algorithm by employing the circumcenter of intermediate iterates to solve certain best approximation problems.

The paper is organized as follows. Various auxiliary results are collected in Section 2 to ease subsequent proofs. Based on the circumcenter, we introduce our main actor, the *circumcenter operator*, in Section 3. Explicit formulae for the circumcenter are provided in Sections 4 and 5 while Section 6 records some basic properties. In Section 7, we turn to the behaviour of the circumcenter when sequences of sets are considered. Section 8 deals with the case when the set contains three points which yields particularly pleasing results. The importance of the circumcenter in

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the algorithmic work of Behling et al. is explained in Section 9. In the final Section 10, we return to more classical roots of the circumcenter and discuss formulae involving cross products when $\mathcal{H} = \mathbb{R}^3$.

The notation employed is standard and largely follows [2].

2. Auxiliary results

In this section, we provide various results that will be useful in the sequel.

2.1. **Affine sets.** Recall that a nonempty subset S of $\mathcal{H}$ is an *affine subspace* of $\mathcal{H}$ if $(\forall \rho \in \mathbb{R})\ \rho S + (1-\rho)S = S$; moreover, the smallest affine subspace containing S is the *affine hull* of S, denoted $\operatorname{aff} S$.

Fact 2.1. [11, page 4] Let $S \subseteq \mathcal{H}$ be an affine subspace and let $a \in \mathcal{H}$. Then the *translate* of S by a, which is defined by

$$S + a = \{x + a \mid x \in S\},$$

is another affine subspace.

Definition 2.2. An affine subspace S is said to be *parallel* to an affine subspace M if $S = M + a$ for some $a \in \mathcal{H}$.

Fact 2.3. [11, Theorem 1.2] Every affine subspace S is parallel to a unique linear subspace L, which is given by

$$(\forall y \in S) \quad L = S - y = S - S.$$

Definition 2.4. [11, page 4] The *dimension* of an affine subspace is defined to be the dimension of the linear subspace parallel to it.

Fact 2.5. [11, page 7] Let $x_1, \ldots, x_m \in \mathcal{H}$. Then

$$\operatorname{aff}\{x_1, \ldots, x_m\} = \Big\{\lambda_1 x_1 + \cdots + \lambda_m x_m \;\Big|\; \lambda_1, \ldots, \lambda_m \in \mathbb{R} \text{ and } \sum_{i=1}^{m} \lambda_i = 1\Big\}.$$

Some algebraic calculations and Fact 2.5 yield the next result.

Lemma 2.6. *Let $x_1, \ldots, x_m \in \mathcal{H}$. Then for every $i_0 \in \{2, \ldots, m\}$, we have*

$$\begin{aligned}\operatorname{aff}\{x_1, \ldots, x_m\} &= x_1 + \operatorname{span}\{x_2 - x_1, \ldots, x_m - x_1\} \\ &= x_{i_0} + \operatorname{span}\{x_1 - x_{i_0}, \ldots, x_{i_0-1} - x_{i_0}, x_{i_0+1} - x_{i_0}, \ldots, x_m - x_{i_0}\}.\end{aligned}$$

Definition 2.7. [11, page 6] Let $x_0, x_1, \ldots, x_m \in \mathcal{H}$. The $m+1$ vectors $x_0, x_1, \ldots, x_m$ are said to be affinely independent if $\operatorname{aff}\{x_0, x_1, \ldots, x_m\}$ is m-dimensional.

Fact 2.8. [11, page 7] Let $x_1, x_2, \ldots, x_m \in \mathcal{H}$. Then $x_1, x_2, \ldots, x_m$ are affinely independent if and only if $x_2 - x_1, \ldots, x_m - x_1$ are linearly independent.

Lemma 2.9. *Let $x_1, \ldots, x_m$ be affinely independent vectors in $\mathcal{H}$. Let $p \in \operatorname{aff}\{x_1, \ldots, x_m\}$. Then there exists a unique vector $\begin{pmatrix}\alpha_1 & \cdots & \alpha_m\end{pmatrix}^\intercal \in \mathbb{R}^m$ with $\sum_{i=1}^m \alpha_i = 1$ such that*

$$p = \alpha_1 x_1 + \cdots + \alpha_m x_m.$$

The following lemma will be useful later.

Lemma 2.10. *Let*

$$\mathcal{O} = \Big\{ (x_1, \ldots, x_{m-1}, x_m) \in \mathcal{H}^m \ \Big| \ x_1, \ldots, x_{m-1}, x_m \text{ are affinely independent} \Big\}.$$

Then $\mathcal{O}$ is open.

Proof. Assume to the contrary that there exist $(x_1, \ldots, x_{m-1}, x_m) \in \mathcal{O}$ such that for every $k \in \mathbb{N} \smallsetminus \{0\}$, there exist $(x_1^{(k)}, \ldots, x_{m-1}^{(k)}, x_m^{(k)}) \in B\Big((x_1, \ldots, x_{m-1}, x_m); \frac{1}{k}\Big)$ such that $x_1^{(k)}, \ldots, x_{m-1}^{(k)}, x_m^{(k)}$ are affinely dependent. By Fact 2.8, for every k, there exists $b^{(k)} = (\beta_1^{(k)}, \beta_2^{(k)}, \ldots, \beta_{m-1}^{(k)}) \in \mathbb{R}^{m-1} \smallsetminus \{0\}$ such that

$$\beta_1^{(k)}(x_2^{(k)} - x_1^{(k)}) + \cdots + \beta_{m-1}^{(k)}(x_m^{(k)} - x_1^{(k)}) = 0. \tag{2.1}$$

Without loss of generality we assume

$$(\forall k \in \mathbb{N} \smallsetminus \{0\}) \quad \|b^{(k)}\|^2 = \sum_{i=1}^{m-1} (\beta_i^{(k)})^2 = 1, \tag{2.2}$$

and there exists $\bar{b} = (\beta_1, \ldots, \beta_{m-1}) \in \mathbb{R}^{m-1}$ such that

$$\lim_{k \to \infty} (\beta_1^{(k)}, \ldots, \beta_{m-1}^{(k)}) = \lim_{k \to \infty} b^{(k)} = \bar{b} = (\beta_1, \ldots, \beta_{m-1}).$$

Let k go to infinity in (2.2), we get

$$\|\bar{b}\|^2 = \beta_1^2 + \cdots + \beta_{m-1}^2 = 1,$$

which yields that $(\beta_1, \ldots, \beta_{m-1}) \neq 0$.

Let k go to infinity in (2.1), we obtain

$$\beta_1(x_2 - x_1) + \cdots + \beta_{m-1}(x_m - x_1) = 0,$$

which means that $x_2 - x_1, \ldots, x_m - x_1$ are linearly dependent. By Fact 2.8, it contradicts with the assumption that $x_1, \ldots, x_{m-1}, x_m$ are affinely independent. Hence $\mathcal{O}$ is indeed an open set. $\square$

Fact 2.11. [7, Theorem 9.26] Let V be an affine subset of $\mathcal{H}$, say $V = M + v$, where M is a linear subspace of $\mathcal{H}$ and $v \in V$. Let $x \in \mathcal{H}$ and $y_0 \in \mathcal{H}$. Then the following statements are equivalent:

(i) $y_0 = P_V(x)$.
(ii) $x - y_0 \in M^{\perp}$.
(iii) $\langle x - y_0, y - v \rangle = 0$ for all $y \in V$.

Moreover,

$$P_V(x + e) = P_V(x) \text{ for all } x \in X, e \in M^{\perp}.$$

2.2. **The Gram matrix.**

Definition 2.12. Let $a_1, \dots, a_m \in \mathcal{H}$. Then

$$G(a_1, \dots, a_m) = \begin{pmatrix} \|a_1\|^2 & \langle a_1, a_2\rangle & \cdots & \langle a_1, a_m\rangle \\ \langle a_2, a_1\rangle & \|a_2\|^2 & \cdots & \langle a_2, a_m\rangle \\ \vdots & \vdots & & \vdots \\ \langle a_m, a_1\rangle & \langle a_m, a_2\rangle & \cdots & \|a_m\|^2 \end{pmatrix}$$

is called the *Gram matrix* of $a_1, \dots, a_m$.

Fact 2.13. [8, Theorem 6.5-1] Let $a_1, \dots, a_m \in \mathcal{H}$. Then the Gram matrix $G(a_1, \dots, a_m)$ is invertible if and only if the vectors $a_1, \dots, a_m$ are linearly independent.

Remark 2.14. Let x, y, z be affinely independent vectors in $\mathbb{R}^3$. Set $a = y - x$ and $b = z - x$. Then, by Fact 2.8 and Fact 2.13, $\|a\|^2\|b\|^2 - \langle a, b\rangle^2 \neq 0$ and $\|a\| \neq 0$, $\|b\| \neq 0$.

Proposition 2.15. *Let $x_1, \dots, x_m \in \mathcal{H}$. Then for every $k \in \{2, \dots, m\}$, we have*

$$\det\Big(G(x_2 - x_1, \dots, x_m - x_1)\Big)$$
$$= \det\Big(G(x_1 - x_k, \dots, x_{k-1} - x_k, x_{k+1} - x_k, \dots, x_m - x_k)\Big).$$

Proof. By Definition 2.12, $G(x_1 - x_k, \dots, x_{k-1} - x_k, x_{k+1} - x_k, \dots, x_m - x_k)$ is

(2.3)
$$\begin{pmatrix} \langle x_1 - x_k, x_1 - x_k\rangle & \cdots & \langle x_1 - x_k, x_{k-1} - x_k\rangle & \langle x_1 - x_k, x_{k+1} - x_k\rangle & \cdots & \langle x_1 - x_k, x_m - x_k\rangle \\ \langle x_2 - x_k, x_1 - x_k\rangle & \cdots & \langle x_2 - x_k, x_{k-1} - x_k\rangle & \langle x_2 - x_k, x_{k+1} - x_k\rangle & \cdots & \langle x_2 - x_k, x_m - x_k\rangle \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \langle x_{k-1} - x_k, x_1 - x_k\rangle & \cdots & \langle x_{k-1} - x_k, x_{k-1} - x_k\rangle & \langle x_{k-1} - x_k, x_{k+1} - x_k\rangle & \cdots & \langle x_{k-1} - x_k, x_m - x_k\rangle \\ \langle x_{k+1} - x_k, x_1 - x_k\rangle & \cdots & \langle x_{k+1} - x_k, x_{k-1} - x_k\rangle & \langle x_{k+1} - x_k, x_{k+1} - x_k\rangle & \cdots & \langle x_{k+1} - x_k, x_m - x_k\rangle \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \langle x_m - x_k, x_1 - x_k\rangle & \cdots & \langle x_m - x_k, x_{k-1} - x_k\rangle & \langle x_m - x_k, x_{k+1} - x_k\rangle & \cdots & \langle x_m - x_k, x_m - x_k\rangle \end{pmatrix}.$$

In (2.3), we perform the following elementary row and column operations: For every $i \in \{2, 3, \dots, m-1\}$, subtract the 1^{st} row from the i^{th} row, and then subtract the 1^{st} column from the i^{th} column. Then multiply 1^{st} row and 1^{st} column by -1, respectively. It follows that the determinant of (2.3) equals the determinant of

(2.4)
$$\begin{pmatrix} \langle x_k - x_1, x_k - x_1\rangle & \cdots & \langle x_k - x_1, x_{k-1} - x_1\rangle & \langle x_k - x_1, x_{k+1} - x_1\rangle & \cdots & \langle x_k - x_1, x_m - x_1\rangle \\ \langle x_2 - x_1, x_k - x_1\rangle & \cdots & \langle x_2 - x_1, x_{k-1} - x_1\rangle & \langle x_2 - x_1, x_{k+1} - x_1\rangle & \cdots & \langle x_2 - x_1, x_m - x_1\rangle \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \langle x_{k-1} - x_1, x_k - x_1\rangle & \cdots & \langle x_{k-1} - x_1, x_{k-1} - x_1\rangle & \langle x_{k-1} - x_1, x_{k+1} - x_1\rangle & \cdots & \langle x_{k-1} - x_1, x_m - x_1\rangle \\ \langle x_{k+1} - x_1, x_k - x_1\rangle & \cdots & \langle x_{k+1} - x_1, x_{k-1} - x_1\rangle & \langle x_{k+1} - x_1, x_{k+1} - x_1\rangle & \cdots & \langle x_{k+1} - x_1, x_m - x_1\rangle \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \langle x_m - x_1, x_k - x_1\rangle & \cdots & \langle x_m - x_1, x_{k-1} - x_1\rangle & \langle x_m - x_1, x_{k+1} - x_1\rangle & \cdots & \langle x_m - x_1, x_m - x_1\rangle \end{pmatrix}.$$

In (2.4), we interchange i^{th} row and $(i+1)^{\text{th}}$ successively for $i = 1, \dots, k-2$. In addition, we interchange j^{th} column and $(j+1)^{\text{th}}$ column successively for $j = 1, \dots, k-2$. Then the resulting matrix is just $G(x_2 - x_1, \dots, x_m - x_1)$. Because the number of interchange we performed is even, the determinant is unchanged. Therefore, we obtain

$$\det\Big(G(x_1 - x_k, \ldots, x_{k-1} - x_k, x_{k+1} - x_k, \ldots, x_m - x_k)\Big) = \det\Big(G(x_2 - x_1, \ldots, x_m - x_1)\Big)$$

as claimed. □

Fact 2.16. [12, page 16] Let $S = \{A \in \mathbb{R}^{n\times n} \mid A \text{ is invertible }\}$. Then the mapping $S \to S : A \mapsto A^{-1}$ is continuous.

Fact 2.17 (Cramer's rule)**.** [10, page 476] If $A \in \mathbb{R}^{n\times n}$ is invertible and $Ax = b$, then for every $i \in \{1, \ldots, n\}$, we have

$$x_i = \frac{\det(A_i)}{\det(A)},$$

where $A_i = [A_{*,1}|\cdots|A_{*,i-1}|b|A_{*,i+1}|\cdots|A_{*,n}]$. That is, A_i is identical to A except that column $A_{*,i}$ has been replaced by b.

Corollary 2.18. *Let $\{x_1, \ldots, x_m\} \subseteq \mathcal{H}$ with $x_1, \ldots, x_m$ being affinely independent. Let $\big((x_1^{(k)}, \ldots, x_m^{(k)})\big)_{k\in\mathbb{N}} \subseteq \mathcal{H}^m$ such that*

$$\lim_{k\to\infty} (x_1^{(k)}, \ldots, x_m^{(k)}) = (x_1, \ldots, x_m).$$

Then

$$G(x_2 - x_1, \ldots, x_m - x_1)^{-1} = \lim_{k\to\infty} G(x_2^{(k)} - x_1^{(k)}, \ldots, x_m^{(k)} - x_1^{(k)})^{-1}.$$

Proof. By Lemma 2.10, we know there exists $K \in \mathbb{N}$ such that

$$(\forall k \geq K) \quad x_1^{(k)}, \ldots, x_m^{(k)} \text{ are affinely independent.}$$

Using Fact 2.8, we know

$$x_2 - x_1, \ldots, x_m - x_1 \text{ are linearly independent,}$$

and

$$(\forall k \geq K) \quad x_2^{(k)} - x_1^{(k)}, \ldots, x_m^{(k)} - x_1^{(k)} \text{ are linearly independent.}$$

Hence Fact 2.13 tells us that $G(x_2 - x_1, \ldots, x_m - x_1)^{-1}$ and $(\forall k \geq K)$ $G(x_2^{(k)} - x_1^{(k)}, \ldots, x_m^{(k)} - x_1^{(k)})^{-1}$ exist. Therefore, the required result follows directly from Fact 2.16. □

3. The circumcenter

Before we are able to define the main actor in this paper, the circumcenter operator, we shall require a few more results.

Proposition 3.1. *Let $p, x, y \in \mathcal{H}$, and set $U = \operatorname{aff}\{x, y\}$. Then the following are equivalent:*

(i) $\|p - x\| = \|p - y\|$.
(ii) $\langle p - x, y - x\rangle = \frac{1}{2}\|y - x\|^2$.

(iii) $P_U(p) = \frac{x+y}{2}$.
(iv) $p \in \frac{x+y}{2} + (U-U)^{\perp}$.

Proof. It is clear that

$$\begin{aligned}\|p-x\| = \|p-y\| &\iff \|p-x\|^2 = \|(p-x)+(x-y)\|^2 \\ &\iff \|p-x\|^2 = \|p-x\|^2 + 2\langle p-x, x-y\rangle + \|x-y\|^2 \\ &\iff \langle p-x, y-x\rangle = \frac{1}{2}\|y-x\|^2.\end{aligned}$$

Hence we get (i) $\Leftrightarrow$ (ii).

Notice $\frac{x+y}{2} \in U$. Now

$$\begin{aligned}\frac{x+y}{2} = P_U(p) \iff &(\forall u \in U) \quad \langle p - \frac{x+y}{2}, u - x\rangle = 0 \\ &\text{(by (i) } \Leftrightarrow \text{ (iii) in Fact 2.11)} \\ \iff &(\forall \alpha \in \mathbb{R}) \quad \langle p - \frac{x+y}{2}, (x+\alpha(y-x)) - x\rangle = 0 \\ &\text{(by } U = x + \operatorname{span}\{y-x\}) \\ \iff &\langle p - \frac{x+y}{2}, y-x\rangle = 0 \\ \iff &\langle p - (x - \frac{x-y}{2}), y-x\rangle = 0 \\ \iff &\langle p-x, y-x\rangle + \langle \frac{x-y}{2}, y-x\rangle = 0 \\ \iff &\langle p-x, y-x\rangle = \frac{1}{2}\|y-x\|^2,\end{aligned}$$

which imply that (iii) $\Leftrightarrow$ (ii).

On the other hand, by (i) $\Leftrightarrow$ (ii) in Fact 2.11 and by Fact 2.3,

$$\begin{aligned}\frac{x+y}{2} = P_U(p) \iff &p - \frac{x+y}{2} \in (U-U)^{\perp} \\ \iff &p \in \frac{x+y}{2} + (U-U)^{\perp},\end{aligned}$$

which yield that (iii) $\Leftrightarrow$ (iv).

In conclusion, we obtain (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). $\square$

Corollary 3.2. *Let $x_1, \ldots, x_m$ be in $\mathcal{H}$. Let $p \in \mathcal{H}$. Then*

$$\begin{aligned}&\|p-x_1\| = \cdots = \|p-x_{m-1}\| \\ &\qquad = \|p-x_m\| \iff \begin{cases}\langle p-x_1, x_2-x_1\rangle = \frac{1}{2}\|x_2-x_1\|^2 \\ \vdots \\ \langle p-x_1, x_{m-1}-x_1\rangle = \frac{1}{2}\|x_{m-1}-x_1\|^2 \\ \langle p-x_1, x_m-x_1\rangle = \frac{1}{2}\|x_m-x_1\|^2.\end{cases}\end{aligned}$$

Proof. Set $I = \{2, \ldots, m-1, m\}$, and let $i \in I$. In Proposition 3.1, substitute $x = x_1$ and $y = x_i$ and use (i) $\Leftrightarrow$ (ii). Then we get $\|p-x_1\| = \|p-x_i\| \iff$

$\langle p - x_1, x_i - x_1 \rangle = \frac{1}{2}\|x_i - x_1\|^2$. Hence

$$(\forall i \in I) \quad \|p - x_1\| = \|p - x_i\| \iff \langle p - x_1, x_i - x_1 \rangle = \frac{1}{2}\|x_i - x_1\|^2.$$

Therefore, the proof is complete. □

The next result plays an essential role in the definition of the circumcenter operator.

Proposition 3.3. *Set $S = \{x_1, x_2, \ldots, x_m\}$, where $m \in \mathbb{N} \smallsetminus \{0\}$ and $x_1, x_2, \ldots, x_m$ are in $\mathcal{H}$. Then there is at most one point $p \in \mathcal{H}$ satisfying the following two conditions:*

(i) *$p \in \operatorname{aff}(S)$, and*
(ii) *$\{\|p - s\| \mid s \in S\}$ is a singleton: $\|p - x_1\| = \|p - x_2\| = \cdots = \|p - x_m\|$.*

Proof. Assume both of p, q satisfy conditions (i) and (ii).

By assumption and Lemma 2.6, $p, q \in \operatorname{aff}(S) = \operatorname{aff}\{x_1, \ldots, x_m\} = x_1 + \operatorname{span}\{x_2 - x_1, \ldots, x_m - x_1\}$. Thus $p - q \in \operatorname{span}\{x_2 - x_1, \ldots, x_m - x_1\}$, and so there exist $\alpha_1, \ldots, \alpha_{m-1} \in \mathbb{R}$ such that $p - q = \sum_{i=1}^{m-1} \alpha_i (x_{i+1} - x_1)$. Using the Corollary 3.2 above and using the condition (ii) satisfied by both of p and q, we observe that for every $i \in I = \{2, \ldots, m\}$, we have

$$\begin{aligned} \langle p - x_1, x_i - x_1 \rangle &= \frac{1}{2}\|x_i - x_1\|^2 \quad \text{and} \\ \langle q - x_1, x_i - x_1 \rangle &= \frac{1}{2}\|x_i - x_1\|^2. \end{aligned}$$

Subtracting the above equalities, we get

$$(\forall i \in I) \quad \langle p - q, x_i - x_1 \rangle = 0.$$

Multiplying α_i on both sides of the corresponding i^{th} equality and then summing up the $m - 1$ equalities, we get

$$0 = \left\langle p - q, \sum_{i=1}^{m-1} \alpha_i (x_{i+1} - x_1) \right\rangle = \langle p - q, p - q \rangle = \|p - q\|^2.$$

Hence $p = q$, which implies that if such point satisfying conditions (i) and (ii) exists, then it must be unique. □

We are now in a position to define the circumcenter operator.

Definition 3.4 (circumcenter)**.** The *circumcenter operator* is

$$CC\colon \mathcal{P}(\mathcal{H}) \to \mathcal{H} \cup \{\varnothing\} \colon S \mapsto \begin{cases} p, & \text{if } p \in \operatorname{aff}(S) \text{ and } \{\|p - s\| \mid s \in S\} \text{ is a singleton;} \\ \varnothing, & \text{otherwise.} \end{cases}$$

The *circumradius operator* is

$$CR\colon \mathcal{P}(\mathcal{H}) \to \mathbb{R}\colon S \mapsto \begin{cases} \|CC(S) - s\|, & \text{if } CC(S) \in \mathcal{H} \text{ and } s \in S; \\ +\infty, & \text{if } CC(S) = \varnothing. \end{cases}$$

In particular, when $CC(S) \in \mathcal{H}$, that is, $CC(S) \neq \varnothing$, we say that the circumcenter of S exists and we call $CC(S)$ the circumcenter of S and $CR(S)$ the circumradius of S.

Note that in the Proposition 3.3 above, we have already proved that for every $S \in \mathcal{P}(\mathcal{H})$, there is at most one point $p \in \operatorname{aff}(S)$ such that $\{\|p - s\| \mid s \in S\}$ is a singleton, so the notions are *well defined*. Hence we obtain the following alternative expression of the circumcenter operator:

Remark 3.5. Let $S \in \mathcal{P}(\mathcal{H})$. Then the $CC(S)$ is either $\varnothing$ or the *unique* point $p \in \mathcal{H}$ such that

(i) $p \in \operatorname{aff}(S)$ and,
(ii) $\{\|p - s\| \mid s \in S\}$ is a singleton.

Example 3.6. Let x_1, x_2 be in $\mathcal{H}$. Then

$$CC\big(\{x_1, x_2\}\big) = \frac{x_1 + x_2}{2}.$$

4. Explicit formulae for the circumcenter

We continue to assume that

$$m \in \mathbb{N} \smallsetminus \{0\}, \quad x_1, \ldots, x_m \text{ are vectors in } \mathcal{H}, \quad \text{and} \quad S = \{x_1, \ldots, x_m\}.$$

If S is a singleton, say $S = \{x_1\}$, then, by Definition 3.4, we clearly have $CC(S) = x_1$. So in this section, to deduce the formula of $CC(S)$, we always assume that

$$m \geq 2.$$

We are ready for an explicit formula for the circumcenter.

Theorem 4.1. *Suppose that $x_1, \ldots, x_m$ are affinely independent. Then $CC(S) \in \mathcal{H}$, which means that $CC(S)$ is the unique point satisfying the following two conditions:*

(i) *$CC(S) \in \operatorname{aff}(S)$, and*
(ii) *$\{\|CC(S) - s\| \mid s \in S\}$ is a singleton.*

Moreover,

$$CC(S) = x_1 + \frac{1}{2}(x_2 - x_1, \ldots, x_m - x_1) G(x_2 - x_1, \ldots, x_m - x_1)^{-1} \begin{pmatrix} \|x_2 - x_1\|^2 \\ \vdots \\ \|x_m - x_1\|^2 \end{pmatrix},$$

where $G(x_2 - x_1, \ldots, x_{m-1} - x_1, x_m - x_1)$ is the Gram matrix defined in Definition 2.12:

$$\begin{aligned} &G(x_2 - x_1, \ldots, x_{m-1} - x_1, x_m - x_1) \\ &= \begin{pmatrix} \|x_2 - x_1\|^2 & \langle x_2 - x_1, x_3 - x_1\rangle & \cdots & \langle x_2 - x_1, x_m - x_1\rangle \\ \vdots & \vdots & & \vdots \\ \langle x_{m-1} - x_1, x_2 - x_1\rangle & \langle x_{m-1} - x_1, x_3 - x_1\rangle & \cdots & \langle x_{m-1} - x_1, x_m - x_1\rangle \\ \langle x_m - x_1, x_2 - x_1\rangle & \langle x_m - x_1, x_3 - x_1\rangle & \cdots & \|x_m - x_1\|^2 \end{pmatrix}. \end{aligned}$$

Proof. By assumption and Fact 2.8, we get that $x_2 - x_1, \dots, x_m - x_1$ are linearly independent. Then by Fact 2.13, the Gram matrix $G(x_2 - x_1, x_3 - x_1, \dots, x_m - x_1)$ is invertible. Set

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{m-1} \end{pmatrix} = \frac{1}{2} G(x_2 - x_1, x_3 - x_1, \dots, x_m - x_1)^{-1} \begin{pmatrix} \|x_2 - x_1\|^2 \\ \|x_3 - x_1\|^2 \\ \vdots \\ \|x_m - x_1\|^2 \end{pmatrix},$$

and

$$p = x_1 + \alpha_1(x_2 - x_1) + \alpha_2(x_3 - x_1) + \cdots + \alpha_{m-1}(x_m - x_1).$$

By the definition of $G(x_2 - x_1, x_3 - x_1, \dots, x_m - x_1)$ and by the definitions of $\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{m-1} \end{pmatrix}^\intercal$ and p, we obtain the equivalences

$$\begin{aligned}
& G(x_2 - x_1, x_3 - x_1, \dots, x_m - x_1) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{m-1} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \|x_2 - x_1\|^2 \\ \|x_3 - x_1\|^2 \\ \vdots \\ \|x_m - x_1\|^2 \end{pmatrix} \\
\iff & \begin{cases} \langle \alpha_1(x_2 - x_1) + \cdots + \alpha_{m-1}(x_m - x_1), x_2 - x_1 \rangle = \frac{1}{2}\|x_2 - x_1\|^2 \\ \vdots \\ \langle \alpha_1(x_2 - x_1) + \cdots + \alpha_{m-1}(x_m - x_1), x_m - x_1 \rangle = \frac{1}{2}\|x_m - x_1\|^2 \end{cases} \\
\iff & \begin{cases} \langle p - x_1, x_2 - x_1 \rangle = \frac{1}{2}\|x_2 - x_1\|^2 \\ \vdots \\ \langle p - x_1, x_m - x_1 \rangle = \frac{1}{2}\|x_m - x_1\|^2. \end{cases}
\end{aligned}$$

Hence by Corollary 3.2, we know that p satisfy condition (ii). In addition, it is clear that $p = x_1 + \alpha_1(x_2 - x_1) + \alpha_2(x_3 - x_1) + \cdots + \alpha_{m-1}(x_m - x_1) \in x_1 + \operatorname{span}\{x_2 - x_1, \dots, x_m - x_1\} = \operatorname{aff}(S)$, which is just the condition (i). Hence the point satisfying conditions (i) and (ii) exists.

Moreover, by Proposition 3.3, if the point exists, then it must be unique. ☐

Lemma 4.2. *Suppose that $CC(S) \in \mathcal{H}$, and let $K \subseteq S$ such that $\operatorname{aff}(K) = \operatorname{aff}(S)$. Then*

$$CC(S) = CC(K).$$

Proof. By assumption, $CC(S) \in \mathcal{H}$, that is:

(i) $CC(S) \in \operatorname{aff}(S)$, and
(ii) $\{\|CC(S) - s\| \mid s \in S\}$ is a singleton.

Because $K \subseteq S$, we get $\{\|CC(S) - s\| \mid s \in K\}$ is a singleton, by (ii). Since $\operatorname{aff}(K) = \operatorname{aff}(S)$, by (i), the point $CC(S)$ satisfy

(I) $CC(S) \in \operatorname{aff}(K)$, and
(II) $\{\|CC(S) - u\| \mid u \in K\}$ is a singleton.

Replacing S in Proposition 3.3 by K and combining with Definition 3.4, we know $CC(K) = CC(S)$. ☐

Corollary 4.3. *Suppose that $CC(S) \in \mathcal{H}$. Let $x_{i_1}, \ldots, x_{i_t}$ be elements of S such that $x_1, x_{i_1}, \ldots, x_{i_t}$ are affinely independent, and set $K = \{x_1, x_{i_1}, \ldots, x_{i_t}\}$. Furthermore, assume that $\operatorname{aff}(K) = \operatorname{aff}(S)$. Then*

$$\begin{aligned} CC(S) &= CC(K) \\ &= x_1 + \frac{1}{2}(x_{i_1} - x_1, \ldots, x_{i_t} - x_1) G(x_{i_1} - x_1, \ldots, x_{i_t} - x_1)^{-1} \begin{pmatrix} \|x_{i_1} - x_1\|^2 \\ \vdots \\ \|x_{i_t} - x_1\|^2 \end{pmatrix}. \end{aligned}$$

Proof. By Theorem 4.1, $x_1, x_{i_1}, \ldots, x_{i_t}$ are affinely independent implies that $CC(K) \neq \varnothing$, and

$$CC(K) = x_1 + \frac{1}{2}(x_{i_1} - x_1, \ldots, x_{i_t} - x_1) G(x_{i_1} - x_1, \ldots, x_{i_t} - x_1)^{-1} \begin{pmatrix} \|x_{i_1} - x_1\|^2 \\ \vdots \\ \|x_{i_t} - x_1\|^2 \end{pmatrix}.$$

Then the desired result follows from Lemma 4.2. □

Lemma 4.4. *Let $x_{i_1}, \ldots, x_{i_t}$ be elements of S, and set $K = \{x_1, x_{i_1}, \ldots, x_{i_t}\}$. Then*

$$\begin{aligned} &\operatorname{aff}(K) = \operatorname{aff}(S) \quad \textit{and } x_1, x_{i_1}, \ldots, x_{i_t} \textit{ are affinely independent.} \\ \iff & x_{i_1} - x_1, \ldots, x_{i_t} - x_1 \textit{ is a basis of } \operatorname{span}\{x_2 - x_1, \ldots, x_m - x_1\}. \end{aligned}$$

Proof. Indeed,

$$\begin{aligned} & x_{i_1} - x_1, \ldots, x_{i_t} - x_1 \text{ is a basis of } \operatorname{span}\{x_2 - x_1, \ldots, x_m - x_1\} \\ \iff & \begin{cases} x_{i_1} - x_1, \ldots, x_{i_t} - x_1 \text{ are linearly independent, and} \\ \operatorname{span}\{x_{i_1} - x_1, \ldots, x_{i_t} - x_1\} = \operatorname{span}\{x_2 - x_1, \ldots, x_m - x_1\} \end{cases} \\ \overset{\text{Fact 2.8}}{\iff} & \begin{cases} x_1, x_{i_1}, \ldots, x_{i_t} \text{ are affinely independent, and} \\ x_1 + \operatorname{span}\{x_{i_1} - x_1, \ldots, x_{i_t} - x_1\} = x_1 + \operatorname{span}\{x_2 - x_1, \ldots, x_m - x_1\} \end{cases} \\ \iff & \begin{cases} x_1, x_{i_1}, \ldots, x_{i_t} \text{ are affinely independent, and} \\ \operatorname{aff}(K) = \operatorname{aff}(S), \end{cases} \end{aligned}$$

which completes the proof. □

5. Additional formulae for the circumcenter

Upholding the assumptions of Section 4, we assume additionally that

$$x_1, \ldots, x_m \text{ are affinely independent.}$$

By Theorem 4.1, $CC(S) \in \mathcal{H}$. Let

$$k \in \{2, 3, \ldots, m\} \text{ be arbitrary but fixed.}$$

By Theorem 4.1 again, we know that

$$\begin{aligned} CC(S) &= x_1 + \alpha_1(x_2 - x_1) + \alpha_2(x_3 - x_1) + \cdots + \alpha_{m-1}(x_m - x_1) && \text{(5.1a)} \\ &= \big(1 - \textstyle\sum_{i=1}^{m-1} \alpha_i\big) x_1 + \alpha_1 x_2 + \cdots + \alpha_{m-1} x_m, && \text{(5.1b)} \end{aligned}$$

where

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{m-1} \end{pmatrix} = \frac{1}{2} G(x_2 - x_1, x_3 - x_1, \ldots, x_m - x_1)^{-1} \begin{pmatrix} \|x_2 - x_1\|^2 \\ \|x_3 - x_1\|^2 \\ \vdots \\ \|x_m - x_1\|^2 \end{pmatrix}. \tag{5.2}$$

By the symmetry of the positions of the points $x_1, \ldots, x_k, \ldots, x_m$ in S in Definition 3.4 and by Proposition 3.3, we also get

$$\begin{aligned} CC(S) &= x_k + \beta_1(x_1 - x_k) + \cdots + \beta_{k-1}(x_{k-1} - x_k) \\ &\quad + \beta_k(x_{k+1} - x_k) + \cdots + \beta_{m-1}(x_m - x_k) && (5.3a) \\ &= \beta_1 x_1 + \cdots + \beta_{k-1} x_{k-1} + (1 - \sum_{i=1}^{m-1} \beta_i) x_k + \beta_k x_{k+1} + \cdots + \beta_{m-1} x_m, && (5.3b) \end{aligned}$$

where

(5.4)

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{m-1} \end{pmatrix} = \frac{1}{2} G(x_1 - x_k, \ldots, x_{k-1} - x_k, x_{k+1} - x_k, \ldots, x_m - x_k)^{-1} \begin{pmatrix} \|x_1 - x_k\|^2 \\ \vdots \\ \|x_{k-1} - x_k\|^2 \\ \|x_{k+1} - x_k\|^2 \\ \vdots \\ \|x_m - x_k\|^2 \end{pmatrix}.$$

Proposition 5.1. *The following equalities hold:*

(5.5)
$\left(1 - \sum_{i=1}^{m-1} \alpha_i\right) = \beta_1,$ *(coefficient of* x_1*)*

(5.6)
$\alpha_{k-1} = \left(1 - \sum_{i=1}^{m-1} \beta_i\right),$ *(coefficient of* x_k*)*

(5.7)
$(\forall i \in \{2, \ldots, k-1\}) \quad \alpha_{i-1} = \beta_i$ *and* $(\forall j \in \{k, k+1, \ldots, m-1\}) \quad \alpha_j = \beta_j.$

Proof. Recall that at the beginning of this section we assumed $x_1, \ldots, x_m$ are affinely independent. Combining the equations (5.1b) & (5.3b) and Lemma 2.9, we get the required results. $\square$

To simplify the statements, we use the following abbreviations.

$$\begin{aligned} A &= G(x_2 - x_1, \ldots, x_k - x_1, \ldots, x_m - x_1), \\ B &= G(x_1 - x_k, \ldots, x_{k-1} - x_k, x_{k+1} - x_k, \ldots, x_m - x_k), \end{aligned}$$

and the determinant of matrix A (by Proposition 2.15, it is also the determinant of matrix B) is denoted by:

$$\delta = \det(A).$$

We denote the two column vectors a, b respectively by:

$$a = \big(\|x_2 - x_1\|^2 \quad \cdots \quad \|x_k - x_1\|^2 \quad \cdots \quad \|x_m - x_1\|^2\big)^\intercal,$$
$$b = \big(\|x_1 - x_k\|^2 \quad \cdots \quad \|x_{k-1} - x_k\|^2 \quad \|x_{k+1} - x_k\|^2 \quad \cdots \quad \|x_m - x_k\|^2\big)^\intercal.$$

For every $M \in \mathbb{R}^{n\times n}$, and for every $j \in \{1, 2, \ldots, n\}$,

$$\text{we denote the } j^{\text{th}} \text{ column of the matrix } M \text{ as } M_{*,j}.$$

In turn, for every $i \in \{1, \ldots, m-1\}$,

$$A_i = [A_{*,1}|\cdots|A_{*,i-1}|a|A_{*,i+1}|\cdots|A_{*,m-1}],$$

and

$$B_i = [B_{*,1}|\cdots|B_{*,i-1}|b|B_{*,i+1}|\cdots|B_{*,m-1}].$$

That is, A_i is identical to A except that column $A_{*,i}$ has been replaced by a and B_i is identical to B except that column $B_{*,i}$ has been replaced by b.

Lemma 5.2. *The following statements hold:*

(i) $\big(\alpha_1 \cdots \alpha_{m-1}\big)^\intercal$ *defined in* (5.2) *is the unique solution of the nonsingular system* $Ay = \frac{1}{2}a$ *where* y *is the unknown variable. In consequence, for every* $i \in \{1, \ldots, m-1\}$,

$$\alpha_i = \frac{\det(A_i)}{2\delta}.$$

(ii) $\big(\beta_1 \cdots \beta_{m-1}\big)^\intercal$ *defined in* (5.4) *is the unique solution of the nonsingular system* $By = \frac{1}{2}b$ *where* y *is the unknown variable. In consequence, for every* $i \in \{1, \ldots, m-1\}$,

$$\beta_i = \frac{\det(B_i)}{2\delta}.$$

Proof. By assumption, $x_1, \ldots, x_m$ are affinely independent, and by Proposition 2.15, we know $\det(B) = \det(A) = \delta \neq 0$.

(i): By definition of $\big(\alpha_1 \cdots \alpha_{m-1}\big)^\intercal$,

$$\big(\alpha_1 \cdots \alpha_{m-1}\big)^\intercal = \frac{1}{2}A^{-1}a.$$

Clearly we know it is the unique solution of the nonsingular system $Ay = \frac{1}{2}a$. Hence the desired result follows directly from the Fact 2.17, the Cramer Rule.

(ii): Using the same method of proof of (i), we can prove (ii) . □

Using Theorem 4.1, Lemma 5.2 and the equalities (5.5), (5.6) and (5.7), we readily obtain the following result.

Corollary 5.3. *Suppose that* $x_1, \ldots, x_m$ *are affinely independent. Then*

$$CC(S) = \big(1 - \textstyle\sum_{i=1}^{m-1}\alpha_i\big)x_1 + \alpha_1 x_2 + \cdots + \alpha_{m-1}x_m,$$

where $(\forall i \in \{1, \ldots, m-1\})$ $\alpha_i = \frac{1}{2\delta}\det(A_i)$*. Moreover,*

$$1 - \sum_{i=1}^{m-1}\alpha_i = \frac{1}{2\delta}\det(B_1),\ \ \alpha_{k-1} = 1 - \sum_{i=1}^{m-1}\frac{1}{2\delta}\det(B_i),$$

$$(\forall i \in \{2,\ldots,k-1\}) \quad \alpha_{i-1} = \frac{1}{2\delta}\det(B_i)$$
$$\text{and} \quad (\forall j \in \{k,k+1,\ldots,m-1\}) \quad \alpha_j = \frac{1}{2\delta}\det(B_j).$$

6. Basic properties of the circumcenter

In this section we collect some fundamental properties of the circumcenter operator. Recall that

$$m \in \mathbb{N} \smallsetminus \{0\}, \quad x_1,\ldots,x_m \text{ are vectors in } \mathcal{H}, \quad \text{and} \quad S = \{x_1,\ldots,x_m\}.$$

Proposition 6.1 (scalar multiples)**.** *Let* $\lambda \in \mathbb{R} \smallsetminus \{0\}$*. Then* $CC(\lambda S) = \lambda CC(S)$*.*

Proof. Let $p \in \mathcal{H}$. By Definition 3.4,

$$\begin{aligned} p = CC(S) &\iff \begin{cases} p \in \operatorname{aff}(S) \\ \{\|p-s\| \mid s \in S\} \text{ is a singleton} \end{cases} \\ &\iff \begin{cases} \lambda p \in \operatorname{aff}(\lambda S) \\ \{\|\lambda p - \lambda s\| \mid \lambda s \in \lambda S\} \text{ is a singleton} \end{cases} \\ &\iff \lambda p = CC(\lambda S), \end{aligned}$$

and the result follows. □

The next example below illustrates that we had to exclude the case $\lambda = 0$ in Proposition 6.1.

Example 6.2. Suppose that $\mathcal{H} = \mathbb{R}$ and that $S = \{0,-1,1\}$. Then

$$CC(0 \cdot S) = \{0\} \neq \varnothing = 0 \cdot CC(S).$$

Proposition 6.3 (translations)**.** *Let* $y \in \mathcal{H}$*. Then* $CC(S+y) = CC(S) + y$*.*

Proof. Let $p \in \mathcal{H}$. By Lemma 2.6,

$$\begin{aligned} p \in \operatorname{aff}\{x_1,\ldots,x_m\} &\iff (\exists\ \lambda_1,\ldots,\lambda_m \in \mathbb{R} \text{ with } \sum_{i=1}^m \lambda_i = 1) \quad p = \sum_{i=1}^m \lambda_i x_i \\ &\iff (\exists\ \lambda_1,\ldots,\lambda_m \in \mathbb{R} \text{ with } \sum_{i=1}^m \lambda_i = 1) \quad p + y = \sum_{i=1}^m \lambda_i (x_i + y) \\ &\iff p + y \in \operatorname{aff}\{x_1 + y,\ldots,x_m + y\}, \end{aligned}$$

that is

$$p \in \operatorname{aff}(S) \iff p + y \in \operatorname{aff}(S+y). \tag{6.1}$$

By (6.1) and Remark 3.5, we have

$$\begin{aligned} p = CC(S) \in \mathcal{H} &\iff \begin{cases} p \in \operatorname{aff}(S) \\ \{\|p-s\| \mid s \in S\} \text{ is a singleton} \end{cases} \\ &\iff \begin{cases} p + y \in \operatorname{aff}(S+y) \\ \{\|(p+y)-(s+y)\| \mid s+y \in S+y\} \text{ is a singleton} \end{cases} \\ &\iff p + y = CC(S+y) \in \mathcal{H}. \end{aligned}$$

Moreover, because $\varnothing = \varnothing + y$, the proof is complete. $\square$

7. Circumcenters of sequences of sets

We uphold the assumptions that

$$m \in \mathbb{N} \smallsetminus \{0\}, \quad x_1, \ldots, x_m \text{ are vectors in } \mathcal{H}, \quad \text{and} \quad S = \{x_1, \ldots, x_m\}.$$

In this section, we explore the convergence of the circumcenter operator over a sequence of sets.

Theorem 7.1. *Suppose that $CC(S) \in \mathcal{H}$. Then the following hold:*

(i) *Set $t = \dim\Big(\operatorname{span}\{x_2 - x_1, \ldots, x_m - x_1\}\Big)$, and let $\widetilde{S} = \{x_1, x_{i_1}, \ldots, x_{i_t}\} \subseteq S$ be such that $x_{i_1} - x_1, \ldots, x_{i_t} - x_1$ is a basis of $\operatorname{span}\{x_2 - x_1, \ldots, x_m - x_1\}$. Furthermore, let $\Big((x_1^{(k)}, x_{i_1}^{(k)}, \ldots, x_{i_t}^{(k)})\Big)_{k \geq 1} \subseteq \mathcal{H}^{t+1}$ with $\lim_{k \to \infty} (x_1^{(k)}, x_{i_1}^{(k)}, \ldots, x_{i_t}^{(k)}) = (x_1, x_{i_1}, \ldots, x_{i_t})$, and set $(\forall k \geq 1)$ $\widetilde{S}^{(k)} = \{x_1^{(k)}, x_{i_1}^{(k)}, \ldots, x_{i_t}^{(k)}\}$. Then there exist $N \in \mathbb{N}$ such that for every $k \geq N$, $CC(\widetilde{S}^{(k)}) \in \mathcal{H}$ and*

$$\lim_{k \to \infty} CC(\widetilde{S}^{(k)}) = CC(\widetilde{S}) = CC(S).$$

(ii) *Suppose that $x_1, \ldots, x_{m-1}, x_m$ are affinely independent, and let $\Big((x_1^{(k)}, \ldots, x_{m-1}^{(k)}, x_m^{(k)})\Big)_{k \geq 1} \subseteq \mathcal{H}^m$ satisfy $\lim_{k \to \infty} (x_1^{(k)}, \ldots, x_{m-1}^{(k)}, x_m^{(k)}) = (x_1, \ldots, x_{m-1}, x_m)$. Set $(\forall k \geq 1)$ $S^{(k)} = \{x_1^{(k)}, \ldots, x_{m-1}^{(k)}, x_m^{(k)}\}$. Then*

$$\lim_{k \to \infty} CC(S^{(k)}) = CC(S).$$

Proof. (i): Let l be the cardinality of the set S. Assume first that $l = 1$. Then $t = 0$, and $\widetilde{S} = \{x_1\}$. Let $(x_1^{(k)})_{k \geq 1} \subseteq \mathcal{H}$ satisfy $\lim_{k \to \infty} x_1^{(k)} = x_1$. By Definition 3.4, we know $CC(\{x_1^{(k)}\}) = x_1^{(k)}$ and $CC(\{x_1\}) = x_1$. Hence

$$\lim_{k \to \infty} CC(\widetilde{S}^{(k)}) = \lim_{k \to \infty} x_1^{(k)} = x_1 = CC(S).$$

Now assume that $l \geq 2$. By Corollary 4.3 and Lemma 4.4, we obtain

$$\begin{aligned} CC(S) &= CC(\widetilde{S}) \\ &= x_1 + \frac{1}{2}(x_{i_1} - x_1, \ldots, x_{i_t} - x_1) G(x_{i_1} - x_1, \ldots, x_{i_t} - x_1)^{-1} \begin{pmatrix} \|x_{i_1} - x_1\|^2 \\ \vdots \\ \|x_{i_t} - x_1\|^2 \end{pmatrix}. \end{aligned} \tag{7.1}$$

Using the assumptions and the Lemma 2.10, we know that there exists $N \in \mathbb{N}$ such that

$$(\forall k \geq N) \quad x_1^{(k)}, x_{i_1}^{(k)}, \ldots, x_{i_t}^{(k)} \text{ are affinely independent.}$$

By Theorem 4.1, we know $(k \geq N)$ $CC(\widetilde{S}^{(k)}) \in \mathcal{H}$. Moreover, for every $k \geq N$,
(7.2)

$$CC(\widetilde{S}^{(k)}) = x_1^{(k)} + \frac{1}{2}(x_{i_1}^{(k)} - x_1^{(k)}, \ldots, x_{i_t}^{(k)} - x_1^{(k)})G(x_{i_1}^{(k)} - x_1^{(k)}, \ldots, x_{i_t}^{(k)} - x_1^{(k)})^{-1} \begin{pmatrix} \|x_{i_1}^{(k)} - x_1^{(k)}\|^2 \\ \vdots \\ \|x_{i_t}^{(k)} - x_1^{(k)}\|^2 \end{pmatrix}.$$

Comparing (7.1) with (7.2) and using Corollary 2.18, we obtain

$$\lim_{k\to\infty} CC(\widetilde{S}^{(k)}) = CC(\widetilde{S}) = CC(S).$$

(ii): Let $x_1, \ldots, x_{m-1}, x_m \in \mathcal{H}$ be affinely independent. Then $t = m - 1$ and $\widetilde{S} = S$. Substitute the $\widetilde{S}$ and $\widetilde{S}^{(k)}$ in part (i) by our S and $S^{(k)}$ respectively. Then we obtain

$$\lim_{k\to\infty} CC(S^{(k)}) = CC(S)$$

and the proof is complete. □

Corollary 7.2. *The mapping*

$$\Psi\colon \mathcal{H}^m \to \mathcal{H} \cup \{\varnothing\}\colon (x_1, \ldots, x_m) \mapsto CC(\{x_1, \ldots, x_m\})$$

is continuous at every point $(x_1, \ldots, x_m) \in \mathcal{H}^m$ where $x_1, \ldots, x_m$ are affinely independent.

Proof. This follows directly from Theorem 7.1(ii). □

Let us record the doubleton case explicitly.

Proposition 7.3. *Suppose that $m = 2$. Let $\big((x_1^{(k)}, x_2^{(k)})\big)_{k\geq 1} \subseteq \mathcal{H}^2$ satisfy $\lim_{k\to\infty}(x_1^{(k)}, x_2^{(k)}) = (x_1, x_2)$. Then*

$$\lim_{k\to\infty} CC(\{x_1^{(k)}, x_2^{(k)}\}) = CC(\{x_1, x_2\}).$$

Proof. Indeed, we deduce from Example 3.6 that

$$\lim_{k\to\infty} CC(\{x_1^{(k)}, x_2^{(k)}\}) = \lim_{k\to\infty} \frac{x_1^{(k)} + x_2^{(k)}}{2} = \frac{x_1 + x_2}{2} = CC(\{x_1, x_2\})$$

and the result follows. □

The following example illustrates that the assumption that "$m = 2$" in Proposition 7.3 cannot be replaced by "the cardinality of S is 2".

Example 7.4. Suppose that $\mathcal{H} = \mathbb{R}^2$, that $m = 3$, and that $S = \{x_1, x_2, x_3\}$ with $x_1 = (-1, 0)$, $x_2 = x_3 = (1, 0)$. Then there exists $((x_1^{(k)}, x_2^{(k)}, x_3^{(k)}))_{k\geq 1} \subseteq \mathcal{H}^3$ such that $\lim_{k\to\infty}(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}) = (x_1, x_2, x_3)$ and

$$\lim_{k\to\infty} CC(\{x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\}) \neq CC(S).$$

Proof. For every $k \geq 1$, let $(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}) = \Big((-1,0),(1,0),(1+\frac{1}{k},0)\Big) \in \mathcal{H}^3$. Then $\lim_{k\to\infty}(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}) = (x_1, x_2, x_3)$. Moreover, by Definition 3.4, we know that $(\forall k \geq 1)$, $CC(S^{(k)}) = \varnothing$, since there is no point in $\mathbb{R}^2$ which has equal distance to all of the three points. On the other hand, by Definition 3.4 again, we know $CC(S) = (0,0) \in \mathcal{H}$. Hence $\lim_{k\to\infty} CC(\{x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\}) = \varnothing \neq (0,0) = CC(S)$. $\square$

The following question now naturally arises:

Question 7.1. Suppose that $CC(\{x_1, x_2, x_3\}) \in \mathcal{H}$, and let $\big((x_1^{(k)}, x_2^{(k)}, x_3^{(k)})\big)_{k\geq 1} \subseteq \mathcal{H}^3$ be such that $\lim_{k\to\infty}(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}) = (x_1, x_2, x_3)$. Is it true that the implication

$$(\forall k \geq 1) \quad CC(\{x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\}) \in \mathcal{H} \\ \Longrightarrow \lim_{k\to\infty} CC(\{x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\}) = CC(\{x_1, x_2, x_3\})$$

holds?

When x_1, x_2, x_3 are affinely independent, then Theorem 7.1(ii) gives us an affirmative answer. However, the answer is negative if x_1, x_2, x_3 are not assumed to be affinely independent.

Example 7.5. Suppose that $\mathcal{H} = \mathbb{R}^2$ and $S = \{x_1, x_2, x_3\}$ with $x_1 = (-2,0)$, $x_2 = x_3 = (2,0)$. Then there exists a sequence $\big((x_1^{(k)}, x_2^{(k)}, x_3^{(k)})\big)_{k\geq 1} \subseteq \mathcal{H}^3$ such that

(i) $\lim_{k\to\infty}(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}) = (x_1, x_2, x_3)$,
(ii) $(\forall k \geq 1) \quad CC(\{x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\}) \in \mathbb{R}^2$, and
(iii) $\lim_{k\to\infty} CC(\{x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\}) \neq CC(S)$.

Proof. By Definition 3.4, we know that $CC(S) = (0,0) \in \mathcal{H}$. Set

$$(\forall k \geq 1) \quad S^{(k)} = \{x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\} = \Big\{(-2,0),(2,0),\big(2-\tfrac{1}{k},\tfrac{1}{4k}\big)\Big\}.$$

(i): In this case,

$$\begin{aligned}\lim_{k\to\infty}(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}) &= \lim_{k\to\infty}\Big((-2,0),(2,0),\big(2-\tfrac{1}{k},\tfrac{1}{4k}\big)\Big)\\ &=\big((-2,0),(2,0),(2,0)\big)\\ &=(x_1,x_2,x_3).\end{aligned}$$

(ii): It is clear that for every $k \geq 1$, the vectors $(-2,0),(2,0),(2-\frac{1}{k},\frac{1}{4k})$ are not colinear, that is, $(-2,0),(2,0),(2-\frac{1}{k},\frac{1}{4k})$ are affinely independent. By Theorem 4.1, we see that

$$(\forall k \geq 1) \quad CC(\{x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\}) \in \mathbb{R}^2.$$

(iii): Let $k \geq 1$. By definition of $CC(S^{(k)})$ and (ii), we deduce that $CC(S^{(k)}) = (p_1^{(k)}, p_2^{(k)}) \in \mathbb{R}^2$ and that

$$\|CC(S^{(k)}) - x_1^{(k)}\| = \|CC(S^{(k)}) - x_2^{(k)}\| = \|CC(S^{(k)}) - x_3^{(k)}\|.$$

Because $CC(S^{(k)})$ must be in the intersection of the perpendicular bisector of $x_1^{(k)} = (-2,0), x_2^{(k)} = (2,0)$ and the perpendicular bisector of $x_2^{(k)} = (2,0), x_3^{(k)} = (2 - \frac{1}{k}, \frac{1}{4k})$, we obtain

$$p_1^{(k)} = 0 \quad \text{and} \quad p_2^{(k)} = 4\big(p_1^{(k)} - \frac{2+2-\frac{1}{k}}{2}\big) + \tfrac{1}{8k};$$

thus,

$$CC(S^{(k)}) = (p_1^{(k)}, p_2^{(k)}) = \big(0, -8 + \tfrac{2}{k} + \tfrac{1}{8k}\big). \tag{7.3}$$

(Alternatively, we can use the formula in Theorem 4.1 to get (7.3)). Therefore,

$$\lim_{k\to\infty} CC(S^{(k)}) = \lim_{k\to\infty} \big(0, -8 + \tfrac{2}{k} + \tfrac{1}{8k}\big) = (0,-8) \neq (0,0) = CC(S),$$

and the proof is complete.

As the picture below shows, $(\forall k \geq 1)\ x_3^{(k)} = (2 - \frac{1}{k}, \frac{1}{4k})$ converges to $x_3 = (2,0)$ along the purple line $L = \{(x,y) \in \mathbb{R}^2 \mid y = -\frac{1}{4}(x-2)\}$. In fact, $CC(S^{(k)})$ is just the intersection point between the two lines M_1 and M_2, where M_1 is the perpendicular bisector between the points x_1 and x_2, and M_2 is the perpendicular bisector between the points $x_3^{(k)}$ and x_2. ☐

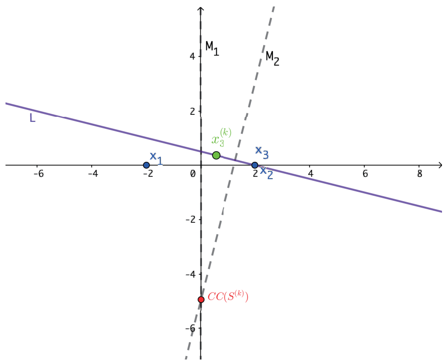

FIGURE 1. Continuity of circumcenter operator may fail even when $(\forall k \geq 1)\ CC(S^{(k)}) \in \mathcal{H}$.

8. The circumcenter of three points

In this section, we study the circumcenter of a set containing three points. We will give a characterization of the existence of circumcenter of three pairwise distinct points. In addition, we shall provide asymmetric and symmetric formulae.

Theorem 8.1. *Suppose that $S = \{x, y, z\} \in \mathcal{P}(\mathcal{H})$ and that $l = 3$ is the cardinality of S. Then x, y, z are affinely independent if and only if $CC(S) \in \mathcal{H}$.*

Proof. If S is affinely independent, then $CC(S) \in \mathcal{H}$ by Theorem 4.1.

To prove the converse implication, suppose that $CC(S) \in \mathcal{H}$, i.e.,

(i) $CC(S) \in \operatorname{aff}\{x, y, z\}$, and

(ii) $\|CC(S)-x\| = \|CC(S)-y\| = \|CC(S)-z\|$.

We argue by contradiction and thus assume that the elements of S are affinely dependent:

$$\dim(\operatorname{span}\{S-x\}) = \dim(\operatorname{span}\{y-x, z-x\}) \leq 1.$$

Note that $y-x \neq 0$ and $z-x \neq 0$. Set

$$U = x + \operatorname{span}\{y-x, z-x\} = x + \operatorname{span}\{y-x\} = x + \operatorname{span}\{z-x\}.$$

Combining with Lemma 2.6, we get

$$U = \operatorname{aff}\{x,y,z\} = \operatorname{aff}\{x,y\} = \operatorname{aff}\{x,z\}. \tag{8.1}$$

By definition of $CC(S)$, we have

$$CC(S) \in \operatorname{aff}\{x,y\} \overset{(8.1)}{=} U \quad \text{and} \quad \|CC(S)-x\| = \|CC(S)-y\|, \tag{8.2}$$

and

$$CC(S) \in \operatorname{aff}\{x,z\} \overset{(8.1)}{=} U \quad \text{and} \quad \|CC(S)-x\| = \|CC(S)-z\|. \tag{8.3}$$

Now using (i) $\Leftrightarrow$ (iii) in Proposition 3.1 and using (8.2), we get

$$CC(S) = P_U\Big(CC(S)\Big) = \frac{x+y}{2}.$$

Similarly, using (i) $\Leftrightarrow$ (iii) in Proposition 3.1 and using (8.3), we can also get

$$CC(S) = P_U\Big(CC(S)\Big) = \frac{x+z}{2}.$$

Therefore,

$$\frac{x+y}{2} = CC(S) = \frac{x+z}{2} \Longrightarrow y = z,$$

which contradicts the assumption that $l = 3$. The proof is complete. □

In contrast, when the cardinality of S is 4, then

$$CC(S) \in \mathcal{H} \not\Rightarrow \text{ elements of } S \text{ are affinely independent}$$

as the following example demonstrates. Thus the characterization of the existence of circumcenter in Theorem 8.1 is generally not true when we consider $l \geq 3$ pairwise distinct points.

Example 8.2. Suppose that $\mathcal{H} = \mathbb{R}^2$, that $m = 4$, and $S = \{x_1, x_2, x_3, x_4\}$, where $x_1 = (0,0)$, $x_2 = (4,0)$, $x_3 = (0,4)$, and $x_4 = (4,4)$ (see Figure 2). Then x_1, x_2, x_3, x_4 are pairwise distinct and affinely dependent, yet $CC(S) = (2,2)$.

In Theorem 4.1 above, where we presented the formula for $CC(S)$, we gave special importance to the first point x_1 in S. We now provide some longer yet symmetric formulae for $CC(S)$.

Remark 8.3. Suppose that $S = \{x,y,z\}$ and that $l = 3$ is the cardinality of S. Assume furthermore that $CC(S) \in \mathcal{H}$, i.e., there is an unique point $CC(S)$ satisfying

(i) $CC(S) \in \operatorname{aff}\{x,y,z\}$ and

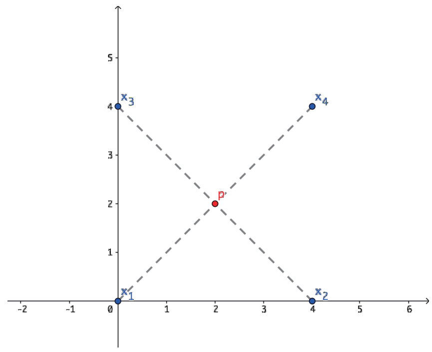

FIGURE 2. Circumcenter of the four affinely dependent points from Example 8.2.

(ii) $\|CC(S) - x\| = \|CC(S) - y\| = \|CC(S) - z\|$.

By Theorem 8.1, the vectors x, y, z must be affinely independent. From Theorem 4.1 we obtain

$$CC(S) = \ x + \frac{1}{2}(y-x, z-x) \begin{pmatrix} \|y-x\|^2 & \langle y-x, z-x\rangle \\ \langle z-x, y-x\rangle & \|z-x\|^2 \end{pmatrix}^{-1} \begin{pmatrix} \|y-x\|^2 \\ \|z-x\|^2 \end{pmatrix}$$

$$= x + \frac{(\|y-x\|^2\|z-x\|^2 - \|z-x\|^2\langle y-x, z-x\rangle)(y-x)}{2(\|y-x\|^2\|z-x\|^2 - \langle y-x, z-x\rangle^2)}$$

$$+ \frac{(\|y-x\|^2\|z-x\|^2 - \|y-x\|^2\langle y-x, z-x\rangle)(z-x)}{2(\|y-x\|^2\|z-x\|^2 - \langle y-x, z-x\rangle^2)}$$

$$= \frac{1}{K_1}\Big(\|y-z\|^2\langle x-z, x-y\rangle x + \|x-z\|^2\langle y-z, y-x\rangle y + \|x-y\|^2\langle z-x, z-y\rangle z\Big),$$

where $K_1 = 2(\|y-x\|^2\|z-x\|^2 - \langle y-x, z-x\rangle^2)$.
Similarly,

$$CC(S)$$
$$= \frac{1}{K_2}\Big(\|y-z\|^2\langle x-z, x-y\rangle x + \|x-z\|^2\langle y-z, y-x\rangle y + \|x-y\|^2\langle z-x, z-y\rangle z\Big),$$

where $K_2 = 2(\|x-y\|^2\|z-y\|^2 - \langle x-y, z-y\rangle^2)$ and

$$CC(S)$$
$$= \frac{1}{K_3}\Big(\|y-z\|^2\langle x-z, x-y\rangle x + \|x-z\|^2\langle y-z, y-x\rangle y + \|x-y\|^2\langle z-x, z-y\rangle z\Big),$$

where $K_3 = 2(\|x-z\|^2\|y-z\|^2 - \langle x-z, y-z\rangle^2)$.

In view of Proposition 3.3 (the uniqueness of the circumcenter), we now average the three formulae from above to obtain the following symmetric formula for p:

$$CC(S)$$
$$= \frac{1}{K}\Big(\|y-z\|^2\langle x-z, x-y\rangle x + \|x-z\|^2\langle y-z, y-x\rangle y + \|x-y\|^2\langle z-x, z-y\rangle z\Big),$$

where

$$K = \frac{1}{6}\Big(\frac{1}{\|y-x\|^2\|z-x\|^2 - \langle y-x, z-x\rangle^2} + \frac{1}{\|x-y\|^2\|z-y\|^2 - \langle x-y, z-y\rangle^2} + \frac{1}{\|x-z\|^2\|y-z\|^2 - \langle x-z, y-z\rangle^2}\Big).$$

In fact, Proposition 2.15 yields $K_1 = K_2 = K_3$.

We now summarize the above discussion so far in the following two pleasing main results.

Theorem 8.4 (nonsymmetric formula for the circumcenter)**.** *Suppose that $S = \{x, y, z\}$ and denote the cardinality of S by l. Then exactly one of the following cases occurs:*

(i) $l = 1$ *and* $CC(S) = x$.

(ii) $l = 2$, *say* $S = \{u, v\}$, *where* $u, v \in S$ *and* $u \neq v$, *and* $CC(S) = \frac{u+v}{2}$.

(iii) $l = 3$ *and exactly one of the following two cases occurs:*

(a) x, y, z *are affinely independent; equivalently,* $\|y-x\|\|z-x\| > \langle y-x, z-x\rangle$, *and*

$$CC(S) = \frac{\|y-z\|^2\langle x-z, x-y\rangle x + \|x-z\|^2\langle y-z, y-x\rangle y + \|x-y\|^2\langle z-x, z-y\rangle z}{2(\|y-x\|^2\|z-x\|^2 - \langle y-x, z-x\rangle^2)}.$$

(b) x, y, z *are affinely dependent; equivalently,* $\|y-x\|\|z-x\| = \langle y-x, z-x\rangle$, *and* $CC(S) = \varnothing$.

Theorem 8.5 (symmetric formula of the circumcenter)**.** *Suppose that $S = \{x, y, z\}$ and denote the cardinality of S by l. Then exactly one of the following cases occurs:*

(i) $l = 1$ *and* $CC(S) = x = y = z = \frac{x+y+z}{3}$.

(ii) $l = 2$ *and* $CC(S) = \frac{\|x-y\|z + \|x-z\|y + \|y-z\|x}{\|x-y\| + \|x-z\| + \|y-z\|}$.

(iii) $l = 3$, *consider* $K = \frac{1}{6}\big(\frac{1}{\|y-x\|^2\|z-x\|^2 - \langle y-x, z-x\rangle^2} + \frac{1}{\|x-y\|^2\|z-y\|^2 - \langle x-y, z-y\rangle^2} + \frac{1}{\|x-z\|^2\|y-z\|^2 - \langle x-z, y-z\rangle^2}\big)$, *and exactly one of the following two cases occurs:*

(a) $K \in \,]0, +\infty[$ *and*

$$CC(S) = \frac{\|y-z\|^2\langle x-z, x-y\rangle x + \|x-z\|^2\langle y-z, y-x\rangle y + \|x-y\|^2\langle z-x, z-y\rangle z}{K}.$$

(b) K *is not defined (because of a zero denominator) and* $CC(S) = \varnothing$.

9. Applications of the circumcenter

In this section, we discuss applications of the circumcenter in optimization.

Let $z \in \mathcal{H}$, and let $U_1, \ldots, U_m$ be closed subspaces of $\mathcal{H}$. The corresponding best approximation problem is to

$$\text{(9.1)} \qquad \text{Find } \bar{u} \in \cap_{i=1}^m U_i \text{ such that } \|z - \bar{u}\| = \min_{u \in \cap_{i=1}^m U_i} \|z - u\|.$$

Clearly, the solution of (9.1) is just $P_{\cap_{i=1}^m U_i} z$.

Now assume that $\mathcal{H} = \mathbb{R}^n$, and let U and V be linear subspaces of $\mathcal{H}$, i.e., we focus on $m = 2$ subspaces. Set

$$\mathcal{S}\colon \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)\colon x \mapsto \{x, R_U x, R_V R_U x\}.$$

Behling, Bello Cruz, and Santos introduced and studied in [4] an algorithm to accelerate the Douglas–Rachford algorithm they termed the *Circumcentered-Douglas-Rachford method (C-DRM)*. Given a current point $x \in \mathbb{R}^n$, the next iterate of the C-DRM is the circumcenter of the triangle with vertices x, $R_U x$ and $R_V R_U x$. Hence, given the initial point $x \in \mathbb{R}^n$, the C-DRM generates the sequence $(x^{(k)})_{k\in\mathbb{N}}$ via

$$x^{(0)} = x, \quad \text{and} \quad (\forall k \in \mathbb{N}) \quad x^{(k+1)} = CC(\mathcal{S}(x^{(k)})). \tag{9.2}$$

Behling et al.'s [4, Lemma 2] guarantees that for every $x \in \mathbb{R}^n$, the circumcenter $CC(\mathcal{S}(x))$ is the projection of any point $w \in U \cap V$ onto the affine subspace $\operatorname{aff}\{x, R_U x, R_V R_U x\}$. Here, the existence of the circumcenter of $\mathcal{S}(x)$ turns out to be a necessary condition for the nonemptiness of $U \cap V$. In fact, $CC(\mathcal{S}(x)) = P_{\operatorname{aff}(\mathcal{S}(x))}(P_{U\cap V} x)$, which means that $CC(\mathcal{S}(x))$ is the closest point to the $P_{U\cap V} x$ among the points in the affine subspace $\operatorname{aff}(\mathcal{S}(x))$. In [4, Theorem 1], the authors proved that if x in (9.2) is replaced by $P_U z$, $P_V z$ or $P_{U+V} z$, where $z \in \mathbb{R}^n$, then the C-DRM sequence defined in (9.2) converges linearly to $P_{U\cap V} z$. Moreover, their rate of convergence is at least the cosine of the Friedrichs angle between U and V, $c_F \in [0, 1[$, which happens to be the sharp rate for the original DRM; see [3, Theorem 4.1] for details.

In [4, Section 3.1], the authors elaborate on how to compute the circumcenter of $\mathcal{S}(x)$ in $\mathbb{R}^n$. They used the fact that the projection of $CC(\mathcal{S}(x))$ onto each vector $R_U x - x$ and $R_V R_U x - x$ has its endpoint at the midpoint of the line segment from x to $R_U x$ and x to $R_V R_U x$. They exhibited a 2×2 linear system of equations to calculate the $CC(\mathcal{S}(x))$ and an expression of the $CC(\mathcal{S}(x))$ with parameters. Their expression of the $CC(\mathcal{S}(x))$ can be deduced from our Remark 8.3. Actually, for every $x \in \mathbb{R}^n$, using Theorem 8.4(iii)(a), we can easily obtain a closed formula for $CC(\mathcal{S}(x))$ allowing us to efficiently calculate the C-DRM sequence.

In [4, Corollary 3], Behling et al. proved that their linear convergence results are applicable to affine subspaces with nonempty intersection using the Friedrichs angle of suitable linear subspaces parallel to the original affine subspaces. Returning to (9.1), we now set

$$\widehat{\mathcal{S}}\colon \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)\colon x \mapsto \big\{x, R_{U_1} x, R_{U_2} R_{U_1} x, \ldots, R_{U_m} \cdots R_{U_2} R_{U_1} x\big\}.$$

In order to minimize the inherent zig-zag behaviour of sequences generated by various reflection and projection methods, Behling et al. generalized the C-DRM in [5] to the so-called *Circumcentered-Reflection Method (CRM)*. Using our notation, it turns out that the underlying CRM operator $C\colon \mathbb{R}^n \to \mathbb{R}^n$ is nothing but the composition $CC \circ \widehat{\mathcal{S}}$. Hence Behling et al.'s CRM sequence is just

$$x^{(0)} = x, \quad \text{and} \quad (\forall k \in \mathbb{N}) \quad x^{(k+1)} = CC(\widehat{\mathcal{S}}(x^{(k)})). \tag{9.3}$$

In [5, Lemma 3.1], they show C is well defined. Moreover, they also obtain

$$(\forall w \in \cap_{i=1}^m U_i) \quad CC(\widehat{\mathcal{S}}(x)) = P_{\operatorname{aff}(\widehat{\mathcal{S}}(x))}(w).$$

In particular, $CC(\widehat{\mathcal{S}}(x)) = P_{\operatorname{aff}(\widehat{\mathcal{S}}(x))}(P_{\cap_{i=1}^m U_i} x)$, which means that the circumcenter of the set $\widehat{\mathcal{S}}(x)$ is the point in $\operatorname{aff}(\widehat{\mathcal{S}}(x))$ that is closest to $P_{\cap_{i=1}^m U_i} x$. Behling et al.'s central convergence result (see [5, Theorem 3.3]) states that the CRM sequence (9.3) converges linearly to $P_{\cap_{i=1}^m U_i} x$.

For the actual computation of the circumcenter of the set $\widehat{\mathcal{S}}(x)$, both [4] and [5] only contain passing references to that the computation "requires the resolution of a suitable $m \times m$ linear system of equations." Concluding this section, let us point out that the explicit formula presented in Corollary 4.3 may be used; after finding a maximally linearly independent subset of $\widehat{\mathcal{S}}(x) - x$ (using Matlab, say) one can directly use the formula in Corollary 4.3 to calculate the circumcenter.

10. The circumcenter in $\mathbb{R}^3$ and the crossproduct

We conclude this paper by expressing the circumcenter and circumradius in $\mathbb{R}^3$ by using the cross product. We start by reviewing some properties of the cross product.

Definition 10.1 (cross product)**.** [1, page 483] Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^3$. The *cross product* $x \times y$ (in that order) is

$$x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

Fact 10.2. [1, Theorem 13.12] and [6, Theorem 17.12] Let x, y, z be in $\mathbb{R}^3$. Then the following hold:

(i) The cross product defined in Definition 10.1 is a bilinear function, that is, for every $\alpha, \beta \in \mathbb{R}$,

$$(\alpha x + \beta y) \times z = \alpha(x \times z) + \beta(y \times z) \quad \text{and} \quad x \times (\alpha y + \beta z) = \alpha(x \times y) + \beta(x \times z).$$

(ii) $x \times y \in (\operatorname{span}\{x, y\})^{\perp}$, that is

$$(\forall \alpha \in \mathbb{R}) \quad (\forall \beta \in \mathbb{R}) \quad \langle x \times y, \alpha x + \beta y \rangle = 0.$$

(iii) We have

$$(x \times y) \times z = \langle x, z\rangle y - \langle y, z\rangle x \quad \text{and} \quad x \times (y \times z) = \langle x, z\rangle y - \langle x, y\rangle z.$$

(iv) **(Lagrange's identity)** $\|x \times y\|^2 = \|x\|^2\|y\|^2 - \langle x, y\rangle^2$.

Definition 10.3. [1, page 458] Let x and y be two nonzero vectors in $\mathbb{R}^n$, where $n \geq 1$. Then the *angle* θ between x and y is defined by

$$\theta = \arccos \frac{\langle x, y\rangle}{\|x\|\|y\|},$$

where $\arccos\colon [-1, 1] \to [0, \pi]$.

Remark 10.4. If x and y are two nonzero vectors in $\mathbb{R}^n$, where $n \geq 1$, then

$$\langle x, y\rangle = \|x\|\|y\| \cos\theta,$$

where θ is the angle between x and y.

Fact 10.5. [1, page 485] Let x and y be two nonzero vectors in $\mathbb{R}^3$, and let θ be the angle between x and y. Then

$$\|x \times y\| = \|x\|\|y\| \sin\theta = \text{ the area of the parallelogram determined by } x \text{ and } y.$$

Now we are ready for the expression of the circumcenter and circumradius by cross product.

Theorem 10.6. *Suppose that $\mathcal{H} = \mathbb{R}^3$, that x, y, z are affinely independent, and that $S = \{x, y, z\}$. Set $a = y - x$, and $b = z - x$ and let the angle between a and b, defined in Definition 10.3, be θ. Then*

(i) $CC(S) = x + \frac{(\|a\|^2 b - \|b\|^2 a)\times(a\times b)}{2\|a\times b\|^2}$.

(ii) [6, 1.54] $CR(S) = \frac{\|a\|\|b\|\|a-b\|}{2\|a\times b\|} = \frac{\|a-b\|}{2\sin\theta}$.

Proof. (i): Using the formula of circumcenter in Theorem 4.1, we have

$$\begin{aligned}
CC(S) &= x + \frac{1}{2}\begin{pmatrix} y-x & z-x\end{pmatrix}\begin{pmatrix}\|y-x\|^2 & \langle y-x, z-x\rangle \\ \langle z-x, y-x\rangle & \|z-x\|^2\end{pmatrix}^{-1}\begin{pmatrix}\|y-x\|^2 \\ \|z-x\|^2\end{pmatrix}\\
&= x + \frac{1}{2}\begin{pmatrix} a & b\end{pmatrix}\begin{pmatrix}\|a\|^2 & \langle a,b\rangle \\ \langle b,a\rangle & \|b\|^2\end{pmatrix}^{-1}\begin{pmatrix}\|a\|^2 \\ \|b\|^2\end{pmatrix}\\
&= x + \frac{1}{2(\|a\|^2\|b\|^2 - \langle a,b\rangle^2)}\begin{pmatrix} a & b\end{pmatrix}\begin{pmatrix}\|b\|^2 & -\langle a,b\rangle \\ -\langle b,a\rangle & \|a\|^2\end{pmatrix}\begin{pmatrix}\|a\|^2 \\ \|b\|^2\end{pmatrix}\\
&= x + \frac{1}{2(\|a\|^2\|b\|^2 - \langle a,b\rangle^2)}\begin{pmatrix} a & b\end{pmatrix}\begin{pmatrix}\|a\|^2\|b\|^2 - \|b\|^2\langle a,b\rangle \\ \|a\|^2\|b\|^2 - \|a\|^2\langle a,b\rangle\end{pmatrix}\\
&= x + \frac{(\|a\|^2\|b\|^2 - \|b\|^2\langle a,b\rangle)a + (\|a\|^2\|b\|^2 - \|a\|^2\langle a,b\rangle)b}{2(\|a\|^2\|b\|^2 - \langle a,b\rangle^2)}\\
&= x + \frac{\langle \|a\|^2 b - \|b\|^2 a, b\rangle a - \langle \|a\|^2 b - \|b\|^2 a, a\rangle b}{2(\|a\|^2\|b\|^2 - \langle a,b\rangle^2)}.
\end{aligned}$$

Using the Fact 10.2(iii) and (iv), we get

$$CC(S) = x + \frac{(\|a\|^2 b - \|b\|^2 a)\times(a\times b)}{2\|a\times b\|^2}.$$

(ii): By Definition 3.4, we have

$$CR(S) = \|CC(S) - x\| = \left\|\frac{(\|a\|^2 b - \|b\|^2 a)\times(a\times b)}{2\|a\times b\|^2}\right\|. \tag{10.1}$$

Using Fact 10.2(iv) and Fact 10.2(ii), we obtain

$$\begin{aligned}
&\Big\|(\|a\|^2 b - \|b\|^2 a)\times(a\times b)\Big\|\\
&= \left(\Big\|\|a\|^2 b - \|b\|^2 a\Big\|^2\|a\times b\|^2 - \langle \|a\|^2 b - \|b\|^2 a, a\times b\rangle^2\right)^{\frac{1}{2}}\\
&= \Big\|\|a\|^2 b - \|b\|^2 a\Big\|\|a\times b\|. \qquad (10.2)
\end{aligned}$$

In addition, by Remarrk 2.14, since $\|a\| \neq 0$, $\|b\| \neq 0$, thus

$$\Big\| \|a\|^2 b - \|b\|^2 a \Big\| = \|a\|\|b\| \left\| \frac{\|a\|}{\|b\|} b - \frac{\|b\|}{\|a\|} a \right\|. \tag{10.3}$$

Now

$$\begin{aligned} \left\| \frac{\|a\|}{\|b\|} b - \frac{\|b\|}{\|a\|} a \right\|^2 &= \left\| \frac{\|a\|}{\|b\|} b \right\|^2 - 2\left\langle \frac{\|a\|}{\|b\|} b, \frac{\|b\|}{\|a\|} a \right\rangle + \left\| \frac{\|b\|}{\|a\|} a \right\|^2 \\ &= \|a\|^2 - 2\langle b, a\rangle + \|b\|^2 \\ &= \|a - b\|^2. \end{aligned} \tag{10.4}$$

Upon combining (10.2), (10.3) and (10.4), we obtain

$$\Big\| (\|a\|^2 b - \|b\|^2 a) \times (a \times b) \Big\| = \|a\|\|b\|\|a - b\|\|a \times b\|.$$

Hence (10.1) yields

$$\begin{aligned} CR(S) &= \frac{1}{2\|a \times b\|^2} \Big\| (\|a\|^2 b - \|b\|^2 a) \times (a \times b) \Big\| \\ &= \frac{1}{2\|a \times b\|^2} \|a\|\|b\|\|a - b\|\|a \times b\| \\ &= \frac{\|a\|\|b\|\|a - b\|}{2\|a \times b\|}. \end{aligned}$$

By Fact 10.5, we know $\|a \times b\| = \|a\|\|b\| \sin\theta$. Thus, we obtain

$$CR(S) = \frac{\|a\|\|b\|\|a - b\|}{2\|a \times b\|} = \frac{\|a - b\|}{2\sin\theta}$$

and the proof is complete. □

Fact 10.7. [9, Theorem I] Suppose that $n \geq 3$, and a cross product is defined which assigns to any two vectors $v, w \in \mathbb{R}^n$ a vector $v \times w \in \mathbb{R}^n$ such that the following three properties hold:

(i) $v \times w$ is a bilinear function of v and w.
(ii) The vector $v \times w$ is perpendicular to both v and w.
(iii) $\|v \times w\|^2 = \|v\|^2\|w\|^2 - \langle v, w\rangle^2$.

Then $n = 3$ or 7.

Remark 10.8. In view of Fact 10.7 and our proof of Theorem 10.6, we cannot generalize the latter result to a general Hilbert space $\mathcal{H}$ — unless the dimension of $\mathcal{H}$ is either 3 or 7.

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Heinz H. Bauschke
Mathematics, University of British Columbia, Kelowna, B.C. V1V 1V7, Canada
E-mail address: `heinz.bauschke@ubc.ca`

Hui Ouyang
Mathematics, University of British Columbia, Kelowna, B.C. V1V 1V7, Canada
E-mail address: `hui.ouyang@ubc.ca`

Xianfu Wang
Mathematics, University of British Columbia, Kelowna, B.C. V1V 1V7, Canada
E-mail address: `shawn.wang@ubc.ca`