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SUBSMOOTH FUNCTIONS AND SETS

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Abstract. Submonotone multimappings have been introduced and studied in 1981 by J. E. Spingarn. This concept allowed D. Aussel, A. Daniilidis and L. Thibault to define subsmooth sets with the submonotonicity of the truncated Clarke normal cone. This survey revises and revisits basic and fundamental properties of subsmooth functions and sets, and shows links with semiconvex and approximate convex functions. Diverse new results are also provided.

1. Introduction

In his 1981 paper [72], as an extension of monotonicity, J. E. Spingarn defined a locally bounded multimapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ to be (strictly) submonotone at a point $x \in \mathbb{R}^n$ if

$$\liminf_{\substack{x_1 \neq x_2 \\ x_i \to x, i=1,2 \\ y_i \in M(x_i), i=1,2}} \frac{\langle y_1 - y_2, x_1 - x_2 \rangle}{\|x_1 - x_2\|} \geq 0; \tag{1.1}$$

we will just say "submonotone". Spingarn showed in [72] that the Clarke subdifferential of a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is submonotone at any point of a neighborhood of $\overline{x} \in \mathbb{R}^n$ (in the sense of (1.1)) if and only if f is lower C^1 near $\overline{x}$, that is, there exist a compact topological space T, an open neighborhood V of $\overline{x}$ and a continuous function $\Phi : V \times T \to \mathbb{R}$ such that $D_1\Phi(\cdot,\cdot)$ exists and is continuous on $V \times T$, and such that $f(x) = \max_{t \in T} \Phi(x,t)$.

Independently, H.V. Ngai, D.T. Luc and M. Théra introduced in their 2000 paper [56], as an extension of Jensen inequality for convexity, a certain class of "approximate convex functions". An extended real-valued function f on a normed space X is declared in [56] to have such a property at a point $\overline{x} \in X$ if for any $\varepsilon > 0$ there exists a neighborhood U of $\overline{x}$ such that, for all $x, y \in U$ and all $t \in]0,1[$ the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon t(1-t)\|x - y\|$$

is satisfied. When X is a Banach space, proper lower semicontinuous functions with such a property have been characterized by the submonotonicity at $\overline{x}$ of their Clarke subdifferentials, independently by A. Daniilidis, F. Jules and M. Lassonde

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[25] and by H.V. Ngai and J.-P. Penot [57]. This characterization was first detected and showed by A. Daniilidis and P. Georgiev [24] for locally Lipschitz functions on normed spaces.

We will see in Section 3 below that a result of L. Vesely and L. Zajíček [81] says that a continuous real-valued function $f : X \to \mathbb{R}$ on a normed space X is strictly Fréchet differentiable at $\overline{x} \in X$ provided that for each $\varepsilon > 0$ there is a neighborhood U of $\overline{x}$ such that for all $u, v \in U$ with $u \neq v$ and all $z \in]u, v[:= \{tu+(1-t)v : t \in]0, 1[\}$

$$-\varepsilon \leq \frac{f(u) - f(z)}{\|u - z\|} + \frac{f(v) - f(z)}{\|v - z\|} \leq \varepsilon. \tag{1.2}$$

The purpose of this survey paper is multiple. First, we will show how the above classes are covered by nonsmooth extended real-valued functions $f : X \to \mathbb{R}\cup\{+\infty\}$ which satisfy (for a point $\overline{x}$ where f is finite) the unilateral left-side inequality in (1.2). Then we will revise and revisit basic and fundamental properties of functions in those classes. We will also continue the survey with the analysis of subsmooth sets of D. Aussel, A. Daniilidis and L. Thibault [8] and metrically subsmooth sets. For functions as well as for sets, diverse new results will be established.

2. Notation and preliminaries

Let $(X, \|\cdot\|)$ be a normed space and X^* be its topological dual. For any $x \in X$ and $r > 0$ we denote by $B(x, r)$ and $B[x, r]$ the open and closed balls centered at x with radius r. The closed unit ball centered at zero will be denoted by $\mathbb{B}$ (or $\mathbb{B}_X$ if there is a risk of confusion), that is $\mathbb{B} := B[0, 1]$, and the unit sphere of X will be denoted by $\mathbb{S}$ (or $\mathbb{S}_X$). Similarly, $\mathbb{U}$ (or $\mathbb{U}_X$) will stand for the open unit ball of X, that is, $\mathbb{U} := B(0, 1)$. We will write $\operatorname{int} S$ (or $\operatorname{int}_X S$) and $\operatorname{cl} S$ ($\overline{S}$ or $\operatorname{cl}_X S$) for the topological interior and closure of a subset S of X. The set S will be said to be *closed near* a point $\overline{x} \in S$ if there is an open neighborhood U of $\overline{x}$ such that $S \cap U$ is closed in U relative to the induced topology. We recall that the distance function from S is given by $d_S(x) = d(x, S) := \inf_{u \in S} \|u - x\|$. We will also use the indicator function ψ_S of S which assigns to any $x \in X$ the extended real

$$\psi_S(x) = 0 \text{ if } x \in S \quad \text{and} \quad \psi_S(x) = +\infty \text{ if } x \in X \setminus S.$$

For a subset Q of X^*, the notation $\overline{\text{co}}^* Q$ will stand for its weak*-closed convex hull. The support function $\sigma(Q, \cdot)$ of Q (resp. $\sigma(S, \cdot)$ of S) is given by (see, e.g., [17])

$$\sigma(Q, x) := \sup_{y^* \in Q} \langle y^*, x\rangle \ \forall x \in X \ (\text{resp. } \sigma(S, x^*) := \sup_{y \in S} \langle x^*, y\rangle \ \forall x^* \in X^*).$$

For $Q' \subset X^*$ (resp. $S' \subset X$) one has $\sigma(Q, \cdot) \leq \sigma(Q', \cdot)$ if and only if $\overline{\text{co}}^* (Q) \subset \overline{\text{co}}^* (Q')$ (resp. $\sigma(S, \cdot) \leq \sigma(S', \cdot)$ if and only if $\overline{\text{co}} (S) \subset \overline{\text{co}} (S')$). We also recall that for a multimapping $M : T \rightrightarrows T'$ between two nonempty sets and for an extended real-valued function $\varphi : T \to \mathbb{R} \cup \{-\infty, +\infty\}$, the effective domains $\operatorname{Dom} M$ and $\operatorname{dom} \varphi$ of M and φ respectively and the graph $\operatorname{gph} M$ of M are defined by

$$\operatorname{Dom} M := \{t \in T : M(t) \neq \emptyset\} \quad \text{and} \quad \operatorname{dom} \varphi := \{t \in T : \varphi(t) < +\infty\},$$
$$\operatorname{gph} M := \{(t, t') \in T \times T' : t' \in M(t)\}.$$

The function φ is said to be *proper* if it does not take the value $-\infty$ and $\operatorname{dom} \varphi \neq \emptyset$. If T is a topological space, we will write $\tau \to_\varphi t$ to mean $(\tau, \varphi(\tau)) \to (t, \varphi(t))$.

Suppose now that T and T' are Hausdorff topological spaces. One says that the multimapping M is *outer semicontinuous or closed* at a point $\bar{t} \in T$ if for every net $(t_j)_j$ in T converging to $\bar{t}$ and every net $(t'_j)_j$ converging to some t' in T' with $t'_j \in M(t_j)$ for all j, one has $t' \in M(\bar{t})$; when the latter property holds with sequences in place of nets, M is said to be *sequentially closed* at $\bar{t}$. One also says that the multimapping M is *upper semicontinuous* at $\bar{t} \in T$ if for any open set W in T' with $W \supset M(\bar{t})$ there exists a neighborhood V of $\bar{t}$ such that $M(t) \subset W$ for all $t \in V$. It is not difficult to see that the upper semicontinuity at $\bar{t}$ implies the outer semicontinuity at $\bar{t}$ whenever $M(\bar{t})$ is closed and the topology of T' is metrizable. The limit inferior $\operatorname*{Lim\,inf}_{t \to \bar{t}} M(t)$ is defined as the set of $t' \in T'$ for which given any neighborhood W of t' there exists a neighborhood V of $\bar{t}$ such that $M(t) \cap W \neq \emptyset$ for every $t \in V$. If T and T' are metric spaces, it is known that $t' \in \operatorname*{Lim\,inf}_{t \to \bar{t}} M(t)$ if and only if for any sequence $(t_n)_n$ in T converging to $\bar{t}$ there exists a sequence $(t'_n)_n$ in T' converging to t' such that $t'_n \in M(t_n)$ for n large enough.

Let U be a nonempty open set of the normed space X and $f : U \to \mathbb{R} \cup \{-\infty, +\infty\}$ be an extended real-valued function. A convenient way to recall the Clarke subdifferential of the function f is to define it through the concept of corresponding normal functionals. Let us thus recall that the *Clarke tangent cone* or *C-tangent cone* $T^C(S; x)$ of the set S at a point $x \in S$ is the set of vectors $v \in X$ such that for every sequence $(t_n)_n$ in $]0, +\infty[$ tending to 0 and every sequence $(x_n)_n$ in S converging to x there exists a sequence $(v_n)_n$ in X convergeng to v such that

$$x_n + t_n v_n \in S \quad \text{for all } n \in \mathbb{N}.$$

This cone $T^C(S; x)$ is closed and convex, and its (negative) polar in X^* is the *Clarke normal cone* or *C-normal cone* $N^C(S; x)$ of S, that is,

$$N^C(S; x) := \big(T^C(S; x)\big)^o := \{x^* \in X^* : \langle x^*, v\rangle \leq 0, \forall v \in T^C(S; x)\}.$$

By convention one puts $T^C(S; x) = \emptyset$ and $N^C(S; x) = \emptyset$ for $x \in X \backslash S$. Given another set Q of a normed space Y, from the definition of C-tangent cone we clearly have for $P := S \times Q$

(2.1)

$$T^C(P; (x, y)) = T^C(S; x) \times T^C(Q; y) \text{ and } N^C(P; (x, y)) = N^C(S; x) \times N^C(Q; y).$$

Considering the subset in $X \times \mathbb{R}$

$$\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} : x \in U, f(x) \leq r\},$$

called the epigraph of f, one defines the *Clarke subdifferential* or *C-subdifferential* $\partial_C f(x)$ of f at a point $x \in U$ by

$$\partial_C f(x) := \{x^* \in X^* : (x^*, -1) \in N^C\big(\operatorname{epi} f; (x, f(x))\big)\}. \tag{2.2}$$

Clearly, $\partial_C f(x)$ is a weak* closed convex subset of X^* which is empty whenever $|f(x)| = +\infty$. If x is a local minimizer of f with $|f(x)| < +\infty$, it is known that $0 \in \partial_C f(x)$. If U is convex and f is convex on U, then $\partial_C f(x)$ coincides with the subdifferential in the sense of convex analysis, that is, with $|f(x)| < +\infty$

$$\partial_C f(x) = \partial f(x) := \{x^* \in X^* : \langle x^*, u - x\rangle + f(x) \leq f(u), \forall u \in U\}.$$

When f is finite on a neighborhood of x and Lipschitz therein (with a constant $\gamma \geq 0$), one has

$$\partial_C f(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq f^o(x;h), \forall h \in X\},$$

where $f^o(x;\cdot)$ is the *Clarke directional derivative* of f defined by

$$f^o(x;h) := \limsup_{u \to x, t \downarrow 0} t^{-1}[f(u+th) - f(u)] \quad \text{for all } h \in X,$$

which gives in particular at this point x around which f is Lipschitz

$$\partial_C(-f)(x) = -\partial_C f(x).$$

Under this Lipschitz assumption, the function $f^o(x;\cdot)$ is sublinear and (globally) Lipschitz on X with γ as a Lipschitz constant, so the two latter equalities give that $\partial_C f(x)$ is a nonempty weak* compact convex set in X^* and

$$\partial_C f(x) \subset \gamma \mathbb{B}_{X^*}.$$

Concerning the second component of C-normal of epigraph one has

$$(x^*, r) \in N^C(\operatorname{epi} f; (x, f(x))) \implies r \leq 0,$$

and if f is Lipschitz near x

$$(x^*, 0) \in N^C(\operatorname{epi} f; (x, f(x))) \iff x^* = 0.$$

In terms of the distance function d_S from the set S (which is Lipschitz on X with constant 1), one has for $x \in S$

$$N^C(S;x) = \mathrm{cl}_{w^*}\left(\mathbb{R}_+ \partial_C d_S(x)\right) \quad \text{and} \quad T^C(S;x) = \{v \in X : d_S^o(x;v) = 0\}, \tag{2.3}$$

where $\mathbb{R}_+ := [0, +\infty[$ and cl_{w^*} stands for the closure operation with respect to the weak* topology in X^*. The class of tangentially regular sets will be considered in many places in the paper. Let us first recall that the *Bouligand tangent cone* $T^B(S;x)$ of S at $x \in S$ is the set of vectors $v \in X$ for which there exist a sequence $(t_n)_n$ in $]0, +\infty[$ tending to 0 and a sequence $(v_n)_n$ in X converging to v such that

$$x + t_n v_n \in S \quad \text{for all } n \in \mathbb{N}.$$

Clearly, the inclusion $T^C(S;x) \subset T^B(S;x)$ always holds true. Then one says that the set S is *tangentially regular* at $x \in S$ whenever the tangent cones $T^C(S;x)$ and $T^B(S;x)$ coincide. When the epigraph $\operatorname{epi} f$ is tangentially regular at $(x, f(x)) \in$ *epif*, one says that the function f is *tangentially regular* at x. It is worth pointing out for $|f(x)| < +\infty$ that $T^C(\operatorname{epi} f; (x, f(x)))$ (resp. $T^B(\operatorname{epi} f; (x, f(x)))$) in $X \times \mathbb{R}$ is the epigraph of some sublinear (resp. positively homogeneous) function from X into $\mathbb{R} \cup \{-\infty, +\infty\}$. The function $f^B(x;\cdot)$ whose $T^B(\operatorname{epi} f; (x, f(x)))$ is the epigraph is given by

$$f^B(x;h) := \liminf_{t \downarrow 0, v \to h} t^{-1}[f(x+tv) - f(x)] \quad \text{for all } h \in X.$$

If the function f is finite near $\overline{x}$ and Lipschitz continuous therein, $f^o(\overline{x};\cdot)$ is the sublinear function whose epigraph is $T^C(\operatorname{epi} f; (\overline{x}, f(\overline{x})))$, so under this Lipschitz assumption property f is tangentially regular at $\overline{x}$ if and only if $f^o(\overline{x};\cdot) = f^B(\overline{x};\cdot)$.

It is also known under this Lipschitz assumption near $\overline{x}$ that f is tangentially regular at $\overline{x}$ if and only if the usual directional derivative $f'(\overline{x};\cdot)$ exists and

$$f^o(\overline{x};h) = f'(\overline{x};h) \quad \text{for all } h \in X,$$

where $f'(\overline{x};h) = \lim_{t\downarrow 0} t^{-1}[f(\overline{x}+th) - f(\overline{x})]$.

In addition to the aforementioned inclusion $T^C(S;x) \subset T^B(S;x)$, it is worth pointing out that for X finite-dimensional and S closed near $\overline{x} \in S$ one has

$$T^C(S;\overline{x}) = \operatorname*{Lim\,inf}_{S\ni x\to\overline{x}} T^B(S;x). \tag{2.4}$$

We recall now the basic subdifferential sum rule theorem as well as some other results for the Clarke subdifferential in the following proposition.

Proposition 2.1. *Let S be any subset of the normed space X, let $f : U \to \mathbb{R} \cup \{-\infty,+\infty\}$ be any function on an open set U of X, and let $g : U \to \mathbb{R}$ be any locally Lipschitz function.*

(a) *For any $x \in S$ and any $v \in X$ one has*

$$d_S^o(x;h) = \limsup_{S\ni u\to x, t\downarrow 0} t^{-1} d_S(u+th).$$

(b) **(Lebourg mean value theorem).** *For any distinct points $x, y \in U$ with $[x,y] \subset U$ there exist $c \in]x,y[:= \{(1-t)x + ty : t \in]0,1[\}$ and $c^* \in \partial_C g(c)$ such that*

$$g(y) - g(x) = \langle c^*, y-x\rangle.$$

(c) *The multimapping $\partial_C g$ is $\|\cdot\|$-to-weak* upper semicontinuous on U.*

(d) **(Rockafellar sum rule theorem for C-subdifferential).** *The following subdifferential sum rule holds true*

$$\partial_C(f+g)(x) \subset \partial_C f(x) + \partial_C g(x) \quad \textit{for all } x \in U.$$

(e) *If $g(\cdot) = \max_{k\in K} g_i(\cdot)$, where $K := \{1,\ldots,m\}$ and $g_k : U \to \mathbb{R}$ are locally Lipschitz functions, then for each $x \in U$*

$$\partial_C g(x) \subset \operatorname{co}\left(\bigcup_{k\in K(x)} \partial_C g_k(x)\right),$$

where $K(x) := \{k \in K : g_k(x) = g(x)\}$.

(f) *If $g(\cdot) = \max_{t\in T} G(\cdot,t)$, where T is a compact topological space and $G : U\times T \to \mathbb{R}$ is upper semicontinuous in its second variable and differentiable in its first variable with $D_1G(\cdot,\cdot)$ continuous on $U \times T$, then for each $x \in U$*

$$\partial_C g(x) = \overline{\operatorname{co}}^*\left(\{D_1G(x,t) : t \in T(x)\}\right),$$

where $T(x) := \{t \in T : G(x,t) = g(x)\}$.

(g) **(Clarke theorem of gradient representation of C-subdifferential).** *If X is finite-dimensional, then*

$$\partial_C g(x) = \operatorname{co}\big(\{\lim_{n\to\infty} \nabla g(x_n) : Q \cap \operatorname{Dom}\nabla g \ni x_n \to x\}\big),$$

for any $x \in U$ and any subset $Q \subset U$ whose Lebesgue measure of $U \setminus Q$ is null.

We refer for example to [18] for the above concepts and results.

The assertion (b) (due to F. Clarke) in the following lemma can be found in [18] while the assertion (a) can be easily verified.

Lemma 2.2. *Assume that* $\overline{x} \in S$.

(a) *Given any* $\delta > 0$ *one has*

$$\mathrm{dist}(x, S \cap B(\overline{x}, 2\delta)) = \mathrm{dist}(x, S) \quad \textit{for all}\, x \in B[\overline{x}, \delta].$$

(b) *Let* $g : X \to \mathbb{R}$ *be a function which is Lipschitz on* X *with constant* $\gamma \geq 0$ *and such that* $\overline{x}$ *is a minimizer of* g *on* S. *Then* $\overline{x}$ *is a minimizer of* $g + \gamma\, d_S$ *on the whole space* X.

In order to state and demonstrate the next result on the C-subdifferential of the distance function, let us recall the following Valadier's theorem in [80].

Theorem 2.3 (Valadier theorem [80]**).** *Let* $(f_t)_{t\in T}$ *be a family of convex functions from* X *into* $\mathbb{R} \cup \{+\infty\}$ *and let* $f(x) := \sup_{t\in T} f_t(x)$ *for all* $x \in X$. *Let* $\overline{x}$ *be a point at which the convex function* f *is finite and continuous and let* $T_\eta(\overline{x}) := \{t \in T : f_t(\overline{x}) \geq f(\overline{x}) - \eta\}$. *Then the equality*

$$\partial f(\overline{x}) = \bigcap_{\eta>0} \overline{\mathrm{co}}^* \left(\bigcup_{t\in T_\eta(\overline{x}), x\in B(\overline{x},\eta)} \partial f_t(x) \right)$$

holds true.

For any real $\varepsilon > 0$, we define the set of ε-nearest points of x in S as

$$\mathrm{Proj}_{S,\varepsilon} x := \{u \in S : \|x - u\| \leq d_S(x) + \varepsilon\}. \tag{2.5}$$

Clearly, $\mathrm{Proj}_{S,\varepsilon} x \neq \emptyset$ for every $\varepsilon > 0$. It is also of great interest to associate with the set S the so-called Asplund function $\varphi_S : X \to \mathbb{R}$ defined by

$$\varphi_S(x) = \sup_{y\in S} (\langle x, y\rangle - \frac{1}{2}\|y\|^2) \quad \text{for all } x \in X.$$

The next proposition provides, in the Hilbert setting, expressions in terms of ε-nearest points for the subdifferential of the convex function φ_S and the Clarke subdifferential of the distance function d_S. The proposition is due to H. Berens [10] and was established therein for its use for properties of Chebyshev sets; its interest in the study of Chebyshev sets was also highlighted by J.-B. Hiriart-Urruty [34]. The idea and the development below of the use of the above Valadier theorem in the proof of the assertion (a) are due to Hiriart-Urruty who applied this theorem in [35, Proposition 3.5 (ii)] for the subdifferential of a similar function related to the farthest distance function. The arguments by Berens [1] [10, Proposition in page 5] are completely different and based on some monotonicity properties of the multimapping given the right-hand side of (a) below.

Proposition 2.4 (Berens). *Assume that* X *is a Hilbert space. The following hold.*

[1]We received a copy of [10] from J.-B. Hiriart-Urruty.

(a) *For any $x \in X$ one has*

$$\partial\varphi_S(x) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}(\mathrm{Proj}_{S,\varepsilon} x).$$

(b) *For any $x \in X \setminus \mathrm{cl}_X S$ one also has*

$$\partial_C d_S(x) = \frac{1}{d_S(x)}\left(x - \bigcap_{\varepsilon>0} \overline{\mathrm{co}}(\mathrm{Proj}_{S,\varepsilon} x)\right) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\left(\frac{x - \mathrm{Proj}_{S,\varepsilon} x}{d_S(x)}\right).$$

Proof. Noting that

$$\|x-y\|^2 = \|x\|^2 - (2\langle x, y\rangle - \|y\|^2),$$

we can write

$$\frac{1}{2}d_S^2(x) = \frac{1}{2}\|x\|^2 - \sup_{y\in S}(\langle x, y\rangle - \frac{1}{2}\|y\|^2).$$

Considering the above function φ_S defined by

$$\varphi_S(u) := \sup_{y\in S}(\langle u, y\rangle - \frac{1}{2}\|y\|^2),$$

it results that

$$\frac{1}{2}d_S^2(x) = \frac{1}{2}\|x\|^2 - \varphi_S(x). \tag{2.6}$$

The function φ_S is obviously convex and the latter equality ensures that it is also finite-valued and locally Lipschitz. Since $\|\cdot\|^2/2$ is C^1 with the identity on X as gradient, the same equality (2.6) gives

$$d_S(x)\partial_C d_S(x) = x + \partial_C(-\varphi_S)(x) = x - \partial_C\varphi_S(x) = x - \partial(\varphi_S)(x). \tag{2.7}$$

Given $\varepsilon > 0$, putting $\eta(\varepsilon, x) := \varepsilon^2 + 2\varepsilon d_S(x)$ we observe that $u \in X$ satisfies $\varphi_S(x) \le \langle x, u\rangle - (1/2)\|u\|^2 + \eta(\varepsilon, x)/2$ if and only if $\|x-u\|^2 \le d_S^2(x) + \eta(\varepsilon, x)$, that is, $\|x-u\| \le d_S(x) + \varepsilon$, which means $u \in \mathrm{Proj}_{S,\varepsilon} x$. Since $\eta(\varepsilon, x) \downarrow 0$ as $\varepsilon \downarrow 0$, applying the above Valadier theorem to the family $(f_s)_{s\in S}$ with $f_s : X \to \mathbb{R}$ defined by $f_s(u) := \langle u, s\rangle - \frac{1}{2}\|s\|^2$ for all $s \in S$, we obtain

$$\partial(\varphi_S)(x) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\{u \in X : u \in \mathrm{Proj}_{S,\varepsilon} x\} = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}(\mathrm{Proj}_{S,\varepsilon} x).$$

So, for $x \notin \mathrm{cl}_X S$ we obtain from (2.7)

$$\partial_C d_S(x) = \frac{1}{d_S(x)}\left(x - \bigcap_{\varepsilon>0} \overline{\mathrm{co}}(\mathrm{Proj}_{S,\varepsilon} x)\right).$$

This finishes the proof. □

We will also use in the development of the paper the concepts of Fréchet normal and Mordukhovich limiting normal. If f is finite at a point $x \in U$, one says that $x^* \in X^*$ is a *Fréchet subgradient* of f at x provided that for any $\varepsilon > 0$ there exits a neighborhood $U_0 \subset U$ of x such that

$$\langle x^*, u - x\rangle \le f(u) - f(x) + \varepsilon\|u - x\| \quad \text{for all } u \in U_0.$$

The set $\partial_F f(x)$ of all Fréchet subgradients of f at x is the *Fréchet subdifferential* or *F-subdifferential* of f at x. If $|f(x)| = +\infty$, as usual we will put $\partial_F f(x) = \emptyset$. If x is a local minimizer of f with $|f(x)| < +\infty$, it is clear that $0 \in \partial_F f(x)$.

The set $\partial_F \psi_S(x)$ is the *Fréchet normal cone* or *F-normal cone* of S at x, and it is usually denoted by $N^F(S;x)$. Then, $N^F(S;x) = \emptyset$ if $x \notin S$, and for $x \in S$, we have $x^* \in N^F(S;x)$ if and only if for any $\varepsilon > 0$ there exists a neighborhood U_0 of x such that

$$\langle x^*, u - x\rangle \le \varepsilon \|u - x\| \quad \text{for all } u \in U_0 \cap S.$$

One always has

$$N^F(S;x) \subset \big(T^B(S;x)\big)^o \subset N^C(S;x) \quad \text{and} \quad \partial_F f(x) \subset \partial_C f(x).$$

It is also known (see [49, 13]) that

$$\partial_F d_S(x) = N^F(S;x) \cap \mathbb{B}_{X^*} \text{ and } N^F(S;x) = \mathbb{R}_+ \partial_F d_S(x) \text{ for all } x \in S, \tag{2.8}$$

and

$$\partial_F d_S(x) = N^F(E_r(S);x) \cap \mathbb{S}_{X^*} \text{ if } x \notin \operatorname{cl} S, \tag{2.9}$$

where $E_r(S)$ denotes the closed r-enlargement of S for $r := d_S(x)$, defined by

$$E_r(S) := \{u \in X : d_S(u) \le r\}.$$

When the normed space X is finite-dimensional one has the equality $N^F(S;x) = \big(T^B(S;x)\big)^o$, where as above

$$\big(T^B(S;x)\big)^o := \{x^* \in X^* : \langle x^*, v\rangle \le 0, \ \forall v \in T^B(S;x)\}$$

is the polar of the cone $T^B(S;x)$.

The *Mordukhovich limiting subdifferential* or *L-subdifferential* $\partial_L f(x)$ of f at a point $x \in U$ with $|f(x)| < +\infty$ can be stated by saying that a functional $x^* \in X^*$ belongs to $\partial_L f(x)$ provided that there exist sequences $(x_n)_n$ in U with $x_n \to_f x$ and $(x_n^*)_n$ in X^* converging weak* to x^* such that $x_n^* \in \partial_F f(x_n)$ for all $n \in \mathbb{N}$. Similarly, for $x \in S$ the *Mordukhovich normal cone* or *L-normal cone* $N^L(S;x)$ is the set of $x^* \in X^*$ for which there are sequences $(x_n)_n$ in S converging to x and $(x_n^*)_n$ in X^* converging weak* to x^* such that $x_n^* \in N^F(S;x_n)$. One puts $\partial_L f(x) = \emptyset$ if $|f(x)| = +\infty$ and $N^L(S;x) = \emptyset$ if $x \in X \setminus S$. One has

$$N^L(S;x) = \partial_L \psi_S(x) \quad \text{and} \quad \partial_L f(x) = \{x^* \in X^* : (x^*, -1) \in N^L\big(\operatorname{epi} f; (x, f(x))\big)\}.$$

From the very definition we see that $\partial_F f(x) \subset \partial_L f(x)$ and $N^F(S;x) \subset N^L(S;x)$.

Most of fundamental properties of the concepts of Fréchet and limiting normals/subgradients require the space X to be an Asplund space. We recall that X is an *Asplund space* if it is a Banach space whose dual space of any separable closed vector subspace is separable. We collect some properties in the following proposition; for (a), (b), (c) , (d), (g) we refer, e.g,. to [52], and for (e), (f), we refer to [8].

Proposition 2.5. *Assume that X is an Asplund space, the function $f : U \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous on the open set U and the set S is closed. Let $g : U \to \mathbb{R}$ be a locally Lipschitz function. The following hold:*

(a) *The set* $\operatorname{Dom}\partial_F f$ *is graphically dense in* $\operatorname{dom} f$ *in the sense that for any* $x \in \operatorname{dom} f$ *there exists a sequence* $(x_n)_n$ *in* $\operatorname{Dom}\partial f$ *such that* $x_n \to_f x$.

(b) *For any* $x \in \operatorname{dom} f$, *any* $x^* \in \partial_F(f+g)(x)$ *and any real* $\varepsilon > 0$, *there exist* $u, v \in U$ *with* $\|u - x\| + |f(u) - f(x)| \le \varepsilon$ *and* $\|v - x\| \le \varepsilon$ *such that*

$$x^* \in \partial_F f(u) + \partial_F g(v) + \varepsilon \mathbb{B}_{X^*}.$$

(c) *For any* $x \in U$ *one has the inclusion*

$$\partial_L(f+g)(x) \subset \partial_L f(x) + \partial_L g(x);$$

if $G : V \to X$ *is a* C^1 *mapping from an open set* V *of an Asplund space* Y *with* $G(V) \subset U$, *one also has for any* $y \in V$

$$\partial_L(g \circ G)(y) \subset DG(y)^*\big(\partial_L g(G(y))\big).$$

(d) *For any* $x \in S$ *one has*

$$N^L(S; x) = \mathbb{R}_+ \partial_L d_S(x).$$

(e) *Let* $u \in X$ *and let* $u^* \in \partial_F d_S(u)$. *Then for every* $\varepsilon > 0$, *there exist* $x \in S$ *and* $x^* \in \partial_F d_S(x)$ *such that*

$$\|x - u\| \le \varepsilon + d_S(u) \quad \text{and} \quad \|x^* - u^*\| \le \varepsilon.$$

(f) *For* $x \in S$ *one has* $x^* \in \partial_L d_S(x)$ *if and only if there are sequences* $(x_n)_n$ *in* S *converging to* x *and* $(x_n^*)_n$ *in* X^* *converging weakly* to* x^* *such that* $x_n^* \in \partial_F d_S(x_n)$ *for all* $n \in \mathbb{N}$.

(g) *For any* $x \in U$ *one has the inclusion* $\partial_L f(x) \subset \partial_C f(x)$ *and the equalities*

$$\partial_C f(x) = \overline{\operatorname{co}}^*\big(\partial_L f(x) + \partial_L^\infty f(x)\big), \ \ \partial_C g(x) = \overline{\operatorname{co}}^*(\partial_L g(x)), \ \ \partial_L^\infty g(x) = \{0\},$$

where $\partial_L^\infty f(x) = \{x^* \in X^* : (x^*, 0) \in N^L\big(\operatorname{epi} f; (x, f(x))\big)\}$.

In addition to the assertion (e) in the above proposition, the next lemma establishes another property of Fréchet subgradients of the distance function d_S at points outside S.

Lemma 2.6. *Let* S *be a nonempty subset of a normed space* X *and* $x^* \in \partial_F d_S(\overline{x})$ *with* $\overline{x} \in X$.

(a) *For any sequence* $(y_n)_n$ *in* S *with* $\|\overline{x} - y_n\| \to d_S(\overline{x})$ *as* $n \to \infty$, *one has*

$$\langle x^*, \overline{x} - y_n \rangle \to d_S(\overline{x}) \quad \text{as } n \to \infty.$$

(b) *In particular, for any* $\overline{y} \in S$ *with* $\|\overline{x} - \overline{y}\| = d_S(\overline{x})$ *(if any) one has*

$$\langle x^*, \overline{x} - \overline{y} \rangle = d_S(\overline{x}).$$

Proof. It is enough to show (a) since (b) follows from (a). Let any real $\varepsilon \in]0,1[$. Choose a real $\delta > 0$ such that

$$\langle x^*, x - \overline{x} \rangle \le d_S(x) - d_S(\overline{x}) + \varepsilon\|x - \overline{x}\| \quad \text{for all } x \in B[\overline{x}, \delta].$$

Setting $\eta_n := \|\overline{x} - y_n\| - d_S(\overline{x})$, we deduce that for every $n \in \mathbb{N}$ and every $x \in B[\overline{x}, \delta]$

$$\langle x^*, x - \overline{x} \rangle \le \|x - y_n\| - \|\overline{x} - y_n\| + \varepsilon\|x - \overline{x}\| + \eta_n. \tag{2.10}$$

On the other hand, noting that the sequence $(y_n)_n$ is bounded, there is a real $r > 0$ such that $y_n \in B[\overline{x}, r]$. Fix $t \in]0,1[$ such that $t(r + 2\|\overline{x}\|) < \delta$, so $x_n := \overline{x} + t(y_n - \overline{x})$

belongs to $B[\overline{x}, \delta]$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, taking x_n in place of x in (2.10), it ensues that

$$t\langle x^*, y_n - \overline{x}\rangle \le \|\overline{x} - y_n + t(y_n - \overline{x})\| - \|\overline{x} - y_n\| + t\varepsilon\|y_n - \overline{x}\| + \eta_n \tag{2.11}$$

$$= (1-t)\|\overline{x} - y_n\| - \|\overline{x} - y_n\| + t\varepsilon\|\overline{x} - y_n\| + \eta_n, \tag{2.12}$$

which means that

$$\langle x^*, \overline{x} - y_n\rangle \ge (1-\varepsilon)\|\overline{x} - y_n\| - t^{-1}\eta_n.$$

This combined with the inequality $\|x^*\| \le 1$ (since d_S is Lipschitz with 1 as Lipschitz constant) yields for any $n \in \mathbb{N}$

$$(1-\varepsilon)\|\overline{x} - y_n\| - t^{-1}\eta_n \le \langle x^*, \overline{x} - y_n\rangle \le \|\overline{x} - y_n\|.$$

Since $\eta_n \to 0$, it follows with $\rho_n := \langle x^*, \overline{x} - y_n\rangle$ that

$$(1-\varepsilon)d_S(\overline{x}) \le \liminf_{n\to\infty} \rho_n \le \limsup_{n\to\infty} \rho_n \le d_S(\overline{x}).$$

This being true for every $\varepsilon \in]0,1[$ we conclude that $\lim_{n\to\infty} \rho_n = d_S(\overline{x})$ as desired. □

Assume now that X is a Hilbert space whose norm $\|\cdot\|$ is associated with the inner product $\langle\cdot,\cdot\rangle$. In addition to the set of ε-nearest points defined in (2.5), for each $y \in X$ denote by $\mathrm{Proj}_S(y)$ the set of nearest points of y in S, that is,

$$\mathrm{Proj}_S(y) := \{u \in S : \|y - u\| = d_S(y)\}.$$

Proximal normals (playing a crucial role in variational analysis) are defined in the Hilbert space X through Proj_S as follows. A vector $v \in X$ is called a *proximal normal* of S at $x \in S$ provided there exists some real $t \ge 0$ and some $y \in X$ such that $x \in \mathrm{Proj}_S(y)$ and $v = t(y - x)$. The set $N^P(S;x)$ of all such vectors v is called the *proximal normal cone* of S at $x \in S$, and by convention one sets $N^P(S;x) = \emptyset$ if $x \notin S$. The *proximal subdifferential* $\partial_P f(x)$ of the function f at a point $x \in U$ is then defined as

$$\partial_P f(x) = \{\zeta \in X : (\zeta, -1) \in N^P(\mathrm{epi}\, f; (x, f(x)))\},$$

where $X \times \mathbb{R}$ is endowed with the canonical Hilbert product structure. As analytical description, it is known for f finite at $x \in U$ that $\zeta \in \partial_P f(x)$ if and only if there exist a real $\sigma \ge 0$ and a neighborhood $U_0 \subset U$ of x such that

$$\langle \zeta, u - x\rangle \le f(u) - f(x) + \sigma\|u - x\|^2 \quad \text{for all } u \in U_0.$$

One also knows that $N^P(S;x) = \partial_P \psi_S(x)$, so by the analytic description of $\partial_P f$

$$\partial_P f(x) \subset \partial_F f(x) \quad \text{and} \quad N^P(S;x) \subset N^F(S;x).$$

Like Fréchet normals, one has (see [19, 13])

$$\partial_P d_S(x) = N^P(S;x) \cap \mathbb{B}_X \text{ and } N^P(S;x) = \mathbb{R}_+ \partial_P d_S(x) \text{ for all } x \in S. \tag{2.13}$$

We state four results of proximal analysis for which we refer, e.g., to [19].

Proposition 2.7. *Assume that X is a Hilbert space, the function $f : U \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous on the open set U and the set S is closed. Let $g : U \to \mathbb{R}$ be a locally Lipschitz function. The following hold.*

(a) *For any* $x \in \operatorname{dom} f$*, any* $x^* \in \partial_P(f+g)(x)$ *and any real* $\varepsilon > 0$*, there exist* $u, v \in U$ *with* $\|u - x\| + |f(u) - f(x)| \le \varepsilon$ *and* $\|v - x\| \le \varepsilon$ *such that*

$$x^* \in \partial_P f(u) + \partial_P g(v) + \varepsilon \mathbb{B}_{X^*}.$$

(b) *For any* $x \in U$ *a vector* $\zeta \in \partial_L f(x)$ *if and only if there exist sequences* $(x_n)_n$ *in* U *with* $x_n \to_f x$ *and* $(\zeta_n)_n$ *in* X *converging weakly to* ζ *such that* $\zeta_n \in \partial_P f(x_n)$ *for all* $n \in \mathbb{N}$*.*

(c) *For any* $x \in S$ *a vector* $\zeta \in N^L(S; x)$ *if and only if there exist sequences* $(x_n)_n$ *in* S *converging to* x *and* $(\zeta_n)_n$ *in* X *converging weakly to* ζ *such that* $\zeta_n \in N^P(S; x_n)$ *for all* $n \in \mathbb{N}$*.*

(d) *For any* $x \in S$ *a vector* $\zeta \in \partial_L d_S(x)$ *if and only if there exist sequences* $(x_n)_n$ *in* S *converging to* x *and* $(\zeta_n)_n$ *in* X *converging weakly to* ζ *such that* $\zeta_n \in \partial_P d_S(x_n)$ *for all* $n \in \mathbb{N}$*.*

3. Definition and first properties of subsmooth functions

Recall that a mapping $G : U \to Y$ from an open set U of a normed space X into a normed space Y is *strictly Fréchet differentiable* at a point $\overline{x} \in U$ provided that there is a continuous linear mapping $\Lambda : X \to Y$ such that for any $\varepsilon > 0$ there exists a neighborhood $V \subset U$ of $\overline{x}$ such that

$$\|G(x) - G(u) - \Lambda(x - u)\| \le \varepsilon \|x - u\| \quad \text{for all } x, u \in V.$$

Given a continuous convex function $f : U \to \mathbb{R}$ on an open convex set U of the normed space X, it is known (see, e.g., [12, Proposition 4.2.7]) that f is strictly Fréchet differentiable at $\overline{x} \in U$ if and only if

$$\lim_{t \downarrow 0} \frac{f(\overline{x} + th) + f(\overline{x} - th) - 2f(\overline{x})}{t} = 0$$

uniformly with respect to $h \in \mathbb{B}_X$ (or equivalently with respect to $h \in \mathbb{S}_X$). In the case of a general mapping, a characterization in the same line holds true as proved in the following result of L. Vesely and L. Zajíček [81].

Proposition 3.1 (Vesely-Zajíček). *Let* U *be a nonempty open set of a normed space* X *and* $G : U \to Y$ *be a mapping from* U *into a Banach space* Y *which is continuous at* $\overline{x} \in U$*. The following assertions are equivalent:*

(a) *The mapping* G *is stricly Fréchet differentiable at the point* $\overline{x}$*.*

(b) *For every* $\varepsilon > 0$ *there is* $\delta > 0$ *with* $B(\overline{x}, \delta) \subset U$ *such that for all* $h \in \mathbb{B}_X$*,* $x \in U$*,* $r, s > 0$ *with* $x + rh \in B(\overline{x}, \delta)$*,* $x - sh \in B(\overline{x}, \delta)$ *one has*

$$\left\| \frac{G(x + rh) - G(x)}{r} - \frac{G(x) - G(x - sh)}{s} \right\| \le \varepsilon.$$

(c) *For every* $\varepsilon > 0$ *there is* $\delta > 0$ *with* $B(\overline{x}, \delta) \subset U$ *such that for all* $h \in \mathbb{S}_X$*,* $x \in U$*,* $r, s > 0$ *with* $x + rh \in B(\overline{x}, \delta)$*,* $x - sh \in B(\overline{x}, \delta)$ *one has*

$$\left\| \frac{G(x + rh) - G(x)}{r} - \frac{G(x) - G(x - sh)}{s} \right\| \le \varepsilon.$$

(d) *For every $\varepsilon > 0$ there is $\delta > 0$ with $B(\overline{x}, \delta) \subset U$ such that for all $u, v \in B(\overline{x}, \delta)$ with $u \neq v$ and all $z \in]u, v[:= \{tu + (1-t)v : t \in]0, 1[\}$ one has*

$$\left\| \frac{G(u) - G(z)}{\|u - z\|} - \frac{G(z) - G(v)}{\|z - v\|} \right\| \leq \varepsilon.$$

We show first the following lemma.

Lemma 3.2. *Let U be a nonempty open set of a normed space X and $G : U \to Y$ be a mapping from U into a Banach space Y. Assume that the property (c) in the above proposition is satisfied. Then*

$$\lim_{t \downarrow 0} t^{-1}\big(G(\overline{x} + th) - G(\overline{x})\big)$$

exists uniformly with respect to $h \in \mathbb{S}_X$.

Proof. Without loss of generality, we may suppose $U = X$. Fix any $\varepsilon > 0$ and take $\delta > 0$ given by the property in (c) of the proposition. Fix any $h \in \mathbb{S}_X$ and consider any $0 < \tau < t < \delta$. Then, putting $Q_\tau(h) := \tau^{-1}\big(G(\overline{x} + \tau h) - G(\overline{x})\big)$ we have

$$\left\| \frac{G(\overline{x} + th) - G(\overline{x} + \tau h)}{t - \tau} - \frac{G(\overline{x} + \tau h) - G(\overline{x})}{\tau} \right\| \leq \varepsilon,$$

or equivalently

$$\|G(\overline{x} + th) - G(\overline{x} + \tau h) - (t - \tau)Q_\tau(h)\| \leq \varepsilon(t - \tau).$$

Observing that $G(\overline{x}+th) - G(\overline{x}+\tau h) - (t-\tau)Q_\tau(h) = G(\overline{x}+th) - G(\overline{x}) - tQ_\tau(h)$, we derive that

$$\|G(\overline{x} + th) - G(\overline{x}) - tQ_\tau(h)\| \leq \varepsilon(t - \tau) \leq \varepsilon t, \text{ hence } \|Q_t(h) - Q_\tau(h)\| \leq \varepsilon.$$

The latter clearly implies the assertion of the lemma by completeness of Y. □

Proof of Proposition 3.1. Suppose again (without loss of generality) that $U = X$. The assertions (b), (c) and (d) are easily seen to be pairwise equivalent, and (a) clearly implies (c).

Suppose that (c) is satisfied. By Lemma 3.2 above, for each $h \in X$ put

$$\Lambda(h) := \lim_{t \downarrow 0} t^{-1}\big(G(\overline{x} + th) - G(\overline{x})\big).$$

Clearly, the equality $\Lambda(rh) = r\Lambda(h)$ holds for all reals $r \geq 0$. Fix any $h_1, h_2 \in X$ with $h_1 \neq h_2$ and fix also any real $\varepsilon > 0$. For $\varepsilon' := 2\varepsilon/\|h_1 - h_2\| > 0$ there exists by (c) some $\delta > 0$ such that for any $t \in]0, \delta[$

$$\left\| \frac{G(\overline{x} + 2th_1) - G(\overline{x} + t(h_1 + h_2))}{t\|h_1 - h_2\|} - \frac{G(\overline{x} + t(h_1 + h_2)) - G(\overline{x} + 2th_2)}{t\|h_1 - h_2\|} \right\| \leq \varepsilon',$$

or equivalently

$$\left\| \frac{G(\overline{x} + 2th_1) - G(\overline{x} + t(h_1 + h_2))}{t} - \frac{G(\overline{x} + t(h_1 + h_2)) - G(\overline{x} + 2th_2)}{t} \right\| \leq 2\varepsilon.$$

The latter amounts to saying, with notation in the proof of the above lemma, that $\|Q_{2t}(h_1)+Q_{2t}(h_2)-Q_t(h_1+h_2)\| \le \varepsilon$. By the above lemma choose some $\delta' \in]0,\delta[$ such that for all $t \in]0,\delta'[$

$$\|Q_t(h_1+h_2)-\Lambda(h_1+h_2)\| \le \varepsilon,\ \|Q_{2t}(h_i)-\Lambda(h_i)\| \le \varepsilon,\ \ i=1,2.$$

Therefore, for all $t \in]0,\delta'[$ we obtain $\|\Lambda(h_1)+\Lambda(h_2)-\Lambda(h_1+h_2)\| \le 3\varepsilon$, which yields that $\Lambda(h_1+h_2)=\Lambda(h_1)+\Lambda(h_2)$. The latter equality combined with the positive homogeneity of Λ easily entails that Λ is linear.

On the other hand, the uniform convergence on $\mathbb{S}_X$ of the family $(Q_t)_{t>0}$ to Λ (as $t \downarrow 0$) is equivalent to the existence of a function $\eta : \mathbb{R} \to [0,+\infty[$ with $t^{-1}\eta(t) \to 0$ (as $t \downarrow 0$) such that $\|G(\overline{x}+h)-G(\overline{x})-\Lambda(h)\| \le \eta(\|h\|)$ for all $h \in X$. This combined with the continuity of G at $\overline{x}$ implies the continuity of Λ, and hence G is Fréchet differentiable at $\overline{x}$

Finally, let us show the strict Fréchet differentiability. Fix any $\varepsilon > 0$ and choose some $\delta > 0$ satisfying (c) and such that (by the Fréchet differentiabilty)

$$\|G(\overline{x}+h)-G(\overline{x})-\Lambda(h)\| \le \varepsilon\|h\| \quad \text{for all } h \in B(0,\delta).$$

Fix any $x, y \in B(\overline{x},\delta/4)$ with $x \ne y$ and put $u := (x-y)/\|x-y\|$, $r := \|x-y\|$, $s := \delta/4$. It ensues that $\|(y-su)-\overline{x}\| < \delta$, hence

$$\|G(y-su)-G(\overline{x})-\Lambda(y-su-\overline{x})\| \le \varepsilon\|y-su-\overline{x}\|,$$

$$\|G(y)-G(\overline{x})-\Lambda(y-\overline{x})\| \le \varepsilon\|y-\overline{x}\|.$$

Both inequalities yield

$$\|G(y)-G(y-su)-\Lambda(su)\| \le \varepsilon(\|y-su-\overline{x}\|+\|y-\overline{x}\|) \le \varepsilon(2\|y-\overline{x}\|+s) \le 3\varepsilon s,$$

or equivalently

$$\|s^{-1}\big(G(y)-G(y-su)\big)-\Lambda(u)\| \le 3\varepsilon.$$

On the other hand, by the choice of δ from (c) we also have

$$\left\|\frac{G(x)-G(y)}{\|x-y\|}-\frac{G(y)-G(y-su)}{s}\right\| \le \varepsilon.$$

It then follows that

$$\left\|\frac{G(x)-G(y)}{\|x-y\|}-\Lambda(u)\right\| \le 4\varepsilon, \text{ or equivalently } \|G(x)-G(y)-\Lambda(x-y)\| \le 4\varepsilon\|x-y\|,$$

which translates the strict Fréchet differentiability of G at $\overline{x}$ and finishes the proof of the proposition. □

In the case of a real-valued function f the inequality (c) in the above proposition characterizing the strict Fréchet differentiability at $\overline{x}$ can be rewritten as

$$-\varepsilon \le \frac{f(x+rh)-f(x)}{r}+\frac{f(x-sh)-f(x)}{s} \le \varepsilon.$$

As we will see along this survey, functions satisfying the left-side inequality alone enjoy remarkable and useful properties. This yields to the following definition.

Definition 3.3. Let U be a nonempty open set of a normed space X. An extended real-valued function $f : U \to \mathbb{R} \cup \{+\infty\}$ is *subsmooth at a point* $\overline{x} \in \operatorname{dom} f$ provided that for every real $\varepsilon > 0$ there is a real $\delta > 0$ (depending on ε and $\overline{x}$) with $B(\overline{x}, \delta) \subset U$ such that, for all $h \in \mathbb{S}_X$, $r, s > 0$ with $x + rh \in B(\overline{x}, \delta) \cap \operatorname{dom} f$, $x - sh \in B(\overline{x}, \delta) \cap \operatorname{dom} f$ one has

$$-\varepsilon \leq \frac{f(x+rh) - f(x)}{r} + \frac{f(x-sh) - f(x)}{s}.$$

When f is subsmooth at any point in $U_0 \cap \operatorname{dom} f$ for an open set $U_0 \subset U$, one says that it is *subsmooth on* U_0. The function f is *subsmooth near a point* if it is subsmooth on an open neighborhood $U_0 \subset U$ of this point.

It is clear from this definition that any extended real-valued convex function is subsmooth at any point where it is finite. It is also worth pointing out that the above definition of subsmoothness of f at $\overline{x}$ is equivalent to requiring that, for every real $\varepsilon > 0$ there exists a real $\delta > 0$ with $B(\overline{x}, \delta) \subset U$ such that, for all $u, v \in B(\overline{x}, \delta) \cap \operatorname{dom} f$ with $u \neq v$ and all $z \in]u, v[$ one has

$$-\varepsilon \leq \frac{f(u) - f(z)}{\|u - z\|} + \frac{f(v) - f(z)}{\|v - z\|}. \tag{3.1}$$

This obviously entails the following property.

Proposition 3.4. *Let U be a nonempty open set of a normed space X and $f : U \to \mathbb{R} \cup \{+\infty\}$ be a proper function. If f is subsmooth at a point $\overline{x} \in \operatorname{dom} f$, then there exists some $\delta > 0$ such that $B(\overline{x}, \delta) \cap \operatorname{dom} f$ is a convex set.*

The characterization (3.1) of subsmoothness also leads to introduce the corresponding uniform and one-sided notions.

Definition 3.5. Let U be a nonempty subset of a normed space X and $f : U \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. The function f is said to be *uniformly subsmooth* on a nonempty open set $U_0 \subset U$ if for every real $\varepsilon > 0$ there exists a real $\delta > 0$ such that, for all $u \neq v$ in $U_0 \cap \operatorname{dom} f$ with $\|u - v\| < \delta$ and all $z \in]u, v[\cap U$ one has

$$-\varepsilon \leq \frac{f(u) - f(z)}{\|u - z\|} + \frac{f(v) - f(z)}{\|v - z\|}.$$

The function f is *uniformly subsmooth near a point* in U if it is uniformly subsmooth on an open neighborhood of this point.

Similarly, one says that f is *one-sided subsmooth* at $\overline{x} \in \operatorname{dom} f$ if for every real $\varepsilon > 0$ there exists a real $\delta > 0$ with $B(\overline{x}, \delta) \subset U$ such that, for every $v \in B(\overline{x}, \delta) \cap \operatorname{dom} f$ with $v \neq \overline{x}$ and for every $z \in]\overline{x}, v[$ one has

$$-\varepsilon \leq \frac{f(\overline{x}) - f(z)}{\|\overline{x} - z\|} + \frac{f(v) - f(z)}{\|v - z\|}.$$

The uniform equi-subsmoothness is defined in a similar way.

Definition 3.6. Given a family $(f_i)_{i \in I}$ of functions from the open set U into $\mathbb{R} \cup \{+\infty\}$ and a family $(U_i)_{i \in I}$ of open subsets of U, one says that this family of functions is *uniformly equi-subsmooth relative to* $(U_i)_{i\in I}$ when for every real $\varepsilon > 0$ there exists a real $\delta > 0$ such that for each $i \in I$ one has

$$-\varepsilon \leq \frac{f_i(u) - f_i(z)}{\|u - z\|} + \frac{f_i(v) - f_i(z)}{\|v - z\|}$$

for all $u \neq v$ in $U_i \cap \operatorname{dom} f_i$ with $\|u - v\| < \delta$ and all $z \in]u, v[\cap U$. If all the sets U_i coincide with a same open set $U_0 \subset U$, one simply says that the family of functions $(f_i)_{i\in I}$ is *uniformly equi-subsmooth on* U_0.

Of course, the uniform subsmoothness implies subsmoothness, which in turn implies one-sided subsmoothness. It is aslo clear that the three properties of subsmoothness, one-sided subsmoothness and uniform subsmoothness are stable under sum of finitely many functions.

Example 3.7. Consider the function $f := -|\cdot|$ on $\mathbb{R}$. For any $v \neq 0$ in $\mathbb{R}$ and any z strictly between 0 and v, we have

$$\frac{f(0) - f(z)}{|z|} + \frac{f(v) - f(z)}{|v - z|} = 1 + \frac{-|v| + |z|}{|v - z|} \geq 0,$$

so f is one-sided subsmooth at 0 (and hence one-sided subsmooth at any point in $\mathbb{R}$).

However, observing that

$$\frac{f(1/n) - f(0)}{1/n} + \frac{f(-1/n) - f(0)}{1/n} = -2,$$

we see that f is not subsmooth at 0.

The following strict differentiabily result follows directly from Definition 3.3 and Proposition 3.1.

Proposition 3.8. *Let U be a nonempty open set of a normed space X and $f : U \to \mathbb{R}$ be a real-valued function which is continuous at $\overline{x} \in U$. Then f is strictly Fréchet differentiable at $\overline{x}$ if and only if both functions f and $-f$ are subsmooth at the point $\overline{x}$.*

The next proposition extends to subsmooth functions the well-known property that a convex function which is Fréchet differentiable at point is (see, e.g., [12, Proposition 4.2.7]) strictly Fréchet differentiable at that point.

Proposition 3.9. *Let U be a nonempty open set of a normed space X and $f : U \to \mathbb{R}$ be a real-valued function which is subsmooth at $\overline{x}$. Then f is strictly Fréchet differentiable at $\overline{x}$ if and only if it is Fréchet differentiable at $\overline{x}$.*

Proof. Only the implication $\Leftarrow$ needs to be justified. Assume that f is Fréchet differentiable at $\overline{x}$. Without loss of generality me may suppose that $U = X$ along with $f(\overline{x}) = 0$ and $Df(\overline{x}) = 0$. Choose a real $\delta > 0$ such that the inequality in Definition 3.3 is satisfied for $\varepsilon' := \varepsilon/2$ in place of ε and such that for all $x \in B(\overline{x}, \delta)$ one has

$$|f(x)| = |f(x) - f(\overline{x}) - Df(\overline{x})(x - \overline{x})| \leq (\varepsilon/8)\|x - \overline{x}\|.$$

Take any $x, y \in B(\overline{x}, \delta/2)$ with $x \neq y$ and set $h := (x-y)/\|x-y\|$. Putting $r := \delta/2$ and noting that $y = x - \|y-x\|\, h$, by Definition 3.3 we have

(3.2)
$$-\varepsilon/2 \leq \frac{f(x+rh)-f(x)}{r} + \frac{f(y)-f(x)}{\|y-x\|}, \; -\varepsilon/2 \leq \frac{f(x)-f(y)}{\|x-y\|} + \frac{f(y-rh)-f(y)}{r}.$$

Further, the above inequality given by the Fréchet differentiability of f at $\overline{x}$ entails that

$$\max\{|f(x)|, |f(y)|, |f(x+rh)|, |f(y-rh)|\} \leq (\varepsilon/8)2r = \varepsilon r/4,$$

which in turn ensures that

$$|r^{-1}\big(f(x+rh)-f(x)\big)| \leq \varepsilon/2 \quad \text{and} \quad |r^{-1}\big(f(y-rh)-f(y)\big)| \leq \varepsilon/2.$$

The latter inequalities combined with the inequalities in (3.2) yield

$$\frac{f(x)-f(y)}{\|x-y\|} \leq \varepsilon \quad \text{and} \quad -\varepsilon \leq \frac{f(x)-f(y)}{\|x-y\|},$$

which translates the strict Fréchet differentiability of f at $\overline{x}$ (with $Df(\overline{x}) = 0$). $\square$

Proposition 3.8 tells us in particular that any $\mathcal{C}^1$ function $f : U \to \mathbb{R}$ on an open set U is subsmooth on U. A similar result provides a first example of famillies of uniformly equi-submooth functions. Given a nonempty open set U of a normed space X, a family of mappings $(G_i)_{i\in I}$ from U into a normed space Y is said to be *uniformly equi-continuous relative to a family* $(U_i)_{i\in I}$ *of open subsets* of U when for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $i \in I$ one has

$$\|G_i(x') - G_i(x)\| \leq \varepsilon \quad \text{for all } x, x' \in U_i \text{ with } \|x'-x\| < \delta. \tag{3.3}$$

Proposition 3.10. *Let U be an open set of a normed space X and $(f_i)_{i\in I}$ be a family of functions from U into $\mathbb{R}$. Let $(U_i)_{i\in I}$ be a family of open convex subsets of U such that for each $i \in I$ the function f_i is differentiable on U_i and such that the family of derivatives $(Df_i)_{i\in I}$ is uniformly equi-continuous relative to $(U_i)_{i\in I}$. Then the family of functions $(f_i)_{i\in I}$ is uniformly equi-subsmooth relative to $(U_i)_{i\in I}$.*

Proof. Take any real $\varepsilon > 0$ and choose a real $\delta > 0$ such that $\|Df_i(x') - Df_i(x)\| < \varepsilon$ for any $i \in I$ and any $x, x' \in U_i$ with $\|x'-x\| < \delta$. Fix any $i \in I$ and take any $u, v \in U_i$ with $\|u-v\| < \delta$. Consider $z \in]u, v[$ and note that with $\nu := (v-u)/\|v-u\|$, $u_t := z + t(u-z)$ and $v_t := z + t(v-z)$

$$\begin{aligned} &\frac{f_i(u)-f_i(z)}{\|u-z\|} + \frac{f_i(v)-f_i(z)}{\|v-z\|} \\ &= \int_0^1 \langle Df_i(u_t), \frac{u-z}{\|u-z\|}\rangle\, dt + \int_0^1 \langle Df_i(v_t), \frac{v-z}{\|v-z\|}\rangle\, dt \\ &= \int_0^1 \langle Df_i(v_t) - Df_i(u_t), \nu\rangle\, dt \geq -\varepsilon, \end{aligned}$$

where the latter inequality is due to the fact that for every $t \in [0,1]$ one has $u_t, v_t \in U_i$ with $\|v_t - u_t\| = t\|v-u\| < \delta$. This justifies the desired uniform equi-subsmoothness property. $\square$

Subsmooth functions can be characterized via a Jensen-like inequality.

Proposition 3.11. *Let $f : U \to \mathbb{R} \cup \{+\infty\}$ be a function on a nonempty open set U of a normed space X. The function f is subsmooth (resp. one-sided subsmooth) at $\overline{x} \in \operatorname{dom} f$ if and only if for every real $\varepsilon > 0$ there exists a real $\delta > 0$ with $B(\overline{x}, \delta) \subset U$ such that, for all $x, y \in B(\overline{x}, \delta)$ and all $t \in]0, 1[$ the inequality*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \varepsilon t(1-t)\|x - y\| \tag{3.4}$$

holds (resp. the inequality holds with $y = \overline{x}$ and all $x \in B(\overline{x}, \delta)$).

Similarly, f is uniformly subsmooth on an open set $U_0 \subset U$ if and only if for every real $\varepsilon > 0$ there exists a real $\delta > 0$ such that for any $x, y \in U_0$ with $\|x - y\| < \delta$ and any $t \in]0, 1[$ with $tx + (1-t)y \in U$ the above inequality is satisfied.

Proof. We only justify the equivalence for the subsmoothness property, the case of either one-sided or uniform smoothness is similar. Let $u, v \in \operatorname{dom} f$ with $u \neq v$ and $z \in]u, v[\cap U$ with $z = tu + (1-t)v$ and $t \in]0, 1[$. Since $u - z = (1-t)(u - v)$ and $v - z = t(v - u)$, we note that

$$\frac{f(u) - f(z)}{\|u - z\|} + \frac{f(v) - f(z)}{\|v - z\|} = \frac{tf(u) + (1-t)f(v) - f(z)}{t(1-t)\|u - v\|}.$$

From this and (3.1) the implication $\Leftarrow$ follows. The reverse implication being obtained in an analogous way, the equivalence is justified. □

Remark 3.12. Functions satisfying the inequality (3.4) are called *approximately convex at* $\overline{x}$ by H.V. Ngai, D.T. Luc and M. Thera [56], so the above proposition says that this approximate convexity notion coincides with the submoothness property.

Remark 3.13. Let be given an open set U of a normed space X, a family $(f_i)_{i \in I}$ of functions from U into $\mathbb{R} \cup \{+\infty\}$ and a family of open subsets $(U_i)_{i \in I}$ of U. The above arguments also show that this family of functions is *uniformly equi-subsmooth relative to* $(U_i)_{i \in I}$ if and only if for every real $\varepsilon > 0$ there exists a real $\delta > 0$ such that for each $i \in I$ one has

$$f_i(tx + (1-t)y) \le tf_i(x) + (1-t)f_i(y) + \varepsilon t(1-t)\|x - y\|$$

for any $x, y \in U_i$ with $\|x - y\| < \delta$ and any $t \in]0, 1[$ with $tx + (1-t)y \in U$. In particular the family $(f_i)_{i \in I}$ is uniformly equi-subsmooth relative to $(U_i)_{i \in I}$ whenever any set U_i is convex and any function f_i is convex on U_i.

The next proposition proves that any function which is subsmooth at a point $\overline{x}$ and bounded from above near $\overline{x}$ is Lipschitz near $\overline{x}$.

Proposition 3.14. *Let $f : U \to \mathbb{R}$ be a real-valued function on an open set U of a normed space X which is subsmooth at a point $\overline{x} \in U$ and bounded from above near $\overline{x}$. Then f is Lipschitz continuous near $\overline{x}$.*

Proof. Take $\varepsilon = 1$ and by Proposition 3.11 take a real $\delta_0 > 0$ with $B(\overline{x}, \delta_0) \subset U$ such that f is bounded above on $B(\overline{x}, \delta_0)$ and such that (3.4) is satisfied for all $x, y \in B(\overline{x}, \delta_0)$. Taking any $x \in B(\overline{x}, \delta_0)$ and setting $u := 2\overline{x} - x$, we see that $\overline{x} = (1/2)x + (1/2)u$ with $u \in B(\overline{x}, \delta_0)$, thus

$$f(\overline{x}) \le \frac{1}{2}f(x) + \frac{1}{2}f(u) + \frac{1}{4}\|u - x\|.$$

Since f is bounded above on the ball $B(\overline{x}, \delta_0)$, it follows that f is also bounded from below on this ball. We can then choose an upper bound $\mu > 0$ of $|f|$ on the ball $B(\overline{x}, \delta_0)$.

Now put $\delta := \delta_0/2$ and fix any $x, y \in B(\overline{x}, \delta)$ with $x \neq y$. Putting $t := \frac{\|y-x\|}{\delta+\|y-x\|}$ and $z := y + \delta \frac{y-x}{\|y-x\|}$, and noting that $z \in B(\overline{x}, \delta_0)$, it ensues that

$$f(y) = f(tz + (1-t)x) \leq tf(z) + (1-t)f(x) + t(1-t)\|z - x\|,$$

hence (since $t(1-t) \leq t \leq \|y - x\|/\delta$ and $\|z - x\| \leq 3\delta$)

$$f(y) - f(x) \leq t\big(f(z) - f(x)\big) + 3\|y - x\| \leq (3 + \frac{2\mu}{\delta})\|y - x\|,$$

which translates the Lipschitz property of f on $B(\overline{x}, \delta)$. □

Corollary 3.15. *Let U be a nonempty open set of a Banach space X and $f : U \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. If f is subsmooth at a point $\overline{x} \in \operatorname{int}(\operatorname{dom} f)$, then f is Lipschitz continuous near $\overline{x}$.*

Proof. Without loss of generality (putting $g(x) := f(x+\overline{x}) - f(\overline{x})$) we may suppose that $\overline{x} = 0$ and $f(\overline{x}) = 0$. Let $\delta_0 > 0$ be such that $B(0, \delta_0) \subset \operatorname{dom} f$ and such that the condition (3.4) holds with $\varepsilon := 1$. Let $\delta := \delta_0/2$ and for each integer n put $V_n := \{x \in W : f(x) \leq n\}$ with $W := B(0, \delta_0)$. Noting that $W = \bigcup_{n\in\mathbb{N}} V_n$ (and keeping in mind that $B(0, \delta_0)$ is open in the complete space X), Baire theorem tells us that $\operatorname{int} V_k \neq \emptyset$ for some $k \in \mathbb{N}$. Choose $a \in W$ and $r \in]0, \delta_0[$ such that $B(a, 2r) \subset V_k$. We have $-a \in W$ and for each $x \in B(0, r)$ there is some $y_x \in B(a, 2r)$ such that $x = (1/2)(-a) + (1/2)y_x$, hence

$$f(x) \leq \frac{1}{2}f(-a) + \frac{1}{2}f(y_x) + \frac{1}{4}\|y_x + a\| \leq \frac{1}{2}\big(f(-a) + k + 2\delta_0\big).$$

The function f is then bounded from above near the point $\overline{x} = 0$, so it is Lipschitz continuous near this point according to Proposition 3.14. □

Consider now the case of subsmooth functions over intervals of the real line.

Proposition 3.16. *Let I be an open interval of $\mathbb{R}$ and $f : I \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous subsmooth function. Then the restriction of f to $[r, s]$ is continuous for any interval $[r, s] \subset \operatorname{dom} f$ with $r < s$.*

Proof. Let $[r, s] \subset I$ with $r < s$. We already know by Corollary 3.15 above that f is (locally Lipschitz) continuous on $]r, s[$. Let us prove, for example, that f is continuous on the right at r. Taking $\varepsilon = 1$, choose $\delta > 0$ with $B(r, \delta) \subset I$ such that

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y) + \theta(1-\theta)|x - y|$$

for all $x, y \in B(r, \delta)$, and $\theta \in]0, 1[$. Putting $\sigma := \min\{s, r + \delta\}$ we obtain for all $t \in]r, \sigma[$

$$f(t) \leq \frac{t-r}{\sigma-r}f(\sigma) + \frac{\sigma-t}{\sigma-r}f(r) + \frac{(t-r)(\sigma-t)}{(\sigma-r)^2}|\sigma - r|,$$

and hence $\limsup_{t\downarrow r} f(t) \leq f(r)$. This and the lower semicontinuity of f guarantee that f is continuous on the right at r as desired. □

4. Directional derivatives and subdifferentials of subsmooth functions

This section analyzes directional derivatives and subdifferentials of subsmooth functions.

4.1. **General properties of derivatives and subdifferentials.** Let us begin with some properties of the differential quotient. Let U be a nonempty open set of a normed space X and $f : U \to \mathbb{R} \cup \{+\infty\}$ be a function which is subsmooth at a point $\overline{x} \in \operatorname{dom} f$. Fix any real $\varepsilon > 0$. Let $\delta > 0$ with $B(\overline{x}, \delta) \subset U$ for which condition (3.4) is fulfilled. Fix any $x \in B(\overline{x}, \delta)$ and any $h \in X$. Let $0 < s < t$ with $t\|h\| < \delta - \|x - \overline{x}\|$ and let any $r > 0$ with $r\|h\| < \delta - \|x - \overline{x}\|$. Observing that

$$x = (r + s)^{-1}s(x - rh) + (r + s)^{-1}r(x + sh),$$

we have

$$f(x) \leq \frac{s}{r+s} f(x - rh) + \frac{r}{r+s} f(x + sh) + \frac{\varepsilon rs}{r+s}\|h\|,$$

which is equivalent to the following first *slope ε-inequality*:

$$-r^{-1}[f(x - rh) - f(x)] \leq s^{-1}[f(x + sh) - f(x)] + \varepsilon\|h\| \tag{4.1}$$

for $r, s > 0$ with $\|h\| \max\{r, s\} < \delta - \|x - \overline{x}\|$.

Similarly, from the equality

$$x + sh = \frac{s}{t}(x + th) + (1 - \frac{s}{t})x$$

we obtain

$$f(x + sh) \leq \frac{s}{t} f(x + th) + (1 - \frac{s}{t}) f(x) + \varepsilon s(1 - \frac{s}{t})\|h\|,$$

which in turn is equivalent the following second *slope ε-inequality*:

$$s^{-1}[f(x + sh) - f(x)] \leq t^{-1}[f(x + th) - f(x)] + \varepsilon(1 - \frac{s}{t})\|h\| \tag{4.2}$$

for reals $0 < s < t$ with $t\|h\| < \delta - \|x - \overline{x}\|$.

Proposition 4.1. *Let U be a nonempty open set of a normed space X and $f : U \to \mathbb{R} \cup \{+\infty\}$ be a function which is subsmooth at a point $\overline{x} \in U$ at which f is finite. The following hold:*

(a) *For each real $\varepsilon > 0$ there is $\delta > 0$ with $B(\overline{x}, 2\delta) \subset U$ such that for any $x \in B(\overline{x}, \delta) \cap \operatorname{dom} f$, any $t > 0$ and any $h \in X$ with $t\|h\| < \delta$*

$$f^B(x; h) \leq \limsup_{s \downarrow 0} s^{-1}[f(x + sh) - f(x)] \leq t^{-1}[f(x + th) - f(x)] + \varepsilon\|h\|.$$

(b) *The directional derivative*

$$f'(\overline{x}; h) := \lim_{t \downarrow 0} t^{-1}[f(\overline{x} + th) - f(\overline{x})]$$

exists in $\mathbb{R} \cup \{-\infty, +\infty\}$ for any direction $h \in X$, and the function $f'(\overline{x}; \cdot)$ is convex and positively homogeneous.

Proof. Let any $\varepsilon > 0$ and let $\delta_0 > 0$ with $B(\overline{x}, \delta_0) \subset U$ such that the condition (3.4) is fulfilled. Set $\delta := \delta_0/2$ and fix any $x \in B(\overline{x}, \delta) \cap \operatorname{dom} f$ and any $h \in X$. Take $0 < s < t$ with $t\|h\| < \delta$ and write according to (4.2) that

$$s^{-1}[f(x + sh) - f(x)] \leq t^{-1}[f(x + th) - f(x)] + \varepsilon(1 - \frac{s}{t})\|h\|.$$

Fixing t we deduce as $s \downarrow 0$ that

$$f^B(x; h) \leq \limsup_{s\downarrow 0} s^{-1}[f(x + sh) - f(x)] \leq t^{-1}[f(x + th) - f(x)] + \varepsilon\|h\|, \tag{4.3}$$

which justifies (a). On the other hand, keeping $h \in X$ and $t > 0$ with $t\|h\| < \delta$ in the second inequality in (4.3) and choosing $x = \overline{x}$, we obtain by passing to the limit inferior as $t \to 0$ that

$$\limsup_{s\downarrow 0} s^{-1}[f(\overline{x} + sh) - f(\overline{x})] \leq \liminf_{t\downarrow 0} t^{-1}[f(\overline{x} + th) - f(\overline{x})] + \varepsilon\|h\|.$$

This being true for all $\varepsilon > 0$, the desired limit giving $f'(\overline{x}; h)$ exists in $\mathbb{R}\cup\{-\infty, +\infty\}$.

The positive homogeneity being obvious, it remains to show the convexity of $f'(\overline{x}; \cdot)$. Fix any (h, α) and (h', β) in $X\times\mathbb{R}$ and satisfying $f'(\overline{x}; h) < \alpha$ and $f'(\overline{x}; h') < \beta$. Take any $\varepsilon > 0$ and choose $\delta > 0$ such that the condition (3.4) holds and such that for all $0 < t < \delta$

$$(2t)^{-1}[f(\overline{x} + 2th) - f(\overline{x})] < \alpha \quad \text{and} \quad (2t)^{-1}[f(\overline{x} + 2th') - f(\overline{x})] < \beta.$$

Take any $t > 0$ with $t \max\{\|h\|, \|h'\|\} < \delta/2$. It ensues that

$$f(\overline{x} + th + th') \leq \frac{1}{2}f(\overline{x} + 2th) + \frac{1}{2}f(\overline{x} + 2th') + \frac{\varepsilon t}{2}\|h - h'\|,$$

or otherwise written

$$\begin{aligned} t^{-1}[f(\overline{x} + th + th') - f(\overline{x})] \leq{}& (2t)^{-1}[f(\overline{x} + 2th) - f(\overline{x})] \\ &+ (2t)^{-1}[f(\overline{x} + 2th') - f(\overline{x})] + (\varepsilon/2)\|h - h'\|, \end{aligned}$$

which entails

$$t^{-1}[f(\overline{x} + th + th') - f(\overline{x})] < \alpha + \beta + (\varepsilon/2)\|h - h'\|.$$

Consequently, $f'(\overline{x}; h + h') \leq \alpha + \beta$, so $f'(\overline{x}; \cdot)$ is convex. □

Before stating the result concerning the subdifferential, we need a lemma.

Lemma 4.2. *Let U be a nonempty open set of a normed space X and $f : U \to \mathbb{R} \cup \{+\infty\}$ be a function.*

(a) *If f is subsmooth at a point $\overline{x} \in U$ where it is finite, then for any real $\varepsilon > 0$ there exists a real $\delta > 0$ with $B(\overline{x}, \delta) \subset U$ such that, for each $x \in B(\overline{x}, \delta) \cap \operatorname{dom} f$ and for each $(u, r) \in \operatorname{epi} f$ with $\|u - x\| < \delta$, one has*

$$\big(u - x, r - f(x) + \varepsilon\|u - x\|\big) \in T^C\big(\operatorname{epi} f; (x, f(x))\big). \tag{4.4}$$

(b) *If f is uniformly subsmooth on an open subset $U_0 \subset U$, then for any real $\varepsilon > 0$ there exists a real $\delta > 0$ such that the inclusion (4.4) holds for any $x, u \in U_0 \cap \operatorname{dom} f$ with $\|u - x\| < \delta$ and any real $r \geq f(u)$.*

Proof. Fix any real $\varepsilon > 0$. Under the assumption in (a) (resp. in (b)) choose a real $\delta_0 > 0$ satisfying $B(\overline{x}, \delta_0) \subset U$ as well the condition (3.4) in Proposition 3.11 and put $\delta := \delta_0/2$ (resp. choose a real $\delta > 0$ satisfying the condition for uniform subsmoothness in Proposition 3.11 similar to (3.4)). Take any $x \in B(\overline{x}, \delta)$ with $f(x)$ finite and take any $(u, r) \in \operatorname{epi} f$ with $\|u - x\| < \delta$ (resp. take any $x, u \in U_0 \cap \operatorname{dom} f$ and any real $r \geq f(u)$). If $u = x$, the result is obvious since $T^C(\operatorname{epi} f; (x, f(x))$ is an epigraph set containing $(0, 0)$. Suppose that $\|u - x\| > 0$. Take any sequence $(x_n, r_n)_n$ in $\operatorname{epi} f$ converging to $(x, f(x))$ and any sequence $(t_n)_n$ in $]0, +\infty[$ tending to 0. Fix an integer N such that $t_n < 1$ and $\|x_n - x\| < \delta$ for all $n \geq N$ (resp. $t_n < 1$, $\|x_n - u\| < \delta$ and $x_n + t_n(u - x_n) \in U$ for all $n \geq N$). Putting $z_n := x_n + t_n(u - x_n)$, the condition (3.4) in Proposition 3.11 (resp. the condition for uniform subsmoothness in Proposition 3.11) tells us that, for all $n \geq N$

$$f(z_n) - \varepsilon t_n \|u - x_n\| \leq t_n r + (1 - t_n) r_n,$$

and from this we get that

$$\big(x_n, r_n\big) + t_n\big(u - x_n, r - r_n + \varepsilon\|u - x_n\|\big) \in \operatorname{epi} f.$$

This implies the desired inclusion

$$\big(u - x, r - f(x) + \varepsilon\|u - x\|\big) \in T^C\big(\operatorname{epi} f; (x, f(x))\big)$$

and finishes the proof. □

Remark 4.3. The above proof also shows for $S \subset X$ that if $f : X \to \mathbb{R} \cup \{+\infty\}$ satisfies with $\varepsilon \geq 0$ the inequality

$$f(tx + (1 - t)y) \leq t f(x) + (1 - t) f(y) + \varepsilon t \|x - y\|$$

for all $x, y \in S$ and $t \in]0, 1[$, then

$$\big(u - x, r - f(x) + \varepsilon\|u - x\|\big) \in T^C\big((S \times \mathbb{R}) \cap \operatorname{epi} f; (x, f(x))\big),$$

for every $x \in S \cap \operatorname{dom} f$ and every $(u, r) \in \operatorname{epi} f$ with $u \in S$.

Remark 4.4. Let $(f_i)_{i \in I}$ be a family of functions from an open set U of a normed space X into $\mathbb{R} \cup \{+\infty\}$ and let $(U_i)_{i \in I}$ be a family of open subsets of U. Assume that this family of functions is uniformly equi-subsmooth relative to $(U_i)_{i \in I}$. Using Remark 3.13 in place of Proposition 3.11 in the proof of Lemma 4.2 it is not difficult to see that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each $i \in I$ the inclusion

$$\big(u - x, r - f_i(x) + \varepsilon\|u - x\|\big) \in T^C\big(\operatorname{epi} f_i; (x, f_i(x))\big)$$

holds for any $x, u \in U_i \cap \operatorname{dom} f_i$ with $\|u - x\| < \delta$ and any real $r \geq f_i(u)$.

We can now establish the result showing in particular the coincidence of Fréchet and Clarke subdifferentials of f at any point where the subsmoothness property is satisfied.

Theorem 4.5 (**Ngai-Luc-Théra: Coincidence of subdifferentials of subsmooth function).** *Let U be a nonempty open set of a normed space X and $f : U \to \mathbb{R} \cup \{+\infty\}$ be a function which is subsmooth at a point $\overline{x} \in U$ where it is finite. The following hold:*

(a) *For each real* $\varepsilon > 0$ *there exists a real* $\delta > 0$ *with* $B(\overline{x}, 2\delta) \subset U$ *such that for every* $(x, x^*) \in \operatorname{gph} \partial_C f$ *with* $\|x - \overline{x}\| < \delta$ *one has*

$$\langle x^*, h\rangle \le f(x+h) - f(x) + \varepsilon\|h\| \quad \text{for all } h \in B(0,\delta). \tag{4.5}$$

(b) *The following subdifferential regularity*

$$\partial_C f(\overline{x}) = \partial_F f(\overline{x}) = \{x^* \in X^* : \langle x^*, h\rangle \le f'(\overline{x}; h), \quad \forall h \in X\}$$

also holds true at the point $\overline{x}$.

Proof. Take any real $\varepsilon > 0$. Choose a real $\delta_0 > 0$ given by Lemma 4.2(a) above and put $\delta := \delta_0/2$. Consider any $x \in B(\overline{x}, \delta) \cap \operatorname{Dom} \partial_C f$ and any $x^* \in \partial_C f(x)$, which is equivalent to $\langle x^*, h\rangle - r \le 0$ for all $(h, r) \in T^C\big(\operatorname{epi} f; (x, f(x))\big)$. For any $h \in B(0, \delta)$ with $x + h \in \operatorname{dom} f$, Lemma 4.2 yields

$$\big(h, f(x+h) - f(x) + \varepsilon\|h\|\big) \in T^C\big(\operatorname{epi} f; (x, f(x))\big),$$

thus (keeping in mind that $f(x + h) = +\infty$ when $x + h \notin \operatorname{dom} f$) we obtain

$$\langle x^*, h\rangle \le f(x+h) - f(x) + \varepsilon\|h\| \quad \text{for all } h \in B(0,\delta),$$

which translates the desired first property (a) of the theorem.

This latter property also tells us in particular with $x = \overline{x}$ and $x^* \in \partial_C f(\overline{x})$ that, for every real $\varepsilon > 0$ there is a real $\delta > 0$ such that $\langle x^*, h\rangle \le f(\overline{x} + h) - f(\overline{x}) + \varepsilon\|h\|$ for all $h \in B(0, \delta)$, so $x^* \in \partial_F f(\overline{x})$. We derive that $\partial_C f(\overline{x}) = \partial_F f(\overline{x})$ since the inclusion $\partial_F f(\overline{x}) \subset \partial_C f(\overline{x})$ always holds.

On the other hand, setting $\Delta := \{x^* \in X^* : \langle x^*, h\rangle \le f'(\overline{x}; h), \ \forall h \in X\}$ it is obvious that $\partial_F f(\overline{x}) \subset \Delta$ (keep in mind that $f'(\overline{x}; \cdot)$ exists by Proposition 4.1(b)). Conversely, let $x^* \in \Delta$. Fix any $\varepsilon > 0$. Taking $\delta > 0$ given by Proposition 4.1(a) we obtain that

$$\langle x^*, h\rangle \le f(\overline{x}+h) - f(\overline{x}) + \varepsilon\|h\|$$

for all $h \in X$ with $\|h\| < \delta$, which means that $x^* \in \partial_F f(\overline{x})$. This justifies the inclusion $\Delta \subset \partial_F f(\overline{x})$, so (b) is established and the proof is complete. □

Theorem 4.5(b) directly ensures the following tangential regularity.

Corollary 4.6. *Let* U *be a nonempty open set of a normed space* X *and* $f : U \to \mathbb{R}$ *be a locally Lipschitz function. If* f *is subsmooth at a point* $\overline{x} \in U$*, then it is tangentially regular at* $\overline{x}$.

Remark 4.7. Unlike locally Lipschitz subsmooth functions, a locally Lipschitz function which is one-sided subsmooth at $\overline{x}$ may fail to be tangentially regular at $\overline{x}$. The same function $f := -|\cdot|$ on $\mathbb{R}$ in Example 3.7 is one-sided subsmooth at $\overline{x} := 0$ according to this example, but it is not tangentially regular at $\overline{x} = 0$.

Under the uniform subsmoothness of f, instead of the property in Theorem 4.5(a), we have:

Proposition 4.8. *Let* U *be a nonempty open set of a normed space* X *and* $f : U \to \mathbb{R} \cup \{+\infty\}$ *be a function which is uniformly subsmooth on an open set* $U_0 \subset U$. *Then for each real* $\varepsilon > 0$ *there exists a real* $\delta > 0$ *such that for every* $y \in U_0$ *and every* $(x, x^*) \in \operatorname{gph} \partial_C f$ *with* $x \in U_0$ *and* $\|x - y\| < \delta$ *one has*

$$\langle x^*, y - x\rangle \le f(y) - f(x) + \varepsilon\|y - x\|. \tag{4.6}$$

Proof. Let any real $\varepsilon > 0$. Choose a real $\delta > 0$ given by Lemma 4.2(b). Take any $y \in U_0$ and any $(x, x^*) \in \operatorname{gph} \partial_C f$ with $x \in U_0$ and $\|x - y\| < \delta$. The inclusion $x^* \in \partial_C f(x)$ means $(x^*, -1) \in N^C(\operatorname{epi} f; (x, f(x))$. If $f(y) < +\infty$, Lemma 4.2(b) tells us that

$$\big(y - x, f(y) - f(x) + \varepsilon\|y - x\|\big) \in T^C\big(\operatorname{epi} f; (x, f(x))\big),$$

hence we obtain

$$\langle x^*, y - x\rangle \le f(y) - f(x) + \varepsilon\|y - x\|.$$

Trivially, the latter inequality still holds if $f(y) = +\infty$. □

Remark 4.9. Let U be a nonempty open set of a normed space X and $(f_i)_{i\in I}$ be a family of functions from U into $\mathbb{R} \cup \{+\infty\}$ which is uniformly equi-subsmooth relative to a family of open subsets $(U_i)_{i\in I}$ of U. Using Remark 4.4 instead of Lemma 4.2(b) we obtain that, for every real $\varepsilon > 0$ there exists $\delta > 0$ such that for each $i \in I$ one has

$$\langle x^*, y - x\rangle \le f_i(y) - f_i(x) + \varepsilon\|y - x\|$$

for any $y \in U_i$ and any $(x, x^*) \in \operatorname{gph} \partial_C f_i$ with $x \in U_i$ and $\|x - y\| < \delta$.

Proposition 3.1 established a characterization of strict differentiability of a mapping G at a point $\overline{x}$ through, for $h \in \mathbb{B}_X$, the difference of ratios

$$\frac{G(x + rh) - G(x)}{r} - \frac{G(x) - G(x - sh)}{s}$$

involving x near $\overline{x}$ and both r and s. When G is a continuous function f which is subsmooth at $\overline{x}$, the next lemma provides a similar characterization of the Fréchet differentiability (or equivalently, strict differentiability by Proposition 3.9) of f at $\overline{x}$ through the ratio $t^{-1}[f(\overline{x} + th) - f(\overline{x} - th) - 2f(\overline{x})]$ involving merely the reference point $\overline{x}$, extending in this way the known property for convex functions recalled at the beginning of this section. The lemma will be used in Theorem 4.11 below.

Lemma 4.10. *Let U be a nonempty open set of a normed space X and $f : U \to \mathbb{R}$ be a real-valued function which is subsmooth at $\overline{x} \in U$ and continuous at $\overline{x}$. The following assertions hold.*

(a) *The function f is Fréchet differentiable at $\overline{x}$ if and only if*

$$\lim_{t\downarrow 0} \frac{f(\overline{x} + th) + f(\overline{x} - th) - 2f(\overline{x})}{t} = 0$$

uniformly with respect to $h \in \mathbb{B}_X$ (or equivalently, with respect to $h \in \mathbb{S}_X$).

(b) *The function f is Gâteaux differentiable at $\overline{x}$ if and only if for each $\overline{h} \in X$ with $\|\overline{h}\| = 1$*

$$\lim_{t\downarrow 0} \frac{f(\overline{x} + t\overline{h}) + f(\overline{x} - t\overline{h}) - 2f(\overline{x})}{t} = 0.$$

Proof. (a) Suppose first that f is Frćhet differentiable at $\overline{x}$ and fix $r > 0$ such that $B(\overline{x}, r) \subset U$. There exists a function $\eta :]0, r[\times X \to \mathbb{R}$ with $\sup_{h\in\mathbb{B}_X} |\eta(t, h)| \to 0$ as

$t \downarrow 0$ such that $f(\overline{x}+th) - f(\overline{x}) = tDf(\overline{x})h + t\eta(t,h)$. Consequently, for all $t \in]0,r[$ and $h \in \mathbb{B}_X$

$$t^{-1}|f(\overline{x}+th) + f(\overline{x}-th) - 2f(\overline{x})| = |\eta(t,h) + \eta(t,-h)| \le 2 \sup_{v\in\mathbb{B}_X} |\eta(t,v)|,$$

which justifies the implication $\Rightarrow$ of the lemma.

Now, let us suppose that $\lim\limits_{t\downarrow 0} t^{-1}\rho(t) = 0$, where

$$\rho(t) := \sup_{h\in\mathbb{B}_X} |f(\overline{x}+th) + f(\overline{x}-th) - 2f(\overline{x})|.$$

Since f is subsmooth at $\overline{x}$ and continuous at this point, it is Lipschitz on some ball $B(\overline{x},r) \subset U$ (see Proposition 3.14). Fix some $\overline{x}^* \in \partial_C f(\overline{x})$. Take any real $\varepsilon > 0$ and, by Theorem 4.5(a) choose a positive real $\delta < r$ such that

$$\langle \overline{x}^*, x-\overline{x}\rangle \le f(x) - f(\overline{x}) + \varepsilon\|x-\overline{x}\|, \quad \text{for all } x \in B(\overline{x},\delta). \tag{4.7}$$

Choose a positive real $\delta_0 < \delta$ such that $t^{-1}\rho(t) \le \varepsilon$ for all $t \in]0,\delta_0[$. Considering any $t \in]0,\delta_0[$, we derive from (4.7) that, for all $h \in \mathbb{B}_X$

$$\rho(t) \ge f(\overline{x}+th) + f(\overline{x}-th) - 2f(\overline{x}) \ge f(\overline{x}+th) - f(\overline{x}) - t\langle \overline{x}^*, h\rangle - \varepsilon t,$$

and hence by (4.7) again

$$-\varepsilon \le \frac{f(\overline{x}+th) - f(\overline{x}) - t\langle \overline{x}^*, h\rangle}{t} \le t^{-1}\rho(t) + \varepsilon \le 2\varepsilon.$$

This tells us that f is Fréchet differentiable at $\overline{x}$ (with $\overline{x}^*$ as Fréchet derivative at $\overline{x}$), so the proof of (a) is finished.

(b) For any fixed $\overline{h} \in X$ with $\|\overline{h}\| = 1$, a slight modification of the above arguments with the use of $K := \{-\overline{h}, \overline{h}\}$ in place of $\mathbb{B}_X$ and $\overline{x} + t\overline{h}$ in place of x establishes the assertion (b). □

Given a multimapping $M : U \rightrightarrows Y$ between two sets U and Y, recall that a mapping $\zeta : S \to Y$ is a *selection* of M on a set $S \subset \operatorname{Dom} M$ whenever $\zeta(x) \in M(x)$ for all $x \in S$. When $S = \operatorname{Dom} M$, one just says that ζ is a selection of M.

Theorem 4.11 (**Differentiability of subsmooth function via continuous selection of subdifferential).** *Let U be a nonempty open set of a normed space X and $f : U \to \mathbb{R}$ be a function which is subsmooth at $\overline{x} \in U$ and continuous at $\overline{x}$. The following are equivalent:*

(a) *The function f is Fréchet (resp. Gâteaux) differentiable at $\overline{x}$.*
(b) *Any selection $\zeta(\cdot)$ of $\partial_C f$ is norm-norm (resp. norm-weak*) continuous at $\overline{x}$.*
(c) *There exists an open neighborhood $U_0 \subset U$ of $\overline{x}$ and a selection $\zeta : U_0 \to X^*$ of $\partial_C f$ on U_0 which is norm-norm (resp. norm-weak*) continuous at $\overline{x}$.*

Proof. We prove the theorem only for the Fréchet differentiabilty; the other case is similar. We note first that the implication (b)$\Rightarrow$(c) is obvious.

Let us show (c)$\Rightarrow$(a). Let U_0 and ζ be given by (c), and take any real $\varepsilon > 0$. By continuity of ζ at $\overline{x}$ and by Theorem 4.5(a) there exists a real $\delta > 0$ with $B(\overline{x},2\delta) \subset U_0$ such that $\|\zeta(x) - \zeta(\overline{x})\| < \varepsilon$ for any $x \in B(\overline{x},\delta)$ and such that

$$\langle x^*, h\rangle \le f(x+h) - f(x) + \varepsilon\|h\|$$

for any $x \in B(\overline{x}, \delta)$, any $x^* \in \partial_C f(x)$ and any $h \in B(0, \delta)$. Fixing any $x \in B(\overline{x}, \delta)$, we deduce that

$$(4.8)\quad \langle \zeta(\overline{x}), x - \overline{x} \rangle \le f(x) - f(\overline{x}) + \varepsilon \|x - \overline{x}\|, \ \ \langle \zeta(x), \overline{x} - x \rangle \le f(\overline{x}) - f(x) + \varepsilon \|x - \overline{x}\|.$$

The latter inequality along with the fact that $\|\zeta(x) - \zeta(\overline{x})\| < \varepsilon$ yields

$$f(x) - f(\overline{x}) \le \langle \zeta(\overline{x}), x - \overline{x} \rangle + \langle \zeta(x) - \zeta(\overline{x}), x - \overline{x} \rangle + \varepsilon \|x - \overline{x}\| \le \langle \zeta(\overline{x}), x - \overline{x} \rangle + 2\varepsilon \|x - \overline{x}\|,$$

which combined with the first inequality in (4.8) gives

$$| f(x) - f(\overline{x}) - \langle \zeta(\overline{x}), x - \overline{x} \rangle | \le 2\varepsilon \|x - \overline{x}\|.$$

This translates the Fréchet differentiability of f at $\overline{x}$, that is, (a) holds.

Finally, let us prove (a)⇒(b). By Proposition 3.14 the function f is Lipschitz near $\overline{x}$, so without loss of generality we may suppose that f is Lipschitz on U with constant $\gamma > 0$. Let $\zeta : U \to X^*$ be any selection of $\partial_C f$ on U. Fix any real $\varepsilon > 0$. By Theorem 4.5 choose a real $\delta > 0$ with $B(\overline{x}, 2\delta) \subset U$ such that for any $x, y \in B(\overline{x}, \delta)$ and $x^* \in \partial_C f(x)$

$$(4.9)\qquad \langle x^*, y - x \rangle \le f(y) - f(x) + \varepsilon \|y - x\|.$$

Since f is strictly Fréchet differentiable at $\overline{x}$ by Proposition 3.9 and Proposition 3.14, we have $Df(\overline{x}) = \zeta(\overline{x})$, so $t^{-1}\eta(t) \to 0$ as $t \downarrow 0$, where

$$\eta(t) := \sup_{h \in \mathbb{B}_X} |f(\overline{x} + th) - f(\overline{x}) - t\langle \zeta(\overline{x}), h \rangle| \quad \text{for all } t \in]0, \delta[.$$

Now choose some positive real $r < \delta$ such that $r^{-1}\eta(r) < \varepsilon$. Taking any $x \in B(\overline{x}, \delta)$ and any $h \in \mathbb{B}_X$ we derive from (4.9) and from the definition of $\eta(\cdot)$ that

$$\begin{aligned}\langle \zeta(x), \overline{x} + rh - x \rangle &\le f(\overline{x} + rh) - f(x) + \varepsilon \|\overline{x} + rh - x\| \\ &\le f(\overline{x}) + r\langle \zeta(\overline{x}), h \rangle + \eta(r) - f(x) + \varepsilon \|x - \overline{x}\| + \varepsilon r,\end{aligned}$$

which gives

$$r\langle \zeta(x) - \zeta(\overline{x}), h \rangle \le \langle \zeta(x), x - \overline{x} \rangle + f(\overline{x}) - f(x) + \eta(r) + \varepsilon \|x - \overline{x}\| + \varepsilon r,$$

and hence

$$\langle \zeta(x) - \zeta(\overline{x}), h \rangle \le r^{-1}[\gamma \|x - \overline{x}\| + |f(x) - f(\overline{x})|] + r^{-1}\eta(r) + \varepsilon r^{-1} \|x - \overline{x}\| + \varepsilon.$$

Choosing a positive real $\delta_0 < \delta$ such that $r^{-1}[\gamma \|x - \overline{x}\| + |f(x) - f(\overline{x})|] < \varepsilon$ and $r^{-1}\|x - \overline{x}\| < 1$ for all $x \in B(\overline{x}, \delta_0)$, it ensues that $\langle \zeta(x) - \zeta(\overline{x}), h \rangle \le 4\varepsilon$ for all $x \in B(\overline{x}, \delta_0)$ and all $h \in \mathbb{B}_X$. It results that $\|\zeta(x) - \zeta(\overline{x})\| \le 4\varepsilon$ for all $x \in B(\overline{x}, \delta_0)$, which confirms the norm-norm continuity of $\zeta(\cdot)$ at $\overline{x}$. The proof of the theorem is then complete. □

4.2. **Submonotonicity of subdifferentials.** Let any real $\varepsilon > 0$. Taking with $\varepsilon' := \varepsilon/2$ a real $\delta > 0$ given by Theorem 4.5(a), we see that, for all $(x_i, x_i^*) \in \operatorname{gph} \partial_C f$, $i = 1, 2$, with $x_i \in B(\overline{x}, \delta)$

$$\langle x_1^*, x_2 - x_1 \rangle \le f(x_2) - f(x_1) + \varepsilon' \|x_2 - x_1\|, \ \langle x_2^*, x_1 - x_2 \rangle \le f(x_1) - f(x_2) + \varepsilon' \|x_1 - x_2\|,$$

and hence $\langle x_1^* - x_2^*, x_1 - x_2 \rangle \ge -\varepsilon \|x_1 - x_2\|$. This property of the multimapping $\partial_C f$ for such a subsmooth function f is clearly weaker than the usual monotonicity property. We formalize it as a definition.

Definition 4.12. Let U be a nonempty open set of a normed space X and $M : X \rightrightarrows X^*$ be a multimapping from U into the topological dual X^* of X. One says that M is *submonotone* at a point $\overline{x} \in U$ provided that for any real $\varepsilon > 0$ there exists a real $\delta > 0$ with $B(\overline{x}, \delta) \subset U$ such that for all $x, y \in B(\overline{x}, \delta) \cap \operatorname{Dom} M$, $x^* \in M(x)$ and $y^* \in M(y)$ one has

$$\langle y^* - x^*, y - x \rangle \geq -\varepsilon \|y - x\|. \tag{4.10}$$

When M is submonotone at any point of a nonempty open set $U_0 \subset U$, one says that M is *submonotone on* U_0.

When the above inequality holds true with $x = \overline{x} \in U \cap \operatorname{Dom} M$ and all $y \in B(\overline{x}, \delta) \cap \operatorname{Dom} M$, $y^* \in M(y)$, $x^* \in M(\overline{x})$, one says that M is *one-sided submonotone* at $\overline{x}$. The multimapping M is one-sided submonotone on an open set $U_0 \subset U$ if it is one-sided submonotone at any point in $U_0 \cap \operatorname{Dom} M$.

We say that M is *uniformly submonotone on an open set* $U_0 \subset U$ when for any real $\varepsilon > 0$ there exists a real $\delta > 0$ such that the inequality (4.10) is fulfilled for all $x, y \in U_0 \cap \operatorname{Dom} M$ with $\|y - x\| < \delta$ and all $x^* \in M(x)$ and $y^* \in M(y)$. The multimapping M is *uniformly submonotone near a point* in U if it is uniformly submonotone on an open neighborhood of this point.

Remark 4.13. Given an open U of a normed space X, a family of multimappings $(M_i)_{i \in I}$ from U into X^* is called *uniformly equi-submonotone relative to a family of open subsets* $(U_i)_{i \in I}$ *of* U provided that for every $\varepsilon > 0$ there is $\delta > 0$ such that for each $i \in I$ the inequality (4.10) is satisfied for all $x, y \in U_i \cap \operatorname{Dom} M_i$ with $\|y - x\| < \delta$ and all $x^* \in M_i(x)$ and $y^* \in M_i(y)$. When all the sets U_i coincide with a same open set U_0, one simply says that the family of multimappings is *uniformly equi-submonotone on* U_0.

With notation of the above definition, let $M_0 : U \rightrightarrows X^*$ be another multimapping whose graph is included and sequentially $\|\cdot\| \times w(X^*, X)$ dense in $\operatorname{gph} M$. It is clear that M is submonotone at $\overline{x} \in U$ if and only if M_0 is submonotone at $\overline{x}$. It is also worth pointing out that the sum of two multimappings from $U \subset X$ into X^* is clearly submonotone (resp. one-sided submonotone) at a point whenever both are submonotone (resp. one-sided submonotone) at that point. Further, submonotonicity obviously implies one-sided submonotonicity.

We will focus our analysis on the submonotonicity (resp. one-sided submonotonicity) of subdifferentials. The next theorem says that the subsmoothness at $\overline{x}$ of lower semicontinuous functions on a Banach space is characterized by the submonotonocity at $\overline{x}$ of their subdifferentials. Before stating the theorem let us give an example pointing out the difference between submonotonicity and one-sided submonotonicity even for subdifferential.

Example 4.14 (Spingarn example [72]). Consider the locally Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(s,t) = \begin{cases} |t| & \text{if } s \leq 0 \\ |t| - s^2 & \text{if } s \geq 0 \text{ and } |t| \geq s^2 \\ (s^4 - t^2)/2s^2 & \text{if } s > 0 \text{ and } |t| \leq s^2. \end{cases}$$

Put $x_n := (1/n, 1/n^2)$ and $y_n := (1/n, -1/n^2)$. By the gradient representation of Clarke subdifferential (see Proposition 2.1) one easily sees (through the third line in the definition of f) that

$$x_n^* := (2/n, -1) \in \partial_C f(x_n) \quad \text{and} \quad y_n^* := (2/n, 1) \in \partial_C f(y_n).$$

It follows that

$$\frac{\langle x_n^* - y_n^*, x_n - y_n \rangle}{\|x_n - y_n\|} = -2 \quad \text{for all } n \in \mathbb{N},$$

so $\partial_C f$ is not submonotone at $\overline{x} := (0, 0)$.

However, noting via the gradient representation of Clarke subdifferential again that

$$\partial_C f(\overline{x}) = [(0, -1), (0, 1)]$$

(the line segment between $(0, -1)$ and $(0, 1)$) one can verify that $\partial_C f$ is one-sided submonotone at $\overline{x}$.

In order to state in a unified way the theorem of subdifferential characterization of subsmooth functions for the three subdifferentials in Section 2 (and for others), we consider some basic properties common to the subdifferentials in Section 2 in appropriate spaces.

Let U be a nonempty open set of a normed space X and $\mathcal{F}(U)$ be a class of functions from U into $\mathbb{R} \cup \{-\infty, +\infty\}$ which contains the restrictions to U of continuous convex functions on X and is stable by addition with these functions. Given a subdifferential for functions in $\mathcal{F}(U)$, which is in particular an operator ∂ from $\mathcal{F}(U) \times U$ into subsets of the topological dual space X^* (assigning to any every pair $(f, x) \in \mathcal{F}(U) \times U$ a set $\partial f(x) \subset X^*$), consider the following fundamental properties:

Prop.1: $\partial f(x) = \emptyset$ if $|f(x)| = +\infty$ and $0 \in \partial f(x)$ whenever $x \in U$ is a local minimum point of f with $|f(x)| < +\infty$;

Prop.2: $\partial f(x) = \partial g(x)$ whenever f and g coincide on a neighborhood of x;

Prop.3: if f is finite at $x \in U$ and the restriction $f_{|V}$ of f to a convex neighborhood $V \subset U$ of x is lower semicontinuous and convex, then $\partial f(x)$ is equal to the subdifferential in the sense of convex analysis of $f_{|V}$ at x, that is,

$$\partial f(x) = \{x^* \in X^* : \langle x^*, u - x \rangle \leq f(u) - f(x) \ \forall u \in V\};$$

Prop.4: for $f \in \mathcal{F}(U)$ lower semicontinuous near x and for the restriction g to U of a finite-valued, convex, and continuous function on X, if x is a local minimum point for $f + g$, then for any real $\varepsilon > 0$ there are $x', x'' \in U \cap B(x, \varepsilon)$ with $|f(x') - f(x)| < \varepsilon$ and such that

$$0 \in \partial f(x') + \partial g(x'') + \varepsilon \mathbb{B}_{X^*}.$$

When $\mathcal{F}(U)$ is the class of all extended real-valued functions on U, we will just say a subdifferential on U with properties **Prop.1-Prop.4**. If $f : V \to \mathbb{R} \cup \{-\infty, +\infty\}$ is a function defined on a neighborhood $V \subset U$ of x such that $\overline{f} \in \mathcal{F}(U)$, where $\overline{f}$ is the extension of f to U with $\overline{f}(x') = +\infty$ for $x' \in U \setminus V$, we will set $\partial f(x) := \partial \overline{f}(x)$.

If ∂ is a subdifferential on X and S is a subset of X, we will write $N(S; x)$ in place of $\partial \psi_S(x)$ and we will call $N(S; x)$ the normal cone of S at x associated with the subdifferential ∂. When the subdifferential ∂ needs to be emphasized, we will write $N^\partial(S; x)$.

Properties **Prop.1-Prop.4** are fulfilled by the Carke subdifferential in any normed space, by the Ioffe (geometric) approximate subdifferential in any Banach space (see [40] for the definition), and by the Fréchet and the Mordukhovich limiting subdifferentials in any Asplund space. For other cases, we refer the reader to [73].

The subdifferential characterizations in the next theorem (Theorem 4.15) have been independently established by A. Daniilidis, F. Jules and M. Lassonde [25] and by H.V. Ngai and J. P. Penot [57].

Theorem 4.15 (**Subdifferential characterizations of subsmooth function).** *Let U be a nonempty open set of a Banach space X and $f : U \to \mathbb{R} \cup \{+\infty\}$ be a proper function which is lower semicontinuous near $\overline{x} \in \operatorname{dom} f$. Let ∂ be a subdifferential on X with ∂f included in the Clarke one and satisfying the properties* **Prop.1-$\cdots$-Prop.4** *above. The following assertions are equivalent:*

(a) *The function f is subsmooth at $\overline{x}$.*
(b) *For any real $\varepsilon > 0$ there is $\delta > 0$ with $B(\overline{x}, \delta) \subset U$ such that for any $y \in B(\overline{x}, \delta)$, $x \in B(\overline{x}, \delta) \cap \operatorname{Dom} \partial f$, and $x^* \in \partial f(x)$, one has*

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + \varepsilon\|y - x\|.$$

(c) *The multimapping ∂f is submonotone at $\overline{x}$, that is, for any real $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in B(\overline{x}, \delta) \cap \operatorname{Dom} \partial f$, $x^* \in \partial f(x)$, and $y^* \in \partial f(y)$, one has*

$$\langle x^* - y^*, x - y\rangle \geq -\varepsilon\|x - y\|.$$

In the case when f is locally Lipschitz, the completeness of the space X is not needed and much more simpler arguments yield the following.

Proposition 4.16. *Let U be a nonempty open subset of a normed space X and $f : U \to \mathbb{R}$ be a locally Lipschitz function. For $\overline{x} \in U$ the following are equivalent:*

(a) *The function f is subsmooth at $\overline{x}$.*
(b) *For each real $\varepsilon > 0$ there exists a real $\delta > 0$ with $B(\overline{x}, \delta) \subset U$ such that*

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + \varepsilon\|y - x\|,$$

for all $x, y \in B(\overline{x}, \delta)$ and all $x^* \in \partial_C f(x)$.
(c) *The multimapping $\partial_C f$ is submonotone at $\overline{x}$.*

Proof. The implications (a)⇒(b) follows from Theorem 4.5(a) and the implication (b)⇒(c) is evident. Now suppose (c), that is, $\partial_C f$ is submonotone at $\overline{x}$. Fix any real $\varepsilon > 0$ and fix $\delta > 0$ with $V := B(\overline{x}, \delta) \subset U$ such that for any $x_i \in V$ and $x_i^* \in \partial_C f(x_i)$ $i = 1, 2$, one has

$$\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq -\varepsilon\|x_1 - x_2\|.$$

Fix any $x, y \in V$ with $x \neq y$ and any $s, t \in]0, 1[$ with $s + t = 1$. Putting $z := sx + ty$, by the Lebourg mean value equality (see Proposition 2.1(b)) there are $c \in]x, z[$, $d \in]z, y[$, $c^* \in \partial_C f(c)$ and $d^* \in \partial_C f(d)$ such that

$$f(z) - f(x) = \langle c^*, z - x\rangle \quad \text{and} \quad f(z) - f(y) = \langle d^*, z - y\rangle.$$

Noting that $z - x = t(y - x)$ and $z - y = s(x - y)$, multiplying the first equality by s and the second by t and adding together yield

$$\begin{aligned} f(z) - sf(x) - tf(y) &= st\langle c^* - d^*, y - x\rangle \\ &= st\frac{\|y - x\|}{\|d - c\|}\langle c^* - d^*, d - c\rangle \leq \varepsilon st\|y - x\|. \end{aligned}$$

This and Proposition 3.11 tell us that the function f is subsmooth at $\overline{x}$. □

For the uniform subsmoothness (resp. uniform equi-subsmoothness (see Definition 3.6) we have the following similar equivalences.

Proposition 4.17. *Let f and $(f_i)_{i\in I}$ be real-valued locally Lipschitz functions on a nonempty open convex subset U of a normed space X. Given a nonempty open subset $U_0 \subset U$ and a family $(U_i)_{i\in I}$ of open subsets of U the following are equivalent:*

(a) *The function f is uniformly subsmooth on U_0 (resp. the family $(f_i)_{i\in I}$ is uniformly equi-subsmooth relative to $(U_i)_{i\in I}$).*

(b) *For each real $\varepsilon > 0$ there exists a real $\delta > 0$ such that (resp. such that for each $i \in I$)*

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + \varepsilon\|y - x\| \ (\text{ resp. } \langle x^*, y - x\rangle \leq f_i(y) - f_i(x) + \varepsilon\|y - x\| \ )$$

for any $x, y \in U_0$ (resp. any $x, y \in U_i$) with $\|x-y\| < \delta$ and any $x^* \in \partial_C f(x)$ (resp. any $x^* \in \partial_C f_i(x)$).

(c) *The multimapping $\partial_C f$ is uniformly submonotone on U_0 (resp. the family of multimappings $(\partial_C f_i)_{i\in I}$ from U into X^* is uniformly equi-submonotone relative to $(U_i)_{i\in I}$).*

Proof. The implication (a)⇒(b) is a direct consequence of Proposition 4.8 while (b)⇒(c) is obvious. For the remaining implication (c)⇒(a) it suffices to proceed like in the proof of Proposition 4.16. □

Before providing other characterizations of uniform subsmoothness let us recall the following lemma which can be found in [52, Lemma 1.29] or [72, Lemma 3.7].

Lemma 4.18. *Let $\xi : [0, +\infty[\to [0, +\infty]$ be a function continuous at 0 with $\xi(0) = 0$. Then there exist a real $\theta > 0$ and a continuously derivable function $\varphi : [0, \theta] \to [0, +\infty[$ with $\varphi(0) = \varphi'_+(0) = 0$ such that $\varphi(t) \geq t\xi(t)$ for all $t \in [0, \theta]$.*

The two additional characterizations of uniform subsmoothness of locally Lipschitz functions are expressed in terms of modulus functions. Recall that $\omega : [0, +\infty[\to [0, +\infty]$ is a modulus function when ω is continuous at 0 with $\omega(0) = 0$.

Proposition 4.19. *Let U be a nonempty open convex subset of a normed space X and $f : U \to \mathbb{R}$ be a locally Lipschitz function. Given a nonempty open subset $U_0 \subset U$ the following are equivalent:*

(a) *The function f is uniformly subsmooth on U_0.*

(b) *There are a real $\theta > 0$ and a modulus function $\omega : [0, +\infty[\to [0, +\infty[$ of class C^1 on $]0, +\infty[$ with $t\,\omega'(t) \to 0$ as $t \downarrow 0$ such that*

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + \|y - x\|\omega(\|y - x\|)$$

for all $x, y \in U_0$ with $\|x - y\| \leq \theta$ and all $x^* \in \partial_C f(x)$.

(c) *There are a real* $\theta > 0$ *and a modulus function* $\omega : [0, +\infty[\to [0, +\infty[$ *of class* C^1 *on* $]0, +\infty[$ *with* $t\,\omega'(t) \to 0$ *as* $t \downarrow 0$ *such that*

$$\langle x^* - y^*, x - y\rangle \geq -\|x - y\|\omega(\|x - y\|)$$

for all $x, y \in U_0$ *with* $\|x - y\| \leq \theta$, *all* $x^* \in \partial_C f(x)$ *and all* $y^* \in \partial_C f(y)$.

Proof. The implication (b)⇒(c) is evident, and (c) implies (a) according to the implication (c)⇒(a) in Proposition 4.17. It remains to show (a)⇒(b). Assume that (a) is satisfied. For any $x, y \in U_0$ and $x^* \in X^*$ put

$$g(x, y, x^*) := \begin{cases} \big(f(y) - f(x) - \langle x^*, y - x\rangle\big)/\|y - x\| & \text{if } x \neq y \\ 0 & \text{if } x = y, \end{cases}$$

and for every real $t > 0$ put

$$\zeta(t) := \inf\{g(x, y, x^*) : x, y \in U_0, \|x - y\| \leq t, x^* \in \partial_C f(x)\}.$$

Put also $\zeta(0) = 0$. By the implication (a)⇒(b) in Proposition 4.17, for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\zeta(t) \geq -\varepsilon \quad \text{for all } t \in [0, \delta[.$$

Then the function $\xi : [0, \infty[\to [0, +\infty]$ defined by $\xi(t) := \max\{-\zeta(t), 0\}$ is continuous at 0 with $\xi(0) = 0$. By Lemma 4.18 there is a real $\theta > 0$ and a continuously derivable function $\varphi : [0, \theta] \to [0, +\infty[$ with $\varphi(0) = \varphi'_+(0) = 0$ such that

$$\varphi(t) \geq t\xi(t) \quad \text{for all } t \in [0, \theta].$$

Let us extend φ to $[0, +\infty[$ by putting

$$\varphi(t) := \varphi(\theta) + \varphi'_-(\theta)(t - \theta) \quad \text{for all } t \in]\theta, +\infty[.$$

Then $\varphi : [0, +\infty[\to [0, +\infty[$ is of class C^1. Take any $x, y \in U_0$ with $\|x - y\| \leq \theta$ and any $x^* \in \partial_C f(x)$. Suppose $x \neq y$ and put $t = \|x - y\|$. Since $t \in]0, \theta]$ we have

$$g(x, y, x^*) \geq \zeta(t) \geq -\xi(t) \geq -\frac{\varphi(t)}{t} = -\frac{\varphi(\|x - y\|)}{\|x - y\|},$$

so $f(y) - f(x) - \langle x^*, y - x\rangle \geq -\varphi(\|x - y\|)$. This latter inequality is still trivially true when $x = y$. Consequently, to get (b) it suffices to define $\omega : [0, +\infty[\to [0, +\infty[$ by $\omega(0) = 0$ and $\omega(t) = \varphi(t)/t$ for every real $t > 0$. □

The next proposition shows in particular the equivalence in finite dimensions between the subsmootness near a point and the lower C^1 property near that point.

Proposition 4.20. *Let* U *be a nonempty open subset of a normed space* X *and* $f : U \to \mathbb{R}$ *be a locally Lipschitz function. For* $\overline{x} \in U$ *the following are equivalent:*

(a) *The function* f *is uniformly subsmooth near* $\overline{x}$.

(b) *There are an open convex neighborhood* $V \subset U$ *of* $\overline{x}$ *and a modulus function* $\omega : [0, +\infty[\to [0, +\infty[$ *of class* C^1 *on* $]0, +\infty[$ *with* $t\,\omega'(t) \to 0$ *as* $t \downarrow 0$ *such that*

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + \|y - x\|\omega(\|y - x\|)$$

for all $x, y \in V$ *and all* $x^* \in \partial_C f(x)$.

(c) *There are an open convex neighborhood $V \subset U$ of $\overline{x}$ and a modulus function $\omega : [0, +\infty[\to [0, +\infty[$ of class C^1 on $]0, +\infty[$ with $t\,\omega'(t) \to 0$ as $t \downarrow 0$ such that*

$$\langle x^* - y^*, x - y\rangle \geq -\|x - y\|\omega(\|x - y\|)$$

for all $x, y \in V$, all $x^* \in \partial_C f(x)$ and all $y^* \in \partial_C f(y)$.

If X is finite-dimensional, one can add anyone of (d) and (e) below to the list of equivalences:

(d) *The function f is subsmooth near $\overline{x}$.*

(e) *The function f is lower C^1 near $\overline{x}$, that is, there exist (as said in the introduction) a compact metric space T, an open neighborhood $V \subset U$ of $\overline{x}$ and a continuous function $\Phi : V \times T \to \mathbb{R}$ such that $D_1\Phi(\cdot, \cdot)$ exists and is continuous on $V \times T$, and such that*

$$f(x) = \max_{t \in T} \Phi(x, t) \quad \textit{for all } x \in V.$$

Proof. The equivalences (a)⇔(b)⇔(c) follow easily from Proposition 4.19 while the implication (a)⇒(d) is evident. Assume now that X is finite-dimensional and f is subsmooth near $\overline{x}$. There exists a real $r > 0$ and an open set $V \subset U$ containing $B[\overline{x}, r]$ such that f is subsmooth at each point in V. Let any real $\varepsilon > 0$. For each $u \in B[\overline{x}, r]$ choose a real $\delta_u > 0$ with $B(u, 2\delta_u) \subset V$ such that for all $x, y \in B(u, 2\delta_u)$ and all $x^* \in \partial_C f(x)$

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + \varepsilon\|y - x\|. \tag{4.11}$$

By compactness of $B[\overline{x}, r]$ there are $u_1, \ldots, u_m$ in $B[\overline{x}, r]$ such that the balls $B(u_i, \delta_{u_i})$ cover $B[\overline{x}, r]$. Denote $\delta := \min\{\delta_{u_1}, \ldots, \delta_{u_m}\} > 0$ and take any $x, y \in B(\overline{x}, r)$ with $\|x - y\| < \delta$ and any $x^* \in \partial_C f(x)$. Choose $k \in \{1, \ldots, m\}$ such that $x \in B(u_k, \delta_{u_k})$. Then both x, y belong to $B(u_k, 2\delta_{u_k})$, so by (4.11) we have

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + \varepsilon\|y - x\|.$$

This justifies the uniform subsmoothness of f on $B(\overline{x}, r)$, so (d)⇒(a) holds true.

Assume again that X is finite-dimensional and fix an Euclidean norm $\|\cdot\|$ on X associated to an inner product $\langle\cdot, \cdot\rangle$. Let us first show (b)⇒(e). Let V and ω be given by (b), so the function $\xi : [0, +\infty[\to [0, +\infty[$, given by $\xi(t) := t\,\omega(t)$ for all $t \in [0, +\infty[$, is continuously derivable on $[0, +\infty[$ with $\xi(0) = \xi'_+(0) = 0$. Choose a real $r > 0$ such that $B[\overline{x}, r] \subset V$ and put

$$T := \{(y, y^*) \in X \times X : y \in B[\overline{x}, r], y^* \in \partial_C f(y)\}.$$

From the local boundedness of $\partial_C f$ we easily see that T is a (nonempty) compact subset of $X \times X$. Further, the function $\Phi : V \times T \to \mathbb{R}$ defined by

$$\Phi(x, (y, y^*)) := f(y) + \langle y^*, x - y\rangle + \xi(\|x - y\|)$$

is continuous on $V \times T$ and $D_1\varphi(\cdot, \cdot)$ exists and is continuous on $V \times T$ according to the above properties of the function ξ. Since $f(x) = \max\limits_{(y,y^*) \in T} \Phi(x, (y, y^*))$ for all $x \in V$, we have shown (b)⇒(e). Let us finally prove (e)⇒(a). Let T, V, Φ be as given by (e). Choose a real $r > 0$ such that $B[\overline{x}, r] \subset V$. Fix any real $\varepsilon > 0$. The mapping $D_1\Phi(\cdot, \cdot)$ being uniformly continuous on the compact set $B[\overline{x}, r] \times T$, there is a real $\delta > 0$ such that for all $(x, t), (y, \tau)$ in $B[\overline{x}, r] \times T$ with $\|x - y\| + d(t, \tau) < \delta$

one has $\|D_1\Phi(x,t) - D_1\Phi(y,\tau)\| \le \varepsilon$. Fix any $x, y \in B(\overline{x}, r)$ with $\|x - y\| < \delta$ and any $t \in T(x) := \{\tau \in T : \Phi(x,\tau) = f(x)\}$. We note that

$$\begin{aligned}
&\langle D_1\Phi(x,t), y - x\rangle \\
&= \Phi(y,t) - \Phi(x,t) - \int_0^1 \langle D\Phi_1(x + s(y-x), t) - D\phi_1(x,t), y - x\rangle\, ds \\
&\le f(y) - f(x) + \|y - x\| \int_0^1 \|D\phi_1(x + s(y-x), t) - D_1\Phi(x,t)\|\, ds \\
&\le f(y) - f(x) + \varepsilon\|y - x\|.
\end{aligned}$$

From this and Proposition 2.1(f) we deduce that $\langle x^*, y-x\rangle \le f(y) - f(x) + \varepsilon\|y-x|$ for all $x^* \in \partial_C f(x)$, which translates the uniform subsmoothness of f on $B(\overline{x}, r)$. The implication (e)$\Rightarrow$(a) then holds, and the proof is finished. □

Concerning the distance function, given a set S and $\overline{x} \in S$ the next proposition shows that a relative particular Jensen-type inequality of the distance function d_S entails the submonotonicity of $\partial_C d_S$ at $\overline{x}$ relative to S.

Proposition 4.21. *Let S be a subset of a normed space X and $\overline{x} \in S$. Consider the assertions:*

(a) *For every real $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$d_S\big(tx + (1-t)y\big) \le \varepsilon t(1-t)\|x - y\| \quad \text{for all } x, y \in S \cap B(\overline{x}, \delta), t \in]0,1[.$$

(b) *For every real $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\langle x^*, y - x\rangle \le \varepsilon\|y - x\| \quad \text{for all } x, y \in S \cap B(\overline{x}, \delta), x^* \in \partial_C d_S(x).$$

(c) *The multimapping $\partial_C d_S$ is submonotone at $\overline{x}$ relative to S, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\langle x^* - y^*, x - y\rangle \ge -\varepsilon\|x - y\| \text{ for all } x, y \in S \cap B(\overline{x}, \delta), x^* \in \partial_C d_S(x), y^* \in \partial_C d_S(y).$$

The implications (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) *hold.*

Proof. The equivalence (b) $\Leftrightarrow$ (c) is trivial since $0 \in \partial_C d_S(y)$ for all $y \in S$. Suppose that (a) holds and take any $\varepsilon > 0$. Let $\delta > 0$ be given by (a). Fix any $x, y \in S \cap B(\overline{x}, \delta)$. Proposition 2.1(a) along with the Lipschitz property of d_S tells us that

$$d_S^o(x; y - x) = \limsup_{S \ni x' \to x, t \downarrow 0} t^{-1} d_S(x' + t(y - x')).$$

On the other hand, for any $t \in]0,1[$ and any $x' \in S \cap B(x, r)$ with $r := \delta - \|x - \overline{x}\| > 0$, we have $t^{-1} d_S(x' + t(y - x')) \le \varepsilon(1-t)\|y - x'\|$. It results that $d_S^o(x; y-x) \le \varepsilon\|y - x\|$, which is equivalent to $\langle x^*, y - x\rangle \le \varepsilon\|y - x\|$ for all $x^* \in \partial_C d_S(x)$. □

We use Propositions 4.16 and 4.17 to establish the assertion (c) in the next proposition.

Proposition 4.22. *Let X and Y be two normed spaces and U be a nonempty open set in X.*

(a) *For any real* $\lambda > 0$ *and for two functions* $f_1, f_2 : U \to \mathbb{R} \cup \{+\infty\}$ *which are subsmooth at* $\overline{x} \in U$ *(resp. uniformly subsmooth on an open set* $V \subset U$*), the functions* λf_1 *and* $f_1 + f_2$ *are subsmooth at* $\overline{x}$ *(resp. uniformly subsmooth on* V*).*

(b) *If* $A : X \to Y$ *is a continuous linear mapping and* $g : Y \to \mathbb{R} \cup \{+\infty\}$ *is subsmooth at* $A\overline{x}$ *(resp. uniformly subsmooth on an open set* $W \supset A(V)$*, where* V *is an open set of* X*), then* $g \circ A$ *is subsmooth at* $\overline{x}$ *(resp. uniformly subsmooth on* V*).*

(c) *If* $G : U \to Y$ *is of class* C^1 *near* $\overline{x}$ *and if* $g : Y \to \mathbb{R}$ *is Lipschitz near* $G(\overline{x})$ *and subsmooth at* $G(\overline{x})$*, then* $g \circ G$ *is subsmooth at* $\overline{x}$*.*

(d) *If* $G : U \to Y$ *is Lipschitz and differentiable on an open convex set* $V \subset U$ *with* DG *uniformly continuous on* V *and if* $g : Y \to \mathbb{R}$ *is Lipschitz and uniformly subsmooth on an open convex set* $W \supset G(V)$*, then the function* $g \circ G$ *is uniformly subsmooth on* V*.*

Proof. The assertions (a) and (b) follows from Proposition 3.11. Concerning (c) put $\overline{y} := G(\overline{x})$ and choose a real $\delta > 0$ such that g is Lipschitz on $B(\overline{y}, \delta)$ with Lipschitz constant $\gamma > 0$ and G is Lipschitz on $B(\overline{x}, \delta)$ with the same Lipschitz constant γ. Fix any $\varepsilon > 0$ and put $\varepsilon' := \varepsilon/(2\gamma)$. By Proposition 4.16 shrinking δ if necessary, we have $\langle y^*, y' - y\rangle \le g(y') - g(y) + \varepsilon'\|y' - y\|$ for all $y, y' \in B(\overline{y}, \delta)$ and $y^* \in \partial_C g(y)$. Choose a positive real $\delta_0 < \delta$ such that $G(B(\overline{x}, \delta_0)) \subset B(\overline{y}, \delta)$ and $\|DG(u') - DG(u)\| \le \varepsilon'$ for all $u, u' \in B(\overline{x}, \delta_0)$. Take any $x, x' \in B(\overline{x}, \delta_0)$ and any $x^* \in \partial_C(g \circ G)(x)$. There exists $y^* \in \partial_C g(G(x))$ with $x^* = y^* \circ DG(x)$ (see, e.g., [18, Theorem 2.3.10]). It follows that

$$\begin{aligned}\langle x^*, x' - x\rangle &= \langle y^*, DG(x)(x' - x)\rangle \\ &= \langle y^*, G(x') - G(x)\rangle - \langle y^*, \int_0^1 \big(DG(x + t(x' - x)) - DG(x)\big)(x' - x)\, dt \\ &\le g(G(x')) - g(G(x)) + \varepsilon'\|G(x') - G(x)\| + \varepsilon'\|y^*\|\,\|x' - x\|,\end{aligned}$$

which gives $\langle x^*, x' - x\rangle \le (g \circ G)(x') - (g \circ G)(x) + \varepsilon\|x' - x\|$. This tells us by Proposition 4.16 again that $g \circ G$ is subsmooth at $\overline{x}$.

The proof of (d) is similar. □

4.3. **Subdifferential characterizations of one-sided subsmooth functions.** This subsection provides various characterizations similar to those of Proposition 4.16 for one-sided subsmoothness property of functions. The approach requires first two lemmas. Recall that a multimapping M between two metric spaces T and Y is bounded near a point $\overline{t} \in T$ if $M(V)$ is bounded for some neighborhood V of $\overline{t}$.

Lemma 4.23. *Let* U *be a nonempty open subset of a normed space* X *and* $M : U \rightrightarrows X^*$ *be a multimapping which is bounded near a point* $\overline{x} \in \operatorname{Dom} M$ *and* $\|\cdot\| - \mathrm{to} - w^*$ *outer semicontinuous at* $\overline{x}$*.*

(a) *If* M *is one-sided submonotone at* $\overline{x}$*, then for any* $u \in \mathbb{S}_X$*, for any net* $(x_j)_{j\in J}$ *in* $U \setminus \{\overline{x}\}$ *converging to* $\overline{x}$ *with* $\|x_j - \overline{x}\|^{-1}(x_j - \overline{x}) \to u$ *and for any net* $(x_j^*)_j$ *converging weakly* to* x^* *in* X^* *with* $x_j^* \in M(x_j)$ *for all* $j \in J$*, one has*

$$\langle x^*, u\rangle = \sigma\big(M(\overline{x}), u\big),$$

where $\sigma\big(M(\overline{x}),\cdot\big)$ is the support function of $M(\overline{x})$.

(b) *The latter implication is an equivalence whenever X is finite-dimensional.*

Proof. (a) Assume that M is one-sided submonotone at $\overline{x}$ and let u, $(x_j)_j$ and $(x_j^*)_j$ as above. We note that there is some $j_0 \in J$ such that $(x_j^*)_{j\succeq j_0}$ is bounded in X^*. Then, according to the one-sided subsmonotonicity property and the local boundedness of M it ensues that, for any $y^* \in M(\overline{x})$

$$\langle x^* - y^*, u\rangle = \lim_{j\in J}\langle x_j^* - y^*, \frac{x_j - \overline{x}}{\|x_j - \overline{x}\|}\rangle \geq 0.$$

Since $x^* \in M(\overline{x})$ by outer semicontinuity of M at $\overline{x}$, it follows that

$$\langle x^*, u\rangle = \sup_{y^*\in M(\overline{x})}\langle y^*, u\rangle = \sigma\big(M(\overline{x}), u\big)$$

as desired.

(b) Now assume that X is finite-dimensional and that M is not one-sided submonotone at $\overline{x}$. There exists a real $\varepsilon > 0$, a sequence $(x_n)_n$ in $U \setminus \{\overline{x}\}$ converging to $\overline{x}$, sequences $(x_n^*)_n$ and $(y_n^*)_n$ with $x_n^* \in M(x_n)$ and $y_n^* \in M(\overline{x})$ such that

$$\langle x_n^* - y_n^*, \frac{x_n - \overline{x}}{\|x_n - \overline{x}\|}\rangle \leq -\varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Since X is finite-dimensional and M is bounded near $\overline{x}$, we may and do suppose that $\|x_n - \overline{x}\|^{-1}(x_n - \overline{x}) \to u$ with $\|u\| = 1$ and that $x_n^* \to x^*$ and $y_n^* \to y^*$. By outer semicontinuity of M at $\overline{x}$, we have both x^* and y^* in $M(\overline{x})$. It results that

$$\langle x^*, u\rangle \leq \langle y^*, u\rangle - \varepsilon \leq \sigma\big(M(\overline{x}), u\big) - \varepsilon,$$

which contradicts the property in (a). The converse implication in (a) is then justified. □

The second lemma shows the tangential regularity of locally Lipschitz functions with one-sided submonotone Clarke subdifferentials. Its proof uses the above lemma.

Lemma 4.24. *Let $f : U \to \mathbb{R}$ be a locally Lipschitz function on an open set U of a normed space X. If $\partial_C f$ is one-sided submonotone at $\overline{x} \in U$, then f is tangentially regular at $\overline{x}$.*

Proof. Fix any $u \in \mathbb{S}_X$ (if $X = \{0\}$ there is nothing to prove) . Since f is locally Lipschitz, there exists a sequence $(t_n)_n$ tending to 0 with $t_n > 0$ such that $f^B(\overline{x}; u) = \lim_{n\to\infty} t_n^{-1}[f(\overline{x} + t_n u) - f(\overline{x})]$. By the Lebourg mean value equality, for each $n \in \mathbb{N}$, there exists some $\theta_n \in]0,1]$ and $x_n^* \in \partial_C f(\overline{x} + t_n\theta_n u)$ such that $t_n^{-1}[f(\overline{x} + t_n u) - f(\overline{x})] = \langle x_n^*, u\rangle$ (see Proposition 2.1(b)). Take a subnet $(x_{s(j)}^*)_{j\in J}$ converging weakly* to some x^* (keep in mind that $\partial_C f$ is bounded near $\overline{x}$ since f is locally Lipschitz). Then, noting that $z_n := \overline{x} + t_n\theta_n u \to \overline{x}$ with $\|z_n - \overline{x}\|^{-1}(z_n - \overline{x}) \to u$ as $n \to \infty$, Lemma 4.23(a) ensures that

$$f^B(\overline{x}; u) = \lim_{j\in J}\langle x_{s(j)}^*, u\rangle = \langle x^*, u\rangle = \sigma\big(\partial_C f(\overline{x}), u\big) = f^o(\overline{x}; u).$$

This being true for all $u \in \mathbb{S}_X$, it ensues that f is tangentially regular at $\overline{x}$. □

Let us provide an example showing that the reverse implication in the above lemma is false.

Example 4.25 (**Spingarn example** [72]). Consider an even function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ which satisfies the following properties:

(i) $f(1/n) = 1/n - 1/n^2$ for every integer $n \geq 2$;

(ii) for each integer $n \geq 2$ the usual derivative f' exists on $]1/(n+1), 1/n[$ and is continuous and decreasing on $]1/(n+1), 1/n[$, $f'_+(1/(n+1)) = 1$ and $f'_-(1/n) = 0$;

(iii) $f(x) = 1/4$ for all $x \geq 1/2$.

The function f is Lipschitz, and noting that $|x| - x^2 \leq f(x) \leq |x|$ for all x it ensues that

$$f'(0; h) = |h| \quad \text{for all } h \in \mathbb{R}.$$

Further, the gradient representation theorem (see Proposition 2.1(d)) allows us to see that $\partial_C f(0) = [-1, 1]$, which gives $f^o(0; h) = |h|$ for all $h \in \mathbb{R}$. The function f is then tangentially regular at $\overline{x} := 0$.

On the other hand, taking $x_n := 1/n$, $x_n^* := 0 \in \partial_C f(x_n)$ and $\overline{x}^* := 1 \in \partial_C f(0)$ we see that

$$\frac{(x_n^* - \overline{x}^*)(x_n - \overline{x})}{|x_n - \overline{x}|} = -1,$$

so $\partial_C f$ is not one-sided submonotone at $\overline{x}$.

With Lemma 4.24 in particular at hands, we can now establish subdifferential characterizations of one-sided subsmoothness on an open set for tangentially regular locally Lipschitz functions.

Proposition 4.26. *Let U be a nonempty open subset of a normed space X and $f : U \to \mathbb{R}$ be a locally Lipschitz function. Consider the following assertions:*

(a) *The function f is one-sided subsmooth on U and tangentially regular on U.*

(b) *For each $\overline{x} \in U$ and each real $\varepsilon > 0$ there exists a real $\delta > 0$ with $B(\overline{x}, \delta) \subset U$ such that*

$$\langle x^*, y - x \rangle \leq f(y) - f(x) + \varepsilon \|y - x\|,$$

for all $x, y \in B(\overline{x}, \delta)$ with either $x = \overline{x}$ or $y = \overline{x}$ and for all $x^* \in \partial_C f(x)$.

(c) *The multimapping $\partial_C f$ is one-sided submonotone on U.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c), *and both implications are equivalences whenever X is finite-dimensional.*

Proof. (a)$\Rightarrow$(b). Fix any $\overline{x} \in U$ and any real $\varepsilon > 0$, and take $\delta > 0$ such that the property for one-sided subsmoothness in Proposition 3.11 is satisfied. Let any $x \in B(\overline{x}, \delta)$ and any $t \in]0, 1[$. Since $\overline{x} + t(x - \overline{x}) = tx + (1-t)\overline{x}$, we have

$$f(\overline{x} + t(x - \overline{x})) - f(\overline{x}) \leq t[\, f(x) - f(\overline{x}) + \varepsilon(1-t)\|x - \overline{x}\|\,].$$

Dividing by t and taking the limit inferior as $t \downarrow 0$ give

$$f^B(\overline{x}; x - \overline{x}) \leq f(x) - f(\overline{x}) + \varepsilon \|x - \overline{x}\|.$$

Similarly, with $s := 1 - t$ and $s \downarrow 0$ one obtains the same inequality with x and $\overline{x}$ mutually changed. Further, the tangential regularity of f on U tells us that $f^B(x; \cdot) = f^o(x; \cdot)$ and $f^B(\overline{x}; \cdot) = f^o(\overline{x}; \cdot)$. This and both inequalities concerning

$f^B(\cdot;\cdot)$ yield the property in (b).
(b)⇒(c). Fix any $\overline{x} \in U$ and any real $\varepsilon > 0$. Let $\delta > 0$ satisfying the property in (b) with $\varepsilon/2$ in place of ε. Applying the related inequality one time with $y = \overline{x}$ and with x and another time with $x = \overline{x}$ and with $y = x$, and adding the resulting inequalities we obtain

$$\langle x^* - \overline{x}^*, x - \overline{x}\rangle \geq -\varepsilon\|x - \overline{x}\|$$

for all $x^* \in \partial_C f(x)$ and all $\overline{x}^* \in \partial_C f(\overline{x})$. This translates the one-sided submonotonicity of $\partial_C f$ on U.

Now assume that X is finite-dimensional and that (c) holds. The tangential regularity of f follows from Lemma 4.24. Let us show the one-sided subsmoothness of f on U. Fix any $\overline{x} \in U$ and any real $\varepsilon > 0$. By (c) choose $\delta > 0$ with $B(\overline{x}, \delta) \subset U$ such that

$$\langle x^* - u^*, x - \overline{x}\rangle \geq -(\varepsilon/2)\|x - \overline{x}\|, \tag{4.12}$$

for all $x \in B(\overline{x}, \delta)$, all $x^* \in \partial_C f(x)$ and all $u^* \in \partial_C f(\overline{x})$. By the $\|\cdot\| - \|\cdot\|$-upper semicontinuity of $\partial_C f$ (keep in mind that X is finite-dimensional) we may also suppose that

$$\partial_C f(z) \subset \partial_C f(\overline{x}) + B(0, \varepsilon/2) \quad \text{for all } z \in B(\overline{x}, \delta). \tag{4.13}$$

Now fix any $x \in B(\overline{x}, \delta)$ with $x \neq \overline{x}$ and any $t \in]0, 1[$, and set $x_t := tx + (1 - t)\overline{x}$. By the Lebourg mean value equality (see Proposition 2.1(b)) there are $z_1 \in [x, x_t[$ and $z_1^* \in \partial_C f(z_1)$ along with $z_2 \in [\overline{x}, x_t[$ and $z_2^* \in \partial_C f(z_2)$ such that

$$\langle z_1^*, x_t - x\rangle = f(x_t) - f(x) \quad \text{and} \quad \langle z_2^*, x_t - \overline{x}\rangle = f(x_t) - f(\overline{x}).$$

Multiplying the first equality by t and the second by $(1-t)$, and adding the resulting equalities we get (noting that $x_t - x = (1 - t)(\overline{x} - x)$ and $x_t - \overline{x} = t(x - \overline{x})$)

$$tf(x) + (1 - t)f(\overline{x}) - f(x_t) = t(1 - t)\langle z_1^* - z_2^*, x - \overline{x}\rangle. \tag{4.14}$$

By (4.13) choose some $\overline{x}^* \in \partial_C f(\overline{x})$ such that

$$\|\overline{x}^* - z_2^*\| \leq \varepsilon/2, \tag{4.15}$$

and note by (4.12) that

$$\langle z_1^* - \overline{x}^*, \frac{z_1 - \overline{x}}{\|z_1 - \overline{x}\|}\rangle \geq -\varepsilon/2.$$

From the latter inequality and from (4.15) we deduce through the equality

$$\frac{z_1 - \overline{x}}{\|z_1 - \overline{x}\|} = \frac{x - \overline{x}}{\|x - \overline{x}\|}$$

that we have

$$\langle z_1^* - z_2^*, \frac{x - \overline{x}}{\|x - \overline{x}\|}\rangle \geq -\varepsilon.$$

Combining this with (4.14) it results that

$$tf(x) + (1 - t)f(\overline{x}) - f(x_t) \geq -\varepsilon t(1 - t)\|x - \overline{x}\|,$$

which translates (by Proposition 3.11) the one-sided subsmoothness of f on U. □

Remark 4.27. The proof of the above implication (c)$\Rightarrow$(a) shows that the locally Lipschitz function f is one-sided subsmooth at $\overline{x} \in U$ whenever $\partial_C f$ is one-sided submonotone at $\overline{x}$ and X is finite-dimensional.

In addition to the one-sided subsmoothness property, another notion of interest is that of semismoothness for locally Lipschitz functions.

Definition 4.28. Let U be a nonempty open subset of a normed space X and $f : U \to \mathbb{R}$ be a locally Lipschitz function. One says that f is *semismooth* (in the sense of Mifflin) at a point $\overline{x} \in U$ if for any $u \in \mathbb{S}_X$, any sequence $(x_n)_n$ in $U \setminus \{\overline{x}\}$ with

$$\lim_{n\to\infty} x_n = \overline{x} \quad \text{and} \quad \lim_{n\to\infty} \left\| \frac{x_n - \overline{x}}{\|x_n - \overline{x}\|} - u \right\| = 0,$$

one has

$$\langle x_n^*, u\rangle \to f'(\overline{x}; u) \quad \text{as } n \to \infty,$$

for any sequence $(x_n^*)_n$ with $x_n^* \in \partial_C f(x_n)$ for all $n \in \mathbb{N}$.

When f is semismooth at any point in an open set U_0 of U, one says that f is semismooth on U_0.

Remark 4.29. Although the semismoothness of a locally Lipschitz function f requires the existence of $f'(\overline{x}; \cdot)$, such a function f may fail to be tangentially regular at $\overline{x}$. The same Lipschitz function $f := -|\cdot|$ in Example 3.7 and Remark 4.7 is semismooth on $\mathbb{R}$ but not tangentially regular at $\overline{x} = 0$. Therefore, it is both semismooth and one-sided subsmooth on $\mathbb{R}$ but not tangentially regular at $\overline{x} = 0$.

Further, since $\partial_C f$ is obviously submonotone at any point in $\mathbb{R}\setminus\{0\}$, from Proposition 4.26 we derive that $\partial_C f$ is not one-sided submonotone at the origin. This can also be easily checked, taking $x_n := 1/n$, $x_n^* := -1 \in \partial_C f(x_n)$ and $\overline{x}^* := 1 \in \partial_C f(0)$ and noting that $\frac{(x_n^* - \overline{x}^*)(x_n - 0)}{|x_n - 0|} = -2$ for all $n \in \mathbb{N}$.

For a locally Lipschitz function, the semismoothness property is satisfied at a point whenever the Clarke subdifferential of the function is one-sided submonotone at that point.

Proposition 4.30. *Let U be a nonempty open set of a normed space X and $f : U \to \mathbb{R}$ be a locally Lipschitz function. The following hold:*

(a) *The function f is semismooth at $\overline{x} \in U$ and tangentially regular at $\overline{x}$ whenever $\partial_C f$ is one-sided submonotone at $\overline{x}$.*

(b) *If f is one-sided subsmooth on U and tangentially regular on U, then f is semismooth on U.*

Proof. (a) Assume that $\partial_C f$ is one-sided submonotone at $\overline{x}$, so f is tangentially regular at $\overline{x}$ by Lemma 4.24. Take any $u \in \mathbb{S}_X$, any sequence $(x_n)_n$ in $U \setminus \{\overline{x}\}$ converging to $\overline{x}$ with $\|x_n - \overline{x}\|^{-1}(x_n - \overline{x}) \to u$ as $n \to \infty$, and any sequence $(x_n^*)_n$ with $x_n^* \in \partial_C f(x_n)$ for all $n \in \mathbb{N}$. Since $\partial_C f$ is one-sided submonotone at $\overline{x}$, we have

$$\liminf_{n\to\infty} \inf_{z^*\in\partial_C f(x_n)} \langle z^*, \frac{x_n - \overline{x}}{\|x_n - \overline{x}\|}\rangle \geq \limsup_{n\to\infty} \sup_{y^*\in\partial_C f(\overline{x})} \langle y^*, \frac{x_n - \overline{x}}{\|x_n - \overline{x}\|}\rangle.$$

This combined with the boundedness of $(x_n^*)_n$ and the continuity of $f^o(\overline{x};\cdot)$ yields

$$\liminf_{n\to\infty}\langle x_n^*, u\rangle \geq f^o(\overline{x};u).$$

Further, the upper semicontinuity of $f^o(\cdot;u)$ at $\overline{x}$ assures us that

$$\limsup_{n\to\infty}\langle x_n^*, u\rangle \leq \limsup_{n\to\infty} f^o(x_n;u) \leq f^o(\overline{x};u).$$

We deduce that

$$\langle x_n^*, u\rangle \longrightarrow f^o(\overline{x};u) = f'(\overline{x};u),$$

so f is semismooth at $\overline{x}$.
(b) The assertion (b) follows from (a) and from the implication (a)⇒(c) in Propososition 4.26. □

In the context of finite-dimensional normed spaces, the implication in the assertion (a) in the above proposition is an equivalence.

Proposition 4.31. *Let U be a nonempty open set of a finite-dimensional normed space X. Let $f : U \to \mathbb{R}$ be a locally Lipschitz function and let $\overline{x} \in U$. Then f is semismooth at $\overline{x}$ and tangentially regular at $\overline{x}$ if and only if $\partial_C f$ is one-sided submonotone at $\overline{x}$.*

Proof. According to Proposition 4.30(a), we only need to prove the implication ⇒. So, assume that f is semismooth at $\overline{x}$ and tangentially regular at $\overline{x}$. It suffices to show that the sequential property in (a) in Lemma 4.23 is satisfied for the multimapping $\partial_C f$. Take any $u \in \mathbb{S}_X$, any sequence $(x_n)_n$ in $U \setminus \{\overline{x}\}$ converging to $\overline{x}$ with $\|x_n - \overline{x}\|^{-1}(x_n - \overline{x}) \to u$ as $n \to \infty$, and any sequence $(x_n^*)_n$ converging to x^* with $x_n^* \in \partial_C f(x_n)$ for all $n \in \mathbb{N}$. By outer semicontinuity of $\partial_C f$ at $\overline{x}$ we have $x^* \in \partial_C f(\overline{x})$. Further, by the semismoothness of f at $\overline{x}$

$$\langle x^*, u\rangle = \lim_{n\to\infty}\langle x_n^*, u\rangle = f'(\overline{x};u),$$

so $\langle x^*, u\rangle = f^o(\overline{x};u) = \sigma\big(\partial_C f(\overline{x}), u\big)$, which is the desired property. □

The next corollary is a direct consequence of the above proposition and of Proposition 4.26.

Corollary 4.32. *let U be a nonempty open set of a finite-dimensional normed space X and $f : U \to \mathbb{R}$ be a locally Lipschitz function. The following assertions are equivalent:*

(a) *The function f is one-sided subsmooth on U and tangentially regular on U.*
(b) *The subdifferential multimapping $\partial_C f$ is one-sided submonotone on U.*
(c) *The function f is semismooth on U and tangentially regular on U.*

5. Subsmooth sets

Given a nonempty closed set S of a normed space X, the subsmoothness property in (3.1) for its indicator function ψ_S at $\overline{x} \in S$ is evidently equivalent to the convexity of $S \cap B(\overline{x}, \delta)$ for some $\delta > 0$ (see also Proposition 3.4). Now suppose that X is a Banach space. By Theorem 4.15 the subsmoothness of the indicator function of the closed set S at $\overline{x} \in S$ amounts to saying that $\partial_C \psi_C(\cdot) = N^C(S;\cdot)$ is submonotone at $\overline{x}$. This means that, for each $\varepsilon > 0$ there exists a real $\delta > 0$ such that for all $x_i \in S \cap B(\overline{x}, \delta)$ and all $x_i^* \in N^C(S; x_i)$, $i = 1, 2$, one has

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\|,$$

which by the positive homogeneity of the C-normal cone gives for every real $t > 0$

$$\langle tx_1^* - tx_2^*, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\|, \text{ or equivalently } \langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\frac{\varepsilon}{t} \|x_1 - x_2\|,$$

so $\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq 0$ by taking the limit as $t \to +\infty$. Consequently, we also see the convexity of the set $S \cap B(\overline{x}, \delta)$ via the latter inequality, since the monotonicity of the C-subdifferentials is a characterization of the convexity of proper lower semicontinuous functions on Banach spaces (see [22, 23], and the previous paper [60] in finite dimensions).

Clearly, from a geometric point of view, the convexity property of $S \cap B(\overline{x}, \delta)$ is not enough relevant for (locally) nonconvex sets; the property is not fulfilled even for C^2 submanifolds.

5.1. **Definition of subsmooth sets and general properties.** Recall that a characterization of the r-prox-regularity at $\overline{x} \in S$ of a subset of a Hilbert space H which is closed near $\overline{x}$ (resp. the uniform r-prox-regularity of a closed set S of H) is that $N^C(S; \cdot) \cap \mathbb{B}$ is $\frac{1}{r}$-hypomonotone at $\overline{x}$ (resp. over S), that is, for all $x, y \in S \cap V$ with some neighborhood V of $\overline{x}$ (resp. for all $x, y \in S$) one has for all $x^* \in N^C(S; x) \cap \mathbb{B}$ and $y^* \in N^C(S; y) \cap \mathbb{B}$ (see [63])

$$\langle x^* - y^*, x - y \rangle \geq -\frac{1}{r} \|x - y\|^2. \tag{5.1}$$

Taking into account the analysis at the end of the previous section, by virtue of the positive homogeneity of $N^C(S; \cdot)$ and following the foregoing characterization of local prox-regular sets, we define subsmooth sets through the submonotonicity of the truncation of this multimapping $N^C(S; \cdot)$ with the closed unit ball.

Definition 5.1. A subset S of a normed space $(X, \|\cdot\|)$ is called *subsmooth at a point* $\overline{x} \in S$ if the multimapping $N^C(S; \cdot) \cap \mathbb{B}_{X^*}$ is submonotone at $\overline{x}$, or equivalently provided that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle x^* - y^*, x - y \rangle \geq -\varepsilon \|x - y\| \tag{5.2}$$

for all $x, y \in S \cap B(\overline{x}, \delta)$, all $x^* \in N^C(S; x) \cap \mathbb{B}_{X^*}$ and all $y^* \in N^C(S; y) \cap \mathbb{B}_{X^*}$. The set S is called *subsmooth* when it is subsmooth at every point in S.

When for each $\varepsilon > 0$ there is $\delta > 0$ such that (5.2) is satisfied for all $x, y \in S$ with $\|x - y\| < \delta$ and all $x^* \in N^C(S; x) \cap \mathbb{B}_{X^*}$ and all $y^* \in N^C(S; x) \cap \mathbb{B}_{X^*}$, one says that the set S is *uniformly subsmooth.* The set S is *uniformly subsmooth near* $\overline{x} \in S$ if there exists an open neighborhood U of $\overline{x} \in S$ such that the set $S \cap U$ is uniformly submooth.

Clearly, the set S is subsmooth at $\overline{x}$ (resp. the set S is uniformly subsmooth) if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\langle x^*, y - x \rangle \leq \varepsilon \|y - x\| \tag{5.3}$$

for all $x, y \in S \cap B(\overline{x}; \delta)$ and all $x^* \in N^C(S; \overline{x}) \cap \mathbb{B}_{X^*}$ (resp. for all $x, y \in S$ with $\|x - y\| < \delta$ and all $x^* \in N^C(S; \overline{x}) \cap \mathbb{B}_{X^*}$). Similarly, the set S is uniformly subsmooth near $\overline{x}$ if there is some open neighborhood U of $\overline{x}$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ for which (5.3) holds for all $x, y \in S \cap U$ with $\|x - y\| < \delta$ and all $x^* \in N^C(S; \overline{x}) \cap \mathbb{B}_{X^*}$.

From the monotonicity of the normal cone of a convex set and from the hypomonotonicity of the truncated C-normal cone of prox-regular sets (see (5.1)) we directy obtain:

Proposition 5.2. (a) *Any convex set of a normed space is uniformly subsmooth.*
(b) *If a subset S of a Hilbert space is uniformly r-prox-regular (resp.r-prox-regular at $\overline{x} \in S$), then S is uniformly subsmooth (resp. uniformly subsmooth near $\overline{x}$).*

The uniform equi-subsmoothness for families of sets need also to be defined.

Definition 5.3. A family $(S_i)_{i \in I}$ of sets of a normed space X is said to be *uniformly equi-subsmooth* if for any $\varepsilon > 0$ there is some $\delta > 0$ such that for any $i \in I$, any $x, y \in S_i$, any $x^* \in N^C(S_i; x) \cap \mathbb{B}_{X^*}$ and any $y^* \in N^C(S_i; y) \cap \mathbb{B}_{X^*}$ one has

$$\langle x^* - y^*, x - y \rangle \geq -\varepsilon \|x - y\|.$$

Clearly, this is equivalent to require that for any $\varepsilon > 0$ there is $\delta > 0$ such that for each $i \in I$ the inequality

$$\langle x^*, y - x \rangle \leq \varepsilon \|y - x\|$$

holds for all $x, y \in S_i$ and all $x^* \in N^C(S_i; x) \cap \mathbb{B}_{X^*}$.

The following proposition is obvious.

Proposition 5.4. *Let $(S_i)_{i \in I}$ be a family of sets of a normed space X.*
(a) *If all the sets S_i are convex, then $(S_i)_{i \in I}$ is a family of sets uniformly equi-subsmooth.*
(b) *If X is a Hilbert space and all the sets S_i are r-prox-regular with a common constant $r \in]0, +\infty]$, then $(S_i)_{i \in I}$ is a family of sets uniformly equi-subsmooth.*

Now using (5.3) we show that a subsmooth set S at $\overline{x} \in S$ enjoys the property that $N^C(S; \cdot)$ is sequentially norm-to-weak* closed at $\overline{x}$. Such a property is in general desired in the study of dynamical system governed by normal cones.

Proposition 5.5. *Let S be a subset of a Banach space $(X, \|\cdot\|)$ which is subsmooth at $\overline{x} \in S$. Then the multimapping $N^C(S; \cdot)$ is sequentially norm-to-weak* closed at the point $\overline{x}$.*

Proof. Let any sequences $(x_n)_n$ in S converging to $\overline{x}$ and $(x_n^*)_n$ in X^* weak-star converging to $x^* \in X^*$ with $x_n^* \in N^C(S; x_n)$ for all $n \in \mathbb{N}$. Since X is a Banach space, there exists a real $\beta > 0$ such that $\|x_n^*\| \leq \beta$ for all $n \in \mathbb{N}$. Take any real $\varepsilon > 0$. There is a real $\delta > 0$ such that (5.3) is satisfied with $\beta^{-1}\varepsilon$ in place of ε. Let $n_0 \in \mathbb{N}$ be such that for every integer $n \geq n_0$ one has $x_n \in B(\overline{x}, \delta)$. Then for each integer $n \geq n_0$ we see that for every $y \in S \cap B(\overline{x}, \delta)$

$$\langle \beta^{-1} x_n^*, y - x_n \rangle \leq \beta^{-1} \varepsilon \|y - x_n\|,$$

hence $\langle x_n^*, y-x_n\rangle \le \varepsilon\|y-x_n\|$. Taking the limit as $n \to \infty$ ensures that $\langle x^*, y-\overline{x}\rangle \le \varepsilon\|y-\overline{x}\|$ for every $y \in S\cap B(\overline{x},\delta)$. This entails that $x^* \in N^F(S;\overline{x})$, thus in particular $x^* \in N^C(S;\overline{x})$, which justifies the desired closedness property of $N^C(S;\cdot)$ at $\overline{x}$. □

Requiring $y := \overline{x}$ in (5.2) yields with a radial counterpart of the above concept. The exact definition is as follows:

Definition 5.6. A subset S of a normed space $(X, \|\cdot\|)$ is called *one-sided subsmooth at* a point $\overline{x} \in S$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle x^* - \overline{x}^*, x - \overline{x}\rangle \ge -\varepsilon\|x - \overline{x}\| \tag{5.4}$$

for all $x \in S \cap B(\overline{x}, \delta)$, all $x^* \in N^C(S; x) \cap \mathbb{B}_{X^*}$ and all $\overline{x}^* \in N^C(S; \overline{x}) \cap \mathbb{B}_{X^*}$. The set S is called *one-sided subsmooth* when it is one-sided subsmooth at every point in S.

Remark 5.7.
(a) It is clear that the definition of subsmooth (resp. one-sided subsmooth) sets is unchanged if any equivalent norm on X is used in place of $\|\cdot\|$.
(b) If S is subsmooth at $\overline{x}$, then it is one-sided subsmooth at $\overline{x}$.
(c) Any set S is subsmooth at any point in $\operatorname{int} S$.
(d) The natural definition of uniform one-sided subsmoothness of the set S obviously yields to the above notion of uniform subsmoothness for S.

Note that the above definition of one-sided subsmoothness obviously amounts to requiring for any $\varepsilon > 0$ the existence of some $\delta > 0$ such that for all $x \in S \cap B(\overline{x}, \delta)$, all $x^* \in N(S; x) \cap \mathbb{B}_{X^*}$, and all $\overline{x}^* \in N(S; \overline{x}) \cap \mathbb{B}_{X^*}$ both inequalities

$$\langle x^*, \overline{x} - x\rangle \le \varepsilon\|\overline{x} - x\| \quad \text{and} \quad \langle \overline{x}^*, x - \overline{x}\rangle \le \varepsilon\|x - \overline{x}\|$$

are satisfied.

If only the second one of the two latter inequalities is required, we obtain another concept that we call hemi-subsmoothness.

Definition 5.8. A subset S of a normed space $(X, \|\cdot\|)$ is called *hemi-subsmooth at* a point $\overline{x} \in S$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle \overline{x}^*, x - \overline{x}\rangle \le \varepsilon\|x - \overline{x}\|, \tag{5.5}$$

for all $x \in S \cap B(\overline{x}, \delta)$ and all $\overline{x}^* \in N^C(S; \overline{x}) \cap \mathbb{B}_{X^*}$.

Obviously a set S is subsmooth (resp. one-sided subsmooth, hemi-subsmooth) at $\overline{x} \in S$ if and only if there is some neigbborhood W of $\overline{x}$ such that $S \cap W$ enjoys the same property at $\overline{x}$. The same property also holds for the uniform subsmoothness of S near $\overline{x} \in S$.

Taking the positive homogeneity of $N^C(S; x)$ into account, we directly obtain:

Proposition 5.9. *Let S be a nonempty subset of a normed space X and let $\overline{x} \in S$. The following hold:*

(a) *The set S is subsmooth at $\overline{x}$ if and only if for any reals $r > 0$ and $\varepsilon > 0$ there exists a real $\delta > 0$ such that*

$$\langle x^* - y^*, x - y\rangle \ge -\varepsilon\|x - y\|, \tag{5.6}$$

for all $x, y \in S \cap B(\overline{x}, \delta)$, all $x^* \in N^C(S; x) \cap r\mathbb{B}_{X^*}$ and all $y^* \in N^C(S; y) \cap r\mathbb{B}_{X^*}$.

(b) *The set S is one-sided subsmooth at $\overline{x}$ if and only if for any reals $r > 0$ and $\varepsilon > 0$ there exists a real $\delta > 0$ such that*

$$\langle x^* - \overline{x}^*, x - \overline{x} \rangle \geq -\varepsilon \|x - \overline{x}\|, \tag{5.7}$$

for all $x \in S \cap B(\overline{x}, \delta)$, all $x^* \in N^C(S; x) \cap r\mathbb{B}_{X^*}$ and all $\overline{x}^* \in N^C(S; \overline{x}) \cap r\mathbb{B}_{X^*}$.

(c) *A similar equivalence holds for uniform submoothness of S (resp. uniform subsmoothness of S near $\overline{x}$, hemi-subsmoothness of S at $\overline{x}$).*

We already saw that convex sets are subsmooth (even uniformly submooth). In fact, a remarkable class of examples of subsmooth sets is given by inverse images of convex sets with C^1 mappings with surjective derivatives between Banach spaces. The proof will use the next lemma.

Lemma 5.10. *Let $A : X \to Y$ be a continuous linear mapping between two Banach spaces X, Y and let $G : X \to Y$ be a mapping which is of class C^1 near a point $\overline{x} \in X$.*

(a) *If there is a real $s > 0$ satisfying $s\mathbb{U}_Y \subset A(\mathbb{B}_X)$, then for any $x^* \in X^*$ and $y^* \in Y^*$ with $x^* = y^* \circ A$ one has*

$$s\|y^*\| \leq \|x^*\|.$$

(b) *If there is a real $s > 0$ satisfying $s\mathbb{B}_Y \subset DG(\overline{x})(\mathbb{B}_X)$, then for any real $\eta > 0$ there is an open neighborhood U of $\overline{x}$ such that for every $x \in U$ the inclusion $s'\mathbb{B}_Y \subset DG(x)(\mathbb{B}_X)$ holds with $s' := (1 + \eta)^{-1}s$, and hence in particular for all $x \in U$, $x^* \in X^*$ and $y^* \in Y^*$ with $x^* = y^* \circ DG(x)$ one has*

$$s\|y^*\| \leq (1 + \eta)\|x^*\|.$$

Proof. (a) Fix any $x^* \in X^*$ and $y^* \in Y^*$ such that $x^* = y^* \circ A$. Take any $v \in \mathbb{U}_Y$ and choose $u \in \mathbb{B}_X$ such that $sv = A(u)$. We notice that

$$\langle y^*, sv \rangle = \langle y^*, A(u) \rangle = \langle x^*, u \rangle \leq \|x^*\|.$$

This being true for all $v \in \mathbb{U}_Y$, we obtain that $s\|y^*\| \leq \|x^*\|$.
(b) Put $A := DG(\overline{x})$ and $s' := (1 + \eta)^{-1}s$ and choose an open convex neighborhood U of $\overline{x}$ over which G is C^1 and such that $\|DG(x) - DG(\overline{x})\| < s - s'$ for all $x \in U$. Fix any $x \in U$ and put $\Lambda := DG(x)$. Writing

$$s'\mathbb{B}_Y + (s - s')\mathbb{B}_Y \subset A(\mathbb{B}_X) \subset \Lambda(\mathbb{B}_X) + (A - \Lambda)(\mathbb{B}_X) \subset \Lambda(\mathbb{B}_X) + (s - s')\mathbb{B}_Y,$$

we see by taking support functions that $s'\mathbb{B}_Y \subset \mathrm{cl}_Y\big(\Lambda(\mathbb{B}_X)\big)$, and hence $s'\mathbb{U}_X \subset \Lambda(\mathbb{B}_X)$ by the Banach open mapping theorem. Therefore, taking any $x^* \in X^*$ and $y^* \in Y^*$ such that $x^* = y^* \circ DG(x)$, the assertion (a) tells us that $s'\|y^*\| \leq \|x^*\|$, which justifies the assertion (b). □

Proposition 5.11. (a) *Any C^1 submanifold of a Banach space is subsmooth.*

(b) *If $G : X \to Y$ is a mapping between Banach spaces which is of class C^1 near a point $\overline{x} \in X$ with $DG(\overline{x})$ surjective and if C is a convex set of Y containing $G(\overline{x})$, then the set $G^{-1}(C)$ is subsmooth at $\overline{x}$.*

Proof. (b) We begin by proving (b). By the Banach open mapping theorem there is a real $s > 0$ such that $s\mathbb{B}_Y \subset DG(\overline{x})(\mathbb{B}_Y)$. Then by the above lemma there are an open neighborhood U of $\overline{x}$ and a real $\gamma > 0$ such that for each $x \in U$ the continuous linear mapping $DG(x)$ is open and $\|y^*\| \le \gamma\|x^*\|$ for all $x^* \in X^*$ and $y^* \in Y^*$ satisfying $x^* = y^* \circ DG(x)$. Now fix any $\varepsilon > 0$ and choose an open convex neighborhood $U_0 \subset U$ of $\overline{x}$ such that $\|DG(x') - DG(x)\| \le \varepsilon/\gamma$ for all $x, x' \in U_0$. Consider any $x, u \in U_0 \cap G^{-1}(C)$ and $x^* \in N^C\left(G^{-1}(C); x\right) \cap \mathbb{B}_{X^*}$. We know by C-subdifferential calculus (see [18]) that there is $y^* \in N^C(C; G(x))$ such that $x^* = y^* \circ DG(x)$, so $\|y^*\| \le \gamma$ by the choice of γ. Since $\langle y^*, G(u) - G(x)\rangle \le 0$, we deduce that

$$\begin{aligned}\langle x^*, u - x\rangle &= \langle y^*, DG(x)(u - x)\rangle \\ &= \langle y^*, G(u) - G(x)\rangle - \langle y^*, \int_0^1 (DG(x + t(u - x)) - DG(x))(u - x)\, dt\rangle \\ &\le \|y^*\|(\varepsilon/\gamma)\|u - x\|,\end{aligned}$$

hence $\langle x^*, u - x\rangle \le \varepsilon\|u - x\|$. This establishes the subsmoothness of $G^{-1}(C)$ at $\overline{x}$.
(a) Let S be a C^1 submanifold of the Banach space X. We know that there is a closed vector subspace E of X such that for each point $\overline{x} \in S$ there exist open neighborhoods U of $\overline{x}$ in X and V of zero in Y along with a C^1 diffeomorphism $\Phi : U \to V$ with $\Phi(\overline{x}) = 0$ such that $\Phi(S \cap U) = E \cap V$. Shrinking the open neighborhood V of zero if necessary, we may suppose that it is convex. Then by (a) the set S is subsmooth at $\overline{x}$, which finishes the proof. ☐

The properties of uniform subsmoothness and hemi-smoothness for sets can be characterized via modulus functions (whose definition has been recalled before the statement of Proposition 4.19).

Proposition 5.12. *Let S be a nonempty set of a normed space X.*

(a) *The set S is uniformly subsmooth (resp. uniformly subsmooth near a point $\overline{x} \in S$) if and only if there exists a modulus function $\omega : [0, +\infty[\to [0, +\infty]$ (resp. there exist an open neighborhood U of $\overline{x}$ and a modulus function $\omega : [0, +\infty[\to [0, +\infty[$ of class C^1 on $]0, +\infty[$ with $t\,\omega'(t) \to 0$ as $t \downarrow 0$) such that*
$$\langle x^* - y^*, x - y\rangle \ge -\|x - y\|\omega(\|x - y\|)$$
for all $x, y \in S$ (resp. $x, y \in S \cap U$), all $x^* \in N^C(S; x) \cap \mathbb{B}_{X^*}$ and all $y^* \in N^C(S; y) \cap \mathbb{B}_{X^*}$.

(b) *The set S is hemi-subsmooth at a point $\overline{x} \in S$ if and only if there exists a modulus function $\omega : [0, +\infty[\to [0, +\infty]$ such that*
$$\langle \overline{x}^*, x - \overline{x}\rangle \le \|x - \overline{x}\|\omega(\|x - \overline{x}\|)$$
for all $x \in S$ and all $\overline{x}^* \in N^C(S; \overline{x}) \cap \mathbb{B}_{X^*}$.

(c) *If X is finite-dimensional, the set S is uniformly subsmooth near $\overline{x}$ if and only if it is subsmooth near $\overline{x}$.*

Proof. (a) The implication $\Leftarrow$ is obvious. Let us first prove the reverse implication in the case when S is uniformly subsmooth. Define $\omega : [0, +\infty[\to [0, +\infty]$ by $\omega(0) := 0$

and for $t > 0$

$$\omega(t) := \sup\{\frac{\langle x^* - y^*, y - x\rangle^+}{\|x - y\|} : 0 < \|x - y\| \le t, (x, x^*), (y, y^*) \in \operatorname{gph} N^C(S; \cdot) \cap \mathbb{B}\},$$

where (as usual) $r^+ := \max\{0, r\}$ for $r \in \mathbb{R}$ and where we use the convention that the supremum is 0 whenever the set over which it is taken is empty, that is, S is a singleton. Clearly, the definition of uniform subsmoothness of S guarantees that $\omega(t) \to 0$ as $t \downarrow 0$ and by the very definition of ω the inequality in the proposition holds true for all $x, y \in S$, $x^* \in N^C(S; x) \cap \mathbb{B}$ and $y^* \in N^C(S; y) \cap \mathbb{B}$. The case of uniform subsmoothness near $\overline{x}$ follows from what precedes and from Lemma 4.18.
(b) Similarly, for the implication $\Rightarrow$ it suffices to define $\omega : [0, +\infty[\to [0, +\infty]$ by $\omega(0) := 0$ and for $t > 0$

$$\omega(t) := \sup\{\frac{\langle x^*, x - \overline{x}\rangle^+}{\|x - \overline{x}\|} : x \in S, 0 < \|x - \overline{x}\| \le t, x^* \in N^C(S; \overline{x}) \cap \mathbb{B}\},$$

and to argue like in (a).
(c) For the implication (c)$\Rightarrow$(a) when X is finite-dimensional, it suffices to proceed like in the proof of the implication (d)$\Rightarrow$(a) in Proposition 4.20. □

Remark 5.13. It is known (see [63, 21]) that the inequality in (a) of the above proposition with the particular modulus $\sigma|\cdot|^2$ characterizes in Hilbert spaces the fundamental class of closed uniformly (resp. locally) prox-regular sets.

The hemi-subsmoothness of a set entails its tangential regularity.

Proposition 5.14. *Let S be a subset of a normed space X and let $\overline{x} \in S$.*

- (a) *The subsmoothness of S at $\overline{x}$ entails its one-sided subsmoothness at $\overline{x}$, which in turn entails the hemi-subsmoothness at $\overline{x}$.*
- (b) *If the set S is hemi-subsmooth at $\overline{x}$, then it enjoys the normal regularity*

$$N^C(S; \overline{x}) = N^F(S; \overline{x}),$$

and hence it is tangentially regular at $\overline{x}$.

Proof. The assertion (a) is evident. To justify (b), assume that S is hemi-subsmooth at $\overline{x}$. Take any $\overline{x}^* \in N^C(S; \overline{x})$ and let $r > \|\overline{x}^*\|$. Fix any real $\varepsilon > 0$ and choose $\delta > 0$ satisfying the property related to hemi-subsmoothness in Proposition 5.9. Fixing any $x \in S \cap B(\overline{x}, \delta)$ we have

$$\langle \overline{x}^*, x - \overline{x}\rangle \le \varepsilon \|x - \overline{x}\|.$$

This tells us that $\overline{x}^*$ is a Fréchet normal of S at $\overline{x}$, that is, the inclusion $N^C(S; \overline{x}) \subset N^F(S; \overline{x})$ holds. In fact, the latter inclusion is an equality since the reverse inclusion always holds. Using this equality and the inclusion $N^F(S; \overline{x}) \subset \big(T^B(S; \overline{x})\big)^o$ (see Section 2) we obtain

$$\big(T^C(S; \overline{x})\big)^o = N^C(S; \overline{x}) \subset \big(T^B(S; \overline{x})\big)^o,$$

hence $T^B(S; \overline{x}) \subset T^C(S; \overline{x})$ since $T^C(S; \overline{x})$ is a closed convex cone. The latter inclusion justifies the desired tangential regularity of S at $\overline{x}$. □

Converses of above assertions will be discussed later.

Local subsmoothness of sets can be characterized with C-subdifferentials of distance functions. The following lemma will be useful for that.

Lemma 5.15. *Let S be a subset of a Banach space X and let $\overline{x} \in S$. If $\partial_C d_S(\cdot)$ in place of $N^C(S;\cdot)$ satisfies* (5.4) *for all $x \in S \cap B(\overline{x},\delta)$, $x^* \in \partial_C d_S(x)$, $\overline{x}^* \in \partial_C d_S(\overline{x})$, then*

$$\partial_C d_S(\overline{x}) = \partial_F d_S(\overline{x}) \quad \textit{and} \quad N^C(S;\overline{x}) = N^F(S;\overline{x}).$$

Proof. Fix any $\overline{x}^* \in \partial_C d_S(\overline{x})$. Then for any real $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in S \cap B(\overline{x},\delta)$ we have $\langle -\overline{x}^*, x - \overline{x}\rangle \geq -\varepsilon\|x-\overline{x}\|$, since $0 \in \partial_C d_S(x)$. This implies that $\overline{x}^* \in N^F(S;\overline{x})$. Moreover, the inclusion $\overline{x}^* \in \partial_C d_S(\overline{x})$ ensures that $\|\overline{x}^*\| \leq 1$. This combined with the equality $\partial_F d_S(\overline{x}) = N^F(S;\overline{x}) \cap \mathbb{B}_{X^*}$ (see Section 2) entails that $\partial_C d_S(\overline{x}) \subset \partial_F d_S(\overline{x})$. The reverse inclusion being always true, it ensues that the first equality $\partial_C d_S(\overline{x}) = \partial_F d_S(\overline{x})$ of the proposition is established.

Concerning the second equality, observe first that the first equality ensures in particular that $\partial_F d_S(\overline{x})$ is $w(X^*,X)$-closed, and hence for every real $r > 0$ the convex set

$$N^F(S;\overline{x}) \cap r\mathbb{B}_{X^*} = r\partial_F d_S(\overline{x})$$

is $w(X^*,X)$-closed. The space X being a Banach space, the Krein-Šmulian theorem guarantees that the convex set $N^F(S;\overline{x})$ is $w(X^*,X)$-closed. Further, the first equality again combined with the equalities (see Section 2)

$$N^C(S;\overline{x}) = \mathrm{cl}_{w^*}\big(\mathbb{R}_+\partial_C d_S(\overline{x})\big) \quad \text{and} \quad N^F(S;\overline{x}) = \mathbb{R}_+\partial_F d_S(\overline{x})$$

gives the equality

$$N^C(S;\overline{x}) = \mathrm{cl}_{w^*}\big(N^F(S;\overline{x})\big).$$

This equality and the above $w(X^*,X)$-closedness of $N^F(S;\overline{x})$ justifies the desired second equality $N^C(S;\overline{x}) = N^F(S;\overline{x})$. □

Proposition 5.16. *Let S be a nonempty subset of a Banach space X and U be a nonempty open set of X with $U \cap S \neq \emptyset$.*

- (A) *The following are equivalent:*
  - (a) *The set S is subsmooth at each point of $S \cap U$.*
  - (b) *For any $\overline{x} \in S \cap U$ and any $\varepsilon > 0$ there is $\delta > 0$ such that* (5.2) *holds on $S \cap B(\overline{x},\delta)$ with $\partial_C d_S$ in place of $N^C(S;\cdot) \cap \mathbb{B}_{X^*}$.*
- (B) *The set S is uniformly subsmooth if and only if for each $\varepsilon > 0$ there exists a real $\delta > 0$ such that $\langle x^* - y^*, x - y\rangle \geq -\varepsilon\|x-y\|$ for all $x, y \in S$ with $\|x-y\| < \delta$, $x^* \in \partial_C d_S(x)$ and $y^* \in \partial_C d_S(y)$.*

Proof. We prove only (A), since (B) follows in the same way. The implication (a)⇒(b) is evident. To prove the converse, suppose that (b) holds. By Lemma 5.15 we have, for all $x \in S \cap U$

$$\partial_C d_S(x) = \partial_F d_S(x) \quad \text{and} \quad N^C(S;x) = N^F(S;x).$$

Both equalities combined with the equality $\partial_F d_S(x) = N^F(S;x) \cap \mathbb{B}_{X^*}$ (for $x \in S$) yield $N^C(S;x) \cap \mathbb{B}_{X^*} = \partial_C d_S(x)$ for all $x \in S \cap U$. This and the assumption (b) entail that (a) holds true. □

Alternative characterizations of local subsmoothness of sets in Asplund spaces can be established via Fréchet normals or via subdifferentials of distance functions. A lemma is needed first, and it has its own interest.

Lemma 5.17. *Let S be a set of an Asplund space which is closed near $\overline{x} \in S$. The following assertions are equivalent:*

(a) *For any $\varepsilon > 0$ there is $\delta > 0$ such that* (5.2) *holds on $S \cap B(\overline{x}, \delta)$ with $\partial_F d_S$ in place of $N^C(S;\cdot) \cap \mathbb{B}_{X^*}$.*

(b) *For any $\varepsilon > 0$ there is $\delta > 0$ such that* (5.2) *holds on $S \cap B(\overline{x}, \delta)$ with $\partial_L d_S$ in place of $N^C(S;\cdot) \cap \mathbb{B}_{X^*}$.*

(c) *For any $\varepsilon > 0$ there is $\delta > 0$ such that* (5.2) *holds on $S \cap B(\overline{x}, \delta)$ with $\partial_C d_S$ in place of $N^C(S;\cdot) \cap \mathbb{B}_{X^*}$.*

Proof. The implication (c)$\Rightarrow$(a) follows directly from the inclusion $\partial_F d_S(\cdot) \subset \partial_C d_S(\cdot)$. Let us show the implication (a)$\Rightarrow$(b). Fix any real $\varepsilon > 0$ and by (a) choose $\delta > 0$ such that for all $u, v \in S \cap B(\overline{x}, \delta)$, $u^* \in \partial_F d_S(u)$ and $v^* \in \partial_F d_S(v)$

$$\langle u^* - v^*, u - v\rangle \geq -\varepsilon\|u - v\|.$$

Fix any $x, y \in S \cap B(\overline{x}, \delta)$, $x^* \in \partial_L d_S(x)$, $y^* \in \partial_L d_S(y)$. By Proposition 2.5(e) there are $x_n \to_S x$, $y_n \to_S y$, $(x_n^*)_n$ and $(y_n^*)_n$ converging weakly* to x^* and y^* respectively, with $x_n^* \in \partial_F d_S(x_n)$ and $y_n^* \in \partial_F d_S(y_n)$. For n large enough we have $x_n, y_n \in S \cap B(\overline{x}, \delta)$, and hence by the above inequality

$$\langle x_n^* - y_n^*, x_n - y_n\rangle \geq -\varepsilon\|x_n - y_n\|.$$

Passing to the limit as $n \to \infty$, it ensues that

$$\langle x^* - y^*, x - y\rangle \geq -\varepsilon\|x - y\|,$$

which corresponds to (b).

To finish the proof, it remains to prove that (b)$\Rightarrow$(c). Note that for each real $\varepsilon > 0$ and for any fixed $x, y \in S \cap B(\overline{x}, \delta)$, the set of $(x^*, y^*) \in X^* \times X^*$ satisfying the inequality $\langle x^* - y^*, x - y\rangle \geq -\varepsilon\|x - y\|$ is convex and weakly* closed in $X^* \times X^*$. The result in (c) then follows from the equality $\partial_C d_S(x) = \overline{\text{co}}^*\big(\partial_L d_S(x)\big)$ (see Proposition 2.5(f)). □

We are now able to characterize local subsmoothness of sets via the Fréchet normal cone in Asplund space.

Proposition 5.18. *Let S be a nonempty closed set of an Asplund space X and U be a nonempty open set of X with $U \cap S \neq \emptyset$. The following are equivalent:*

(a) *The set S is subsmooth at each point of $S \cap U$.*

(b) *The multimapping $N^L(S\ \cdot) \cap \mathbb{B}_{X^*}$ is submonotone at each point of $S \cap U$.*

(c) *The multimapping $N^F(S\ \cdot) \cap \mathbb{B}_{X^*}$ is submonotone at each point of $S \cap U$.*

Proof. First, we note that the implications (a)$\Rightarrow$(b) and (b)$\Rightarrow$(c) follow directly from the inclusions

$$N^F(S;x) \cap \mathbb{B}_{X^*} \subset N^L(S;x) \cap \mathbb{B}_{X^*} \subset N^C(S;x) \cap \mathbb{B}_{X^*}, \text{ for all } x \in S.$$

On the other hand, (c) implies (a) by Proposition 5.16(A) according to the equality $\partial_F d_S(x) = N^F(S;\cdot) \cap \mathbb{B}_{X^*}$ for any $x \in S$ and to the implication (a)$\Rightarrow$(c) in Lemma 5.17, as easily seen. □

5.2. **Subsmoothness of sets versus Shapiro property.** By Proposition 5.14 we know that a set S is tangentially regular at a point in S whenever it is one-sided subsmooth at this point. Let us now compare the notions of subsmoothness and one-sided subsmoothness of sets with other concepts.

Definition 5.19. Let S be a nonempty subset of a normed space X and let $\overline{x} \in S$.

(a) The set S is said to satisfy the *Shapiro k-order contact property* ($k \in \mathbb{N}$) at $\overline{x}$ (see [70]), if for every real $\varepsilon > 0$ there exists a real $\delta > 0$ such that for all $x_1, x_2 \in S \cap B(\overline{x}, \delta)$ one has

$$\operatorname{dist}(x_2 - x_1, T^B(S; x_1)) \leq \varepsilon \|x_1 - x_2\|^k.$$

(b) The set S is called *nearly radial* at $\overline{x}$ (see [50]) if for every real $\varepsilon > 0$ there exists a real $\delta > 0$ such that for all $x \in S \cap B(\overline{x}, \delta)$ one has

$$\operatorname{dist}(\overline{x} - x, T^B(S; x)) \leq \varepsilon \|x - \overline{x}\|,$$

(that is, the inequality in (a) holds for $k = 1$ with $x_2 = \overline{x}$).

Let us first state the following known lemma.

Lemma 5.20. *Let K be a nonempty convex cone of a normed vector space X. Then for any $u \in X$*

$$d(u, K) = \max_{x^* \in K^o \cap \mathbb{B}_{X^*}} \langle x^*, u \rangle. \tag{5.8}$$

Proof. Fix any $u \in X$. For any $x^* \in K^o \cap \mathbb{B}_{X^*}$ and any $x \in K$ we have

$$\langle x^*, u \rangle \leq \langle x^*, u \rangle - \langle x^*, x \rangle = \langle x^*, u - x \rangle \leq \|u - x\|,$$

thus $\sup_{x^* \in K^o \cap \mathbb{B}_{X^*}} \langle x^*, u \rangle \leq d_K(u)$.

Now observe that the subdifferential $\partial d_K(u)$ of the continuous convex function d_K is nonempty. Further, according to the sublinearity of d_K (with $d_K(0) = 0$) we have

$$\partial d_K(u) = \{x^* \in X^* : \langle x^*, u \rangle = d_K(u) \text{ and } \langle x^*, h \rangle \leq d_K(h) \ \forall h \in X\}.$$

So, taking $u^* \in \partial d_K(u)$ we see that $u^* \in K^o \cap \mathbb{B}_{X^*}$ and $\langle u^*, u \rangle = d_K(u)$. We then conclude that

$$d_K(u) = \langle u^*, u \rangle = \max_{x^* \in K^o \cap \mathbb{B}_{X^*}} \langle x^*, u \rangle.$$

□

The second lemma considers the distance to the Bouligand tangent cone.

Lemma 5.21. *Let S be a subset of a finite-dimensional normed space X and let $\overline{x} \in S$. Then one has*

$$\lim_{S \ni x \to \overline{x}} \operatorname{dist}\Big(\frac{x - \overline{x}}{\|x - \overline{x}\|}, T^B(S; \overline{x})\Big) = 0.$$

Proof. Let $h(x) := \|x - \overline{x}\|^{-1}(x - \overline{x})$ for every $x \in X \setminus \{\overline{x}\}$ and let $(x_n)_{n \in \mathbb{N}}$ be any sequence in $S \setminus \{\overline{x}\}$ converging to $\overline{x}$. Since X is finite-dimensional, for some subsequence $(x_{s(n)})_{n \in \mathbb{N}}$ we have that $(h(x_{s(n)}))_{n \in \mathbb{N}}$ converges to some $h \in X$, so (as easily seen) $h \in T^B(S;\overline{x})$. This gives $\mathrm{dist}\big(h(x_{s(n)}), T^B(S;\overline{x})\big) \to 0$ as $n \to \infty$. This being obtained for any sequence $(x_n)_n$ as above, it results that

$$\lim_{S \ni x \to \overline{x}} \mathrm{dist}\big(h(x), T^B(S;\overline{x})\big) = 0$$

as desired. □

Theorem 5.22 (**Local subsmoothness versus Shapiro property**). *Let S be a nonempty set of a normed space X and U be a nonempty open set of X with $U \cap S \neq \emptyset$. The following hold:*

(a) *The set S is subsmooth at each point in $S \cap U$ if and only if it is tangentially regular at each point in $S \cap U$ and satisfies the Shapiro first order contact property at each point in $S \cap U$.*

(b) *If X is an Asplund space and S is closed, then S is subsmooth at each point in $S \cap U$ if and only if it satisfies the Shapiro first order contact property at each point in $S \cap U$.*

(c) *If S is one-sided subsmooth at each point in $S \cap U$, then it is tangentially regular at each point in $S \cap U$ and nearly radial at each point in $S \cap U$. The converse also holds whenever X is finite-dimensional.*

Proof. (a) To show the "necessity" part, let us suppose that S is subsmooth at each point in $S \cap U$. By Proposition 5.14 we know that S is tangentially regular at each point in $S \cap U$. Fix any $\overline{x} \in S \cap U$ and let us prove that the Shapiro first order contact property is satisfied at $\overline{x}$. Consider any real $\varepsilon > 0$. By property of subsmoothness of sets there exists $\delta > 0$ such that

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\|,$$

for all $x_i \in S \cap B(\overline{x}, \delta)$ and all $x_i^* \in N^C(S; x_i) \cap \mathbb{B}_{X^*}$ for $i = 1, 2$. Fixing $x_1, x_2 \in S \cap B(\overline{x}, \delta)$ and taking $x_2^* = 0$, we obtain

$$\sup_{x_1^* \in N^C(S;x_1) \cap \mathbb{B}_{X^*}} \langle x_1^*, x_2 - x_1 \rangle \leq \varepsilon \|x_1 - x_2\|.$$

Further, from the tangential regularity we have $T^B(S; x_1) = T^C(S; x_1)$. Then, keeping in mind that $N^C(S; x_1)$ is the polar cone of the closed convex cone $T^C(S; x_1)$, the latter inequality combined with Lemma 5.20 yields

$$\mathrm{dist}\left(x_2 - x_1, T^B(S; x_1)\right) \leq \varepsilon \|x_1 - x_2\|,$$

which translates the Shapiro first order contact property for S at $\overline{x}$.

Conversely, to show the "sufficiently" part, let us assume that S is tangentially regular at each point in $S \cap U$ and satisfies the Shapiro first order contact property at each point in $S \cap U$. Fix any $\overline{x} \in S \cap U$ and let us show that S is subsmooth at $\overline{x}$. Consider any real $\varepsilon > 0$. By definition of Shapiro first order property there exists a real $\delta > 0$ with $B(\overline{x}, \delta) \subset U$ such that for all $x_i \in S \cap B(\overline{x}, \delta)$, $i = 1, 2$,

$$\max\{\mathrm{dist}\left(x_2 - x_1, T^B(S; x_1)\right), \mathrm{dist}\left(x_1 - x_2, T^B(S; x_2)\right)\} \leq \frac{\varepsilon}{2} \|x_1 - x_2\|. \tag{5.9}$$

Since $T^B(S; x_i) = T^C(S; x_i)$ for $i = 1, 2$, by virtue of Lemma 5.20 we deduce that

$$\max\left\{ \sup_{x_1^* \in N^C(S;x_1) \cap \mathbb{B}} \langle x_1^*, x_2 - x_1 \rangle, \sup_{x_2^* \in N^C(S;x_2) \cap \mathbb{B}} \langle x_2^*, x_1 - x_2 \rangle \right\} \leq \frac{\varepsilon}{2} \|x_1 - x_2\|.$$

This entails that, for all $x_i^* \in N^C(S; x_i) \cap \mathbb{B}_{X^*}$ with $x_i \in S \cap B(\overline{x}, \delta)$

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\|,$$

hence S is subsmooth at $\overline{x}$.

(b) Assume that X is an Asplund space. According to (a) we only have to show the "sufficiency" part. So, let us suppose that S satisfies the Shapiro first order contact property at each point in $S \cap U$ and let us show that S is subsmooth at each point in $S \cap U$. By virtue of the implication (c)⇒(a) in Proposition 5.18, it is enough to show that the truncated Fréchet normal cone $N^F(S; \cdot) \cap \mathbb{B}_{X^*}$ is submonotone at each point in $S \cap U$. Fix any $\overline{x} \in S \cap U$ and any real $\varepsilon > 0$. As in the proof of "sufficiency" part of (a), there is a real $\delta > 0$ with $B(\overline{x}, \delta) \subset U$ such that (5.9) holds with $\varepsilon/3$ in place of $\varepsilon/2$. Consider any $x_i \in S \cap B(\overline{x}, \delta)$ and any $x_i^* \in N^F(S; x_i) \cap \mathbb{B}_{X^*}$, $i = 1, 2$. From the latter inequality in (5.9) with $\varepsilon/3$ (since its first member is strictly less than $(\varepsilon/2)\|x_1 - x_2\|$ for $x_1 \neq x_2$) we may choose $v_i \in T^B(S; x_i)$ and $e_i \in X$ (for $i = 1, 2$) such that

$$x_2 - x_1 = v_1 + e_1 \quad \text{and} \quad x_1 - x_2 = v_2 + e_2,$$

with $\|e_i\| \leq (\varepsilon/2)\|x_1 - x_2\|$. Note that $\langle x_i^*, v_i \rangle \leq 0$ since the Fréchet normal cone $N^F(S; x)$ is always included in the (negative) polar cone of $T^B(S; x)$ for $x \in S$. Using this and the inequality $\|x_i^*\| \leq 1$ (for $i = 1, 2$), it follows that

$$\langle x_1^*, x_1 - x_2 \rangle \geq -\frac{\varepsilon}{2}\|x_1 - x_2\| \quad \text{and} \quad \langle x_2^*, x_2 - x_1 \rangle \geq -\frac{\varepsilon}{2}\|x_1 - x_2\|.$$

Adding these inequalities give

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\|,$$

which is the desired submonotonicity of $N^F(S; \cdot) \cap \mathbb{B}_{X^*}$.

(c) Assume that S is one-sided subsmooth at each point in $S \cap U$. By Proposition 5.14 the set S is tangentially regular at each point in $S \cap U$. To show that S is nearly radial at any point $\overline{x} \in S \cap U$, it suffices to proceed in the same way as in the above proof of the "necessity" part of (a), setting $x_2 = \overline{x}$ (and $x_1 = x$) to conclude that the property in Definition 5.19 holds true.

Suppose now that X is finite-dimensional and that S is tangentially regular at each point in $S \cap U$ and nearly radial at each point in $S \cap U$. Let us fix any $\overline{x} \in S \cap U$ and show that S is one-sided subsmooth at $\overline{x}$. Take any real $\varepsilon > 0$. By Definition 5.19(b) and by tangential regularity of S at points in $S \cap U$ there is a real $\delta' > 0$ with $B(\overline{x}, \delta') \subset U$ such that for all $x \in S \cap B(\overline{x}, \delta')$

$$\operatorname{dist}\left(\overline{x} - x, T^C(S; x)\right) \leq \frac{\varepsilon}{2}\|x - \overline{x}\|.$$

Since $N^C(S; x)$ is the polar of the convex cone $T^C(S; x)$, Lemma 5.20 gives for all $x \in S \cap B(\overline{x}, \delta')$

$$\sup_{x^* \in N^C(S;x) \cap \mathbb{B}} \langle x^*, \overline{x} - x \rangle \leq \frac{\varepsilon}{2}\|x - \overline{x}\|. \tag{5.10}$$

On the other hand, by Lemma 5.21 there exists $\delta \in]0, \delta'[$ such that for all $x \in S \cap B(\overline{x}, \delta)$ with $x \neq \overline{x}$

$$\operatorname{dist}\left(\frac{x - \overline{x}}{\|x - \overline{x}\|}, T^B(S; \overline{x})\right) \leq \frac{\varepsilon}{2}.$$

Since $T^B(S; \overline{x}) = T^C(S : \overline{x})$, using again Lemma 5.20 we obtain

$$\sup_{u^* \in N^C(S;\overline{x}) \cap \mathbb{B}} \langle u^*, x - \overline{x} \rangle \leq \frac{\varepsilon}{2} \|x - \overline{x}\| \quad \text{for all } x \in S \cap B(\overline{x}, \delta).$$

From this and (5.10) we see that

$$\langle x^* - u^*, x - \overline{x} \rangle \geq -\varepsilon \|x - \overline{x}\|$$

for all $x \in S \cap B(\overline{x}, \delta)$, all $x^* \in N^C(S; x)$ and all $u^* \in N^C(S; \overline{x})$. This means that S is one-sided subsmooth at $\overline{x}$, and the proof is finished. □

The proofs of (a) and (c) in the above theorem also work for the subsmoothness (resp. one-sided subsmoothness) property at a fixed point $\overline{x}$ provided that S is tangentially regular near $\overline{x}$. We state this in the proposition:

Proposition 5.23. *Let S be a set of a normed space X which is tangentially regular near a point $\overline{x} \in S$. The following hold:*

(a) *The set S is subsmooth at $\overline{x}$ if and only if it satisfies the Shapiro first order contact property at $\overline{x}$.*

(b) *If S is one-sided subsmooth at $\overline{x}$, then it is nearly radial at $\overline{x}$. The converse also holds whenever X is finite-dimensional.*

6. Epi-Lipschitz subsmooth sets

In this section we will show that epi-Lipschitz subsmooth sets can be seen as epigraphs of Lipschitz subsmooth functions.

Let S be a subset of the normed space X which is closed near $\overline{x} \in S$ and epi-Lipschitz at this point in a direction $h \in X$ with $h \neq 0$. By [66] there exists a topologically complemented closed vector hyperplane E of $\mathbb{R}h$ (so, $X = E \oplus \mathbb{R}h$), an open neighborhood W of $\overline{x}$ in X, and a function $f : E \to \mathbb{R}$ Lipschitz continuous near $\pi_E \overline{x}$ (with E endowed with the induced norm and $x = \pi_E x + (\pi_h x)h$ with $\pi_E x \in E$ and $\pi_h x \in \mathbb{R}$) such that

$$W \cap S = W \cap \{u + rh : u \in E, r \in \mathbb{R}, f(u) \leq r\}.$$

Such a function f is called a *locally Lipschitz representative of S around $\overline{x}$*. Consider the linear isomorphism $A : E \times \mathbb{R} \to X$ defined by $A(u, s) := u \oplus sh$, so that $\Lambda := A^{-1} : X \to E \times \mathbb{R}$ is given $\Lambda(x) = (\pi_E x, \pi_h x)$ and $W \cap S$ can be rewritten as

$$W \cap S = W \cap A(\operatorname{epi} f).$$

We endow $E \times \mathbb{R}$ with the product norm $(u, s) \mapsto (\|u\|^2 + s^2)^{1/2}$ and we define $F : E \to X$ by $F(u) := u \oplus f(u)h$, so that $F(u) = A(u, f(u))$. If $\gamma > 0$ denotes a Lipschitz constant of f in an open neighborhood V of $\overline{u} := \pi_E \overline{x}$ with $F(V) \subset W$, then it is easily seen that

$$\|F(u_1) - F(u_2)\| \leq \alpha \|u_1 - u_2\| \quad \text{for all } u_1, u_2 \in V, \tag{6.1}$$

where

$$\alpha := \|A\|(\gamma^2 + 1)^{1/2}. \tag{6.2}$$

From the equality $W \cap S = W \cap \Lambda^{-1}(\operatorname{epi} f)$ we also note by the isomorphism property of Λ that, for all $u \in V$

$$T^C(S; F(u)) = \Lambda^{-1}\big(T^C(\operatorname{epi} f; (u, f(u)))\big)$$

and

$$N^C(S; F(u)) = \Lambda^*\big(N^C(\operatorname{epi} f; (u, f(u)))\big). \tag{6.3}$$

Lemma 6.1. *Let X be a normed space and S be a subset which is closed near a point $\overline{x} \in \operatorname{bdry} S$ and epi-Lipschitz at $\overline{x}$ in a direction $h \neq 0$. Assume that $N^C(S; \cdot) \cap \mathbb{B}_{X^*}$ is submonotone (resp. one-sided submonotone) at $\overline{x}$. Then, for every locally Lipschitz representative $f : E \to \mathbb{R}$ of S around $\overline{x}$ (where E is a topological complement of $\mathbb{R}h$ in X), the Clarke subdifferential $\partial_C f$ is submonotone (resp. one-sided submonotone) at $\pi_E \overline{x}$.*

Proof. Keep the above notation and let V be an open neighborhood of $\overline{u} := \pi_E \overline{x}$ over which f is Lipschitz with constant $\gamma > 0$ and such that $F(V) \subset W$.

Let us first establish the result for submonotonicity. From the submonotonicity of $N^C(S; \cdot) \cap \mathbb{B}_{X^*}$ at $\overline{x}$ it is easily seen that $N^C(S; \cdot) \cap r\mathbb{B}_{X^*}$ is subsmonotone at $\overline{x}$ for any real $r > 0$. Take $r := \|\Lambda^*\|(\gamma^2 + 1)^{1/2}$, where as above Λ^* denotes the adjoint of $\Lambda := A^{-1}$. In order to prove that $\partial_C f$ is submonotone at $\overline{u} := \pi_E \overline{x}$, take any real $\varepsilon > 0$ and set $\varepsilon_1 := \alpha^{-1}\varepsilon$, where α is given by (6.2). There exists a real $\delta > 0$ with $B(\overline{x}, \delta) \subset W$ such that for all $x_i \in B(\overline{x}, \delta)$ and all $x_i^* \in N^C(S; x_i) \cap r\mathbb{B}_{X^*}$, $i = 1, 2$,

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon_1 \|x_1 - x_2\|. \tag{6.4}$$

Choose a positive real $\delta_1 \leq \alpha^{-1}\delta$ such that $B(\overline{u}, \delta_1) \subset V$. Then by (6.1) we have $F(B(\overline{u}; \delta_1)) \subset B(\overline{x}, \delta)$. Take any $u_1, u_2 \in B(\overline{u}, \delta_1)$ and any $u_i^* \in \partial_C f(u_i)$ for $i = 1, 2$. Then $(u_i^*, -1) \in N^C\big(\operatorname{epi} f; (u_i, f(u_i))\big)$. Using the equality $W \cap S = W \cap \Lambda^{-1}(\operatorname{epi} f)$ and the points

$$x_i := F(u_i) = A(u_i, f(u_i)) \in B(\overline{x}, \delta), \ \ i = 1, 2,$$

we obtain $x_i^* := \Lambda^*((u_i^*, -1)) \in N^C(S; x_i)$, $i = 1, 2$. Since $\|u_i^*\| \leq \gamma$ for $i = 1, 2$ (according to the γ-Lipschitz property of f on V), it ensues that

$$\|x_i^*\| \leq \|\Lambda^*\|(\|u_i^*\|^2 + 1)^{1/2} \leq r, \quad \text{so } x_i^* \in N^C(S; x_i) \cap r\mathbb{B}_{X^*}.$$

This combined with (6.4) yields

$$\langle \Lambda^*(u_1^*, -1) - \Lambda^*(u_2^*, -1), F(u_1) - F(u_2) \rangle \geq -\varepsilon\alpha^{-1}\|F(u_1) - F(u_2)\|.$$

Since

$$\begin{aligned}\langle \Lambda^*(u_1^*, -1) - \Lambda^*(u_2^*, -1), F(u_1) - F(u_2) \rangle &= \langle (u_1^* - u_2^*, 0), (u_1, f(u_1)) - (u_2, f(u_2)) \rangle \\ &= \langle u_1^* - u_2^*, u_1 - u_2 \rangle,\end{aligned}$$

the above inequality and (6.1) yield

$$\langle u_1^* - u_2^*, u_1 - u_2 \rangle \geq -\varepsilon \|u_1 - u_2\|.$$

This justifies the submonotonicity property of $\partial_C f$ at $\overline{u}$.

The case of one-sided submonotonicity is obtained by replacing u_2 by $\overline{u}$ and $u_2^* \in \partial_C f(u_2)$ by $\overline{u}^* \in \partial_C f(\overline{u})$ in the above arguments. □

A similar lemma holds for uniform submonotonicity near a point.

Lemma 6.2. *Let X be a normed space and S be a subset which is closed near a point $\overline{x} \in \operatorname{bdry} S$ and epi-Lipschitz at $\overline{x}$ in a direction $h \neq 0$. Assume that $N^C(S;\cdot) \cap \mathbb{B}_{X^*}$ is uniformly submonotone near $\overline{x}$. Then, for every locally Lipschitz representative $f : E \to \mathbb{R}$ of S around $\overline{x}$ (where E is a topological complement of $\mathbb{R}h$ in X), the Clarke subdifferential $\partial_C f$ is uniformly submonotone near $\pi_E \overline{x}$.*

Proof. Again keep the above notation with $W \cap S = W \cap A(\operatorname{epi} f)$ and let V be an open neighborhood of $\overline{u} := \pi_E \overline{x}$ over which f is Lipschitz with constant $\gamma > 0$ and such that $F(V) \subset W$. Let $W_0 \subset W$ be an open convex neighborhood of $\overline{x}$ over which the multimapping $N^C(S;\cdot) \cap \mathbb{B}_{X^*}$ is uniformly submonotone and let $V_0 \subset V$ be an open convex neighborhood of $\overline{u}$ such that $F(V_0) \subset W_0$. Recall that $\alpha := \|A\|(\gamma^2+1)^{1/2}$ (see (6.2)), and as in the previous lemma set $r := \|\Lambda^*\|(\gamma^2+1)^{1/2}$. Note that the multimapping $N^C(S;\cdot) \cap r\mathbb{B}_{X^*}$ is uniformly submonotone on W_0. Let us show that $\partial_C f(\cdot)$ is uniformly submonotone on V_0. Let any real $\varepsilon > 0$. There exists a real $\delta > 0$ such that

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\alpha^{-1}\varepsilon \|x_1 - x_2\| \tag{6.5}$$

for all $x_1, x_2 \in W_0 \cap S$ with $\|x_1 - x_2\| < \delta$ and all $x_i^* \in N^C(S;\cdot) \cap r\mathbb{B}_{X^*}$. Set $\delta_0 := \alpha^{-1}\delta$ and take any $u_1, u_2 \in V_0$ with $\|u_1 - u_2\| < \delta_0$ and any $u_i^* \in \partial_C f(u_i)$, $i = 1, 2$, so $(u_i^*, -1) \in N^C(\operatorname{epi} f; (u_i, f(u_i)))$. Putting $x_i := F(u_i) = A(u_i, f(u_i))$, we see that $x_i \in W_0 \cap \Lambda^{-1}(\operatorname{epi} f)$. Observing that $W_0 \cap S = W_0 \cap \Lambda^{-1}(\operatorname{epi} f)$ (since $W_0 \subset W$), it ensues that $x_i \in W_0 \cap S$ and $x_i^* := \Lambda^*(u_i^*, -1) \in N^C(S, x_i)$. Further, the inequality $\|u_i^*\| \leq \gamma$ gives

$$\|x_i^*\| \leq \|\Lambda^*\|(\|u_i^*\|^2 + 1)^{1/2} \leq r.$$

The mapping F being α-Lipschitz on V we also observe that

$$\|x_1 - x_2\| = \|F(u_1) - F(u_2)\| \leq \alpha \|u_1 - u_2\| < \delta.$$

We deduce from (6.5) that

$$\langle \Lambda^*(u_1^*, -1) - \Lambda^*(u_2^*, -1), F(u_1) - F(u_2) \rangle \geq -\varepsilon\alpha^{-1}\|F(u_1) - F(u_2)\|.$$

Writing

$$\begin{aligned} \langle \Lambda^*(u_1^*, -1) - \Lambda^*(u_2^*, -1), F(u_1) - F(u_2) \rangle &= \langle (u_1^* - u_2^*, 0), (u_1, f(u_1)) - (u_2, f(u_2)) \rangle \\ &= \langle u_1^* - u_2^*, u_1 - u_2 \rangle, \end{aligned}$$

and using (6.1) it results that

$$\langle u_1^* - u_2^*, u_1 - u_2 \rangle \geq -\varepsilon \|u_1 - u_2\|,$$

which confirms the uniform submonotonicity property of $\partial_C f$ on V_0. □

Theorem 6.3 (Subsmooth functional representation of epi-Lipschitz subsmooth set). *Let S be a subset of a normed space X which is closed near $\overline{x} \in \operatorname{bdry} S$ and epi-Lipschitz at $\overline{x}$ in a direction $h \neq 0$. The following are equivalent:*

(a) *The set S is subsmooth at $\overline{x}$ (resp. uniformly subsmooth near $\overline{x}$).*

(b) *Every locally Lipschitz representative* $f : E \to \mathbb{R}$ *of* S *is subsmooth at* $\pi_E \overline{x}$ *(resp. uniformly submooth near* $\pi_E \overline{x}$*), where* E *is a topological complement of* $\mathbb{R}h$ *in* X*.*

(c) *Some locally Lipschitz representative* $f : E \to \mathbb{R}$ *of* S *is subsmooth at* $\pi_E \overline{x}$ *(resp. uniformly subsmooth near* $\pi_E \overline{x}$*).*

Proof. The implication (b)⇒(c) is evident and the implication (a)⇒(b) follows from Lemma 6.1 and from Proposition 4.16 (resp. Lemma 6.2 and Proposition 4.17). It remains to prove (c)⇒(a). Suppose that f is subsmooth at $\overline{u} := \pi_E \overline{x}$ (resp. uniformly subsmooth near $\overline{u}$). Let V be an open neighborhood of $\overline{u}$ and W an open neighborhood of $\overline{x}$ as in what precedes Lemma 6.1 and such that f is Lipschitz with constant $\gamma > 0$ on V and $F(V) \subset W$ (and in the case of uniform subsmoothness near $\overline{u}$, choose open convex neighborhoods $V_0 \subset V$ and $W_0 \subset W$ of $\overline{u}$ and $\overline{x}$ respectively with $F(V_0) \subset W_0$ such that f is uniformly subsmooth on V_0, and choose also an open convex neighborhood $W_0' \subset W_0$ of $\overline{x}$ such that $\pi_E(W_0') \subset V_0$). Take any real $\varepsilon > 0$ and set

$$\varepsilon_1 := \frac{\varepsilon}{2\|A^*\| \cdot \|\Lambda\|}.$$

According to Theorem 4.5 or Proposition 4.16 (resp. Proposition 4.17), there exists a real $\delta_1 > 0$ with $B(\overline{x}, \delta_1) \subset V$ such that (resp. a real $\delta_1 > 0$ such that)

$$f(u_2) \geq f(u_1) + \langle u_1^*, u_2 - u_1 \rangle - \varepsilon_1 \|u_2 - u_1\|, \tag{6.6}$$

for all $u_1, u_2 \in B(\overline{u}, \delta_1)$ (resp. $u_1, u_2 \in V_0$ with $\|u_1 - u_2\| < \delta_1$) and all $u_1^* \in \partial_C f(u_1)$. Choose a positive real $\delta \leq \delta_1/\|\Lambda\|$ so that $B(\overline{x}, \delta) \subset W$ (resp. a positive real $\delta \leq \delta_1/\|\Lambda\|$). Take any $x_i \in S \cap B(\overline{x}, \delta)$ (resp. $x_i \in S \cap W_0'$ with $\|x_1 - x_2\| < \delta$) and any $x_i^* \in N^C(S; x_i) \cap \mathbb{B}_{X^*}$, $i = 1, 2$. We claim that

$$\langle x_1^*, x_2 - x_1 \rangle \leq \frac{\varepsilon}{2} \|x_1 - x_2\|.$$

To this end, put $u_i := \pi_E x_i$ and $t_i := \pi_h x_i$, so that $x_i = A(u_i, t_i)$ and $t_i \geq f(u_i)$, $i = 1, 2$. If $t_1 > f(u_1)$, then $x_1 \in \operatorname{int} S$, and hence $N^C(S; x_1) = \{0\}$, so the inequality of the claim holds trivially. We may then suppose that $t_1 = f(u_1)$ and $x_1^* \neq 0$. Note that $x_1^* \in \Lambda^*\big(N^C(\operatorname{epi} f; (x_1, f(x_1)))\big)$ according to the isomorphic property of Λ. There exists (see Section 2) a real $\lambda > 0$ and $u_1^* \in \partial_C f(u_1)$ such that

$$x_1^* = \lambda \Lambda^*(u_1^*, -1). \tag{6.7}$$

Then $\lambda(u_1^*, -1) = A^* x_1^*$, and hence $\lambda \leq \|A^*\|$ since $\|x_1^*\| \leq 1$. Further,

$$\|u_1 - u_2\| \leq \|\Lambda\| \cdot \|x_1 - x_2\|,$$

so by the choice of δ we see that $u_1, u_2 \in B(\overline{u}, \delta_1)$ (resp. $\|u_1 - u_2\| < \delta_1$ and $u_1, u_2 \in V_0$). Therefore, (6.6) along with the inequality $t_2 \geq f(u_2)$ gives

$$\langle (u_1^*, -1), (u_2, t_2) - (u_1, f(u_1)) \rangle \leq \varepsilon_1 \|u_1 - u_2\| \leq \varepsilon_1 \|\Lambda\| \cdot \|x_1 - x_2\|.$$

It ensues that

$$\langle \lambda(u_1^*, -1), \Lambda(x_2) - \Lambda(x_1) \rangle \leq \varepsilon_1 \|A^*\| \cdot \|\Lambda\| \cdot \|x_1 - x_2\| = \frac{\varepsilon}{2} \|x_1 - x_2\|,$$

which by (6.7) yields $\langle x_1^*, x_2 - x_1\rangle \le (\varepsilon/2)\|x_1 - x_2\|$ as stated in the claim. Analogously the inequality $\langle x_2^*, x_1 - x_2\rangle \le (\varepsilon/2)\|x_2 - x_1\|$ also holds true. Adding the two latter inequalities it results that

$$\langle x_1^* - x_2^*, x_1 - x_2\rangle \ge -\varepsilon\|x_1 - x_2\|,$$

which proves the subsmoothness of the set S at $\overline{x}$ (resp. the uniform subsmoothness of $S \cap W_0'$). □

As a direct consequence of the above theorem and Proposition 4.16 we have the following corollary.

Corollary 6.4. *Let X be a normed space and $f : X \to \mathbb{R}$ be a function which is Lipschitz near $\overline{x} \in X$. The following are equivalent:*

(a) *The epigraphical set* epi f *is subsmooth at $(\overline{x}, f(\overline{x}))$ (resp. uniformly subsmooth near $(\overline{x}, f(\overline{x}))$).*
(b) *The function f is subsmooth at $\overline{x}$ (resp. uniformly subsmooth near $\overline{x}$).*
(c) *The multimapping $\partial_C f$ is submonotone at $\overline{x}$ (resp. uniformly submonotone near $\overline{x}$).*

Through Corollary 6.4 we provide an epi-Lipschitz set in $\mathbb{R}^2$ which is tangentially regular at a point and fails to be subsmooth at that point.

Example 6.5. Example 4.25 furnished a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ which is tangentially regular at $\overline{x} := 0$ such that the multimapping $\partial_C f$ is not one-sided submonotone at $\overline{x}$, hence in particular f is not subsmooth at $\overline{x}$ by Proposition 4.16. Then the epi-Lipschitz set epi f is tangentially regular at $(\overline{x}, f(\overline{x}))$ but not subsmooth at $(\overline{x}, f(\overline{x}))$ according to Corollary 6.4.

Proposition 6.6. *Let S be a subset of a normed space X which is closed near $\overline{x} \in$ bdry S and epi-Lipschitz at $\overline{x}$. Then S is uniformly subsmooth near $\overline{x}$ if and only if there exist a nonzero vector $h \in X$, a topological complement vector subspace E with $X = E \oplus \mathbb{R}h$, an open neighborhood W of $\overline{x}$ and a modulus function $\omega : [0, +\infty[\to [0, +\infty[$ of class C^1 on $]0, +\infty[$ with $t\,\omega'(t) \to 0$ as $t \downarrow 0$ such that*

$$\langle x^*, y - x\rangle \le \|\pi_E y - \pi_E x\|\,\omega(\|\pi_E y - \pi_E x\|)$$

for all $x, y \in S \cap W$ and all $x^* \in N^C(S; x)$.

If in addition X is finite-dimensional, then the latter property is also equivalent to the subsmoothness of the set S near $\overline{x}$.

Proof. First let us suppose that the property in the proposition holds. We may suppose that W is convex. Fix any $\varepsilon > 0$. Since ω is continuous at 0 with $\omega(0) = 0$, there is $\eta > 0$ such that $\omega(t) \le \varepsilon/(1 + \|\pi_E\|)$ for all $t \in [0, \eta]$. Choose $\delta > 0$ such that $\|\pi_E z\| \le \eta$ whenever $\|z\| \le \delta$. Take any $x, y \in S \cap W$ with $\|y - x\| < \delta$ and any $x^* \in N^C(S; x) \cap \mathbb{B}_{X^*}$. We have

$$\begin{aligned}\langle x^*, y - x\rangle &\le \|\pi_E y - \pi_E x\|\,\omega(\|\pi_E y - \pi_E x\|) = \|\pi_E(y - x)\|\,\omega(\|\pi_E(y - x)\|) \\ &\le \frac{\varepsilon}{1 + \|\pi_E\|}\|\pi_E(y - x)\| \le \varepsilon\|y - x\|,\end{aligned}$$

which tells us that the set S is uniformly subsmooth near the point $\overline{x}$ and justifies the implication $\Leftarrow$.

To prove the converse implication $\Rightarrow$, let us assume that S is epi-Lipschitz and uniformly subsmooth near $\overline{x}$. Choose a nonzero vector $h \in X$, a complement vector subspace E with $X = E \oplus \mathbb{R}h$, a Lipschitz function $f : E \to \mathbb{R}$ and an open neighborhood W of $\overline{x}$ such that

$$W \cap S = W \cap A(\operatorname{epi} f),$$

where $A : E \times \mathbb{R} \to X$ is given by $A(u,r) = u \oplus rh$ as stated in the beginning of this section. By Theorem 6.3 the function f is uniformly subsmooth near $\pi_E\overline{x}$, so by Proposition 4.20 there exist an open neighborhood V of $\pi_E\overline{x}$ and a modulus function $\omega : [0,+\infty[\to [0,+\infty[$ of class C^1 on $]0,+\infty[$ with $t\,\omega'(t) \to 0$ as $t \downarrow 0$ such that for all $u, v \in V$, $u^* \in \partial f(u)$ and $r \geq f(v)$

$$r \geq f(u) + \langle u^*, v-u\rangle - \|v-u\|\,\omega(\|v-u\|),$$

or equivalently

$$\langle (u^*,-1),(v,r)-(u,f(u))\rangle \leq \|v-u\|\,\omega(\|v-u\|). \tag{6.8}$$

Let $W_0 \subset W$ be an open neighborhood of $\overline{x}$ such that $\pi_E(W_0) \subset V$. Let us show the inequality in the proposition with $\omega_0(\cdot) := \|A\|\,\omega(\cdot)$. Consider any $x, y \in S \cap W_0$ and any $x^* \in N^C(S;x) \cap \mathbb{B}_{X^*}$. Set $u := \pi_E x$ and $v := \pi_E y$. If $\pi_h x > f(u)$, then $x \in \operatorname{int} S$, which entails that $x^* = 0$, so the inequality in the proposition is satisfied since $\omega_0(\cdot) \geq 0$. Suppose that $\pi_h x = f(u)$. We may also suppose that $x^* \neq 0$, for otherwise the desired inequality is trivial. Since A is an isomorphism and f is Lipschitz, by (2.2) there is some real $t > 0$ and $u^* \in \partial_C f(u)$ such that $A^*(x^*) = t(u^*,-1)$, or equivalently $t(u^*,-1) = x^* \circ A$. This gives $t\left(\|u^*\|^2+1\right)^{1/2} \leq \|A\|$, hence $t \leq \|A\|$. Further, by (6.8) we have

$$t^{-1}\langle A^*(x^*),(v,\pi_h y)-(u,f(u))\rangle \leq \|\pi_E y - \pi_E x\|\,\omega(\|\pi_E y - \pi_E x\|),$$

which combined with the above inequality $t \leq \|A\|$ yields

$$\langle x^*, A(\pi_E y, \pi_h y) - A(\pi_E x, \pi_h x)\rangle \leq \|A\|\,\|\pi_E y - \pi_E x\|\,\omega(\|\pi_E y - \pi_E x\|),$$

and this means with $\omega_0(\cdot) := \|A\|\,\omega(\cdot)$ as above

$$\langle x^*, y-x\rangle \leq \|\pi_E y - \pi_E x\|\,\omega_0(\|\pi_E y - \pi_E x\|).$$

This justifies the desired implication.

Finally, the situation when X is finite-dimensional follows from what precedes and from Proposition 5.12(c). □

The next result concerns the functional representation of epi-Lipschitz one-sided subsmooth sets. Comparing with Theorem 6.3, two differences need to be emphasized:

- the statements are local (and not pointwise);
- the equivalence is proved only in finite dimensions.

Proposition 6.7. *Let S be a subset of a finite-dimensional normed space X which is closed near $\overline{x} \in \operatorname{bdry} S$ and epi-Lipschitz at $\overline{x}$ in a direction $h \neq 0$. The following are equivalent:*

(a) *The set S is one-sided subsmooth near $\overline{x}$.*

(b) *Every locally Lipschitz representative $f : E \to \mathbb{R}$ of S is one-sided subsmooth near $\pi_E \overline{x}$ and tangentially regular near $\pi_E \overline{x}$, where E is a topological complement of $\mathbb{R}h$ in the space X.*

(c) *Some locally Lipschitz representative $f : E \to \mathbb{R}$ of S is one-sided subsmooth near $\pi_E \overline{x}$ and tangentially regular near $\pi_E \overline{x}$.*

Proof. The implication (b)⇒(c) is obvious. To prove (a)⇒(b), suppose that S is one-sided subsmooth near $\overline{x}$ and let f be any locally Lipschitz representative of S around $\overline{x}$. Lemma 6.1 tells us that $\partial_C f$ is one-sided submonotone in a neighborhood of $\pi_E \overline{x}$. From (c)⇒(a) in Proposition 4.26 we obtain (b). Finally, the implication (c)⇒(a) follows in the same way as in Theorem 6.3. It suffices to use (b) in Proposition 4.26 in place of (b) in Proposition 4.16 to obtain the desired one-sided subsmoothness near $\overline{x}$ of $N^C(S;\cdot) \cap \mathbb{B}$. □

The next corollary directly follows from the above proposition and Corollary 4.32.

Corollary 6.8. *Let X be a finite-dimensional normed space and $f : X \to \mathbb{R}$ be a function which is Lipschitz near $\overline{x} \in X$. The following are equivalent:*

(a) *The epigraphical set* epi f *is one-sided subsmooth near $(\overline{x}, f(\overline{x}))$.*

(b) *The function f is one-sided subsmooth near $\overline{x}$ and tangentially regular near $\overline{x}$.*

(c) *The multimapping $\partial_C f$ is one-sided submonotone near $\overline{x}$.*

7. Metrically subsmooth sets

Using the Clarke subdifferential of the distance function to S in (5.1) instead of the truncation of the Clarke normal cone with the closed unit ball, we consider the following definition.

Definition 7.1. A set S of a normed space $(X, \|\cdot\|)$ is called *metrically subsmooth* at $\overline{x} \in S$ when for every $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$\langle x^* - y^*, x - y \rangle \geq -\varepsilon \|x - y\|$$

for all $x, y \in S \cap B(\overline{x}, \delta)$, all $x^* \in \partial_C d_S(x)$ and all $y^* \in \partial_C d_S(y)$. When the property holds at any $\overline{x} \in S$ we say that S is *metrically subsmooth.* The set S is called *uniformly metrically subsmooth* if for each $\varepsilon > 0$ there is a real $\delta > 0$ such that the above inequality is satisfied for all $x, y \in S$ with $\|x - y\| < \delta$, all $x^* \in \partial_C d_S(x)$ and all $y^* \in \partial_C d_S(y)$.We also say that S is uniformly metrically subsmooth near $\overline{x} \in S$ whenever there exists a neighborhood U of $\overline{x}$ such that $S \cap U$ is uniformly metrically subsmooth.

Since $0 \in \partial_C d_S(y)$ for any $y \in S$, it is easily seen that S is metrically subsmooth at $\overline{x}$ if and only if for any $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$\langle x^*, y - x \rangle \leq \varepsilon \|y - x\| \tag{7.1}$$

for all $x, y \in B(\overline{x}, \delta) \cap S$ and all $x^* \in \partial_C d_S(x)$.

We also notice, according to the inclusion $\partial_C d_S(x) \subset N^C(S; x) \cap \mathbb{B}_{X^*}$ for any $x \in S$, that any set which is subsmooth (at $\overline{x} \in S$) is metrically subsmooth (at $\overline{x}$). Further, from Proposition 5.16(B) we immediately obtain the equivalence in (b) below.

Proposition 7.2. *Let $(X, \|\cdot\|)$ be a normed space.*

(a) *Any set S of X which is subsmooth (resp. subsmooth at $\overline{x} \in S$) is metrically subsmooth (resp. metrically subsmooth at $\overline{x}$).*

(b) *If X is a Banach space, then a set S in X is uniformly metrically subsmooth (resp. uniformly metrically subsmooth near a point $\overline{x} \in S$) if and only if it is uniformly subsmooth (resp. uniformly metrically subsmooth near $\overline{x}$).*

We will see below in Example 7.10 that the converse of the assertion (a) in the above proposition is not true even in the finite-dimensional setting.

First we show that the concept of metric subsmoothness does not depend on the norm.

Proposition 7.3. *The concept of metrical subsmoothness does not depend on the norm on X in the sense that, for any norm $\|\cdot\|_1$ on X equivalent to $\|\cdot\|$, the set S is metrically subsmooth at $\overline{x} \in S$ with respect to the norm $\|\cdot\|$ if and only if it is metrically subsmooth at $\overline{x}$ with respect to $\|\cdot\|_1$.*

Proof. Fix two constants $\alpha, \beta > 0$ such that $\alpha\|x\|_1 \le \|x\| \le \beta\|x\|_1$ for all $x \in X$. Denote by $\mathrm{dist}_1(S, \cdot)$ the distance function to S with respect to the norm $\|\cdot\|_1$. Then

$$\alpha \, \mathrm{dist}_1(S, x) \le \mathrm{dist}\,(S, x) \le \beta \, \mathrm{dist}_1(S, x) \quad \text{for all } x \in X$$

and these inequalities entail by Proposition 2.1(a) that for each $x \in S$

(7.2)
$$\alpha \, (\mathrm{dist}_1(\cdot, S))^o(x; h) \le (\mathrm{dist}\,(\cdot, S))^o(x; h) \le \beta(\mathrm{dist}_1(\cdot, S))^o(x; h) \quad \text{for all } h \in X.$$

Suppose that S is metrically subsmooth at $\overline{x}$ with respect to the norm $\|\cdot\|$. Fix any $\varepsilon > 0$. By definition of metrical subsmoothness there exists $\delta > 0$ such that for all $u, v \in S \cap B(\overline{x}, \delta)$ and all $u^* \in \partial_C \mathrm{dist}\,(\cdot, S)(u)$ one has

$$\langle u^*, v - u \rangle \le \frac{\alpha\varepsilon}{\beta} \|u - v\|. \tag{7.3}$$

Fix any $x, y \in S \cap B(\overline{x}, \delta)$ and $x^* \in \partial_C \mathrm{dist}_1(\cdot, S)(x)$. According to the first inequality of (7.2) we have $\alpha x^* \in \partial_C \mathrm{dist}\,(\cdot, S)(x)$, and hence by (7.3) we obtain $\langle \alpha x^*, y - x \rangle \le \frac{\alpha\varepsilon}{\beta}\|y - x\|$, which entails $\langle x^*, y - x \rangle \le \varepsilon\|y - x\|_1$. This ensures that S is metrically subsmooth at $\overline{x}$ with respect to the norm $\|\cdot\|_1$.

The reverse implication holds in the same way. $\square$

We saw in Proposition 5.14 that a subsmooth set at $\overline{x}$ is tangentially regular at $\overline{x}$. The same property is shown, in the following proposition, to hold under the metric subsmoothness. In fact it is even true for metrically hemi-subsmooth set. When we require (5.4) (resp. (5.5)) to be satisfied with $\partial_C \mathrm{dist}\,(\cdot, S)(\cdot)$ in place of $N^C(S; \cdot) \cap \mathbb{B}$, we say that the set S is *metrically one-sided subsmooth* (resp. *metrically hemi-subsmooth*) at $\overline{x} \in S$. Like the subsmoothness property, the metric subsmoothness clearly entails the metric one-sided subsmoothness, which implies the metric hemi-subsmoothness.

Proposition 7.4. *Every set S in a normed space X which is metrically hemi-subsmooth (metrically one-sided subsmooth, or metrically subsmooth) at $\overline{x} \in S$ is tangentially regular at the point $\overline{x}$.*

Proof. Fix any $x^* \in \partial_C d_S(\overline{x})$ and any $h \in T^B(S;\overline{x})$. By definition of metric hemi-subsmoothness, for each $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$\langle x^*, x - \overline{x}\rangle \le \varepsilon\|x - \overline{x}\| \quad \text{for all } x \in S \cap B(\overline{x}, \delta). \tag{7.4}$$

Choose, by definition of $T^B(S;\overline{x})$, a sequence $(t_n)_n$ tending to 0 with $t_n > 0$ and a sequence $(h_n)_n$ converging to h such that $\overline{x} + t_n h_n \in S$ for all n. For n large enough we have $\overline{x} + t_n h_n \in B(\overline{x}, \delta)$, and hence according to (7.4) we obtain $\langle x^*, h_n\rangle \le \varepsilon\|h_n\|$, which entails $\langle x^*, h\rangle \le \varepsilon\|h\|$. This being true for every $\varepsilon > 0$, we get $\langle x^*, h\rangle \le 0$. Thus $\partial_C d_S(\overline{x}) \subset (T^B(S;\overline{x}))^o$, where the second member of the inclusion denotes the negative polar cone of $T^B(S;\overline{x})$. The equality $N^C(S;\overline{x}) = \mathrm{cl}_{w^*}\big(\mathbb{R}_+ \partial d_S(\overline{x})\big)$ and the $w(X^*, X)$-closedness of $(T^B(C;\overline{x}))^o$ then yield $N^C(S;\overline{x}) \subset (T^B(S;\overline{x}))^o$. Using the equality $N^C(S;\overline{x}) = \big(T^C(S;\overline{x})\big)^o$ and the $w(X, X^*)$-closedness of $T^C(S;\overline{x})$, we can write $\big(T^B(C;\overline{x})\big)^{oo} \subset T^C(S;\overline{x})$, and hence $T^B(S;\overline{x}) \subset T^C(S;\overline{x})$. This means that the set S is tangentially regular at $\overline{x}$ since the reverse inclusion of the latter inclusion always holds. □

Example 7.11 below will provide an example of a closed set tangentially regular at a point $\overline{x}$ which fails to be metrically subsmooth at $\overline{x}$. In contrast, we will see in Proposition 7.13 that in finite dimensions a closed set is tangentially regular at a point $\overline{x}$ if and only if it is hemi-subsmooth at $\overline{x}$ (or equivalently metrically hemi-subsmooth at $\overline{x}$ by Proposition 7.7).

The next proposition examines when one of the two points x or y in the definition of metric subsmoothness is allowed to be outside S (but near $\overline{x}$).

Proposition 7.5. *Let S be a set of a normed space X which is closed near $\overline{x} \in S$. Then the following assertions (a) and (b) are equivalent.*

(a) *The set S is metrically subsmooth at $\overline{x}$.*

(b) *For any $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $y \in B(\overline{x}, \delta)$, all $x \in S \cap B(\overline{x}, \delta)$, and all $x^* \in \partial_C d_S(x)$*

$$\langle x^*, y - x\rangle \le d_S(y) + \varepsilon\|y - x\|.$$

If X is Asplund, each one of the above assertions is equivalent to:

(c) *for any $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $x \in B(\overline{x}, \delta)$, all $x^* \in \partial_C d_S(x)$, and all $y \in S \cap B(\overline{x}, \delta)$ one has*

$$\langle x^*, y - x\rangle \le d_S(x) + \varepsilon\|y - x\|.$$

Proof. The implication $(b) \Rightarrow (a)$ is obvious. Assume that (a) holds and fix any $\varepsilon > 0$. For $\varepsilon' := \varepsilon/2$ choose by definition of metric subsmoothness some $\delta > 0$ such that for all $u, u' \in S \cap B(\overline{x}, 2\delta)$ and all $u^* \in \partial_C d_S(u)$ one has $\langle u^*, u' - u\rangle \le \varepsilon'\|u' - u\|$. Fix any $y \in B(\overline{x}, \delta)$, any $x \in S \cap B(\overline{x}, \delta)$, and any $x^* \in \partial_C d_S(x)$. Take any $z \in S \cap B(\overline{x}, 2\delta)$. Then

$$\begin{aligned}\langle x^*, y - x\rangle &= \langle x^*, y - z\rangle + \langle x^*, z - x\rangle \\ &\le \|y - z\| + \varepsilon'\|z - x\| \\ &\le (1 + \varepsilon')\|y - z\| + \varepsilon'\|y - x\|.\end{aligned}$$

Taking the infimum over all $z \in S \cap B(\overline{x}, 2\delta)$ we obtain

$$\langle x^*, y - x\rangle \le (1 + \varepsilon')\mathrm{dist}\,(y, S \cap B(\overline{x}, 2\delta)) + \varepsilon'\|y - x\|$$

and observing that $\mathrm{dist}\,(y, S \cap B(\overline{x}, 2\delta)) = \mathrm{dist}\,(y, S)$ by the inclusion $y \in B(\overline{x}, \delta)$ (see Lemma 2.2(a)), we may write

$$\begin{aligned}\langle x^*, y - x\rangle &\le (1 + \varepsilon')d_S(y) + \varepsilon'\|y - x\| \\ &\le d_S(y) + 2\varepsilon'\|y - x\|,\end{aligned}$$

the second inequality being due to the fact that $d_S(y) \le \|y - x\|$ since $x \in S$. The last inequality translates the property (b) since $2\varepsilon' = \varepsilon$, so we have proved (a)⇒(b).

The implication $(c) \Rightarrow (a)$ is obvious. Assume that X is Asplund and that (a) holds. Without loss of generality we may suppose that S is closed. Fix any $\varepsilon > 0$ and take some $\varepsilon' > 0$ such that $2\varepsilon'+\varepsilon'(2+\varepsilon') < \varepsilon$. By definition of metric subsmoothness, choose some $\delta > 0$ such that for all $u, u' \in S \cap B(\overline{x}, 3\delta)$ and $u^* \in \partial_C d_S(u)$ one has

$$\langle u^*, u' - u\rangle \le \varepsilon'\|u' - u\|. \tag{7.5}$$

Fix any $x \in B(\overline{x}, \delta) \setminus S$ and $y \in S \cap B(\overline{x}, \delta)$. Suppose that $\partial_F d_S(x) \ne \emptyset$ and take any $x^* \in \partial_F d_S(x)$. Applying Proposition 2.5(e) with any positive $\varepsilon'' < \min\{\delta, \varepsilon', \varepsilon'\mathrm{dist}\,(x, S)\}$ in place of ε, we obtain some $v \in S$ and $v^* \in \partial_F d_S(v)$ such that

$$\|v - x\| < \varepsilon'' + d_S(x) < (1 + \varepsilon')d_S(x) \quad \text{and} \quad \|v^* - x^*\| < \varepsilon'. \tag{7.6}$$

Observe that by the first inequality of (7.6) we have

$$\|v - \overline{x}\| \le \|v - x\| + \|x - \overline{x}\| < \varepsilon'' + d_S(x) + \|x - \overline{x}\|,$$

and hence by the inclusions $\overline{x} \in S$ and $x \in B(\overline{x}, \delta)$ we obtain

$$\|v - \overline{x}\| < \varepsilon'' + 2\|x - \overline{x}\| < 3\delta. \tag{7.7}$$

Keeping in mind that $y \in S \cap B(\overline{x}, \delta)$, we see that

$$\begin{aligned}\langle x^*, y - x\rangle &\le \langle v^*, y - x\rangle + \varepsilon'\|y - x\| \\ &= \langle v^*, y - v\rangle + \langle v^*, v - x\rangle + \varepsilon'\|y - x\| \\ &\le \varepsilon'\|y - v\| + \|v - x\| + \varepsilon'\|y - x\|,\end{aligned}$$

the first inequality being due to the last inequality of (7.6) and the second one being due to (7.5) and (7.7) and to the fact that $\|v^*\| \le 1$. Taking the second inequality of the first part of (7.6) into account it ensues that

$$\begin{aligned}\langle x^*, y - x\rangle &\le 2\varepsilon'\|y - x\| + (1 + \varepsilon')\|v - x\| \\ &\le 2\varepsilon'\|y - x\| + (1 + \varepsilon')d_S(x) + \varepsilon'(1 + \varepsilon')d_S(x),\end{aligned}$$

which gives (since $y \in S$)

$$\begin{aligned}\langle x^*, y - x\rangle &\le 2\varepsilon'\|y - x\| + d_S(x) + \varepsilon'(2 + \varepsilon')\|y - x\| \\ &= (2\varepsilon' + \varepsilon'(2 + \varepsilon'))\|y - x\| + d_S(x).\end{aligned}$$

The choice of ε' yields

$$\langle x^*, y - x\rangle \le d_S(x) + \varepsilon\|y - x\| \tag{7.8}$$

and this is satisfied for any $x \in B(\overline{x}, \delta)$ since the case when $x \in S \cap B(\overline{x}, \delta)$ follows from (7.5). The definition of any limiting subgradient at x (as the weak-star limit of some sequence of Fréchet subgradients at points x_n converging strongly to x) ensures us that (7.8) continues to hold for any $x \in B(\overline{x}, \delta)$ and any $x^* \in \partial_L d_S(x)$, and of course for any $x^* \in \operatorname{co} \partial_L d_S(x)$. Take now any $y \in S \cap B(\overline{x}, \delta)$, any $x \in B(\overline{x}, \delta)$ and any $x^* \in \partial_C d_S(x)$. Since $\partial_C d_S(x) = \overline{\operatorname{co}}^*(\partial_L d_S(x))$ (see Proposition 2.5(f)) there exists a net $(x_j^*)_{j \in J}$ converging weakly* to x^* with $x_j^* \in \operatorname{co} \partial_L d_S(x)$. Then taking the limit with respect to $j \in J$ in the inequality (7.8), it is easily seen that the inequality is still true for such x^*, x, and y, that is, the property (c) is obtained. This finishes the proof of the proposition. □

The same arguments with $N^C(S; x) \cap \mathbb{B}_{X^*}$ in place of $\partial_C d_S(x)$ in the proof of the implication (a) $\Rightarrow$ (b) in the above proposition yields the following equivalence for subsmooth sets.

Proposition 7.6. *For any set S of a normed space X which is closed near $\overline{x} \in S$, the assertions (a) and (b) below are equivalent:*

(a) *The set S is subsmooth at $\overline{x}$.*
(b) *For any $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $y \in B(\overline{x}, \delta)$, all $x \in S \cap B(\overline{x}, \delta)$, and all $x^* \in N^C(S; x) \cap \mathbb{B}_{X^*}$*

$$\langle x^*, y - x \rangle \le d_S(y) + \varepsilon \|y - x\|.$$

A property similar to (b) of Proposition 7.5 also holds for metric hemi-subsmooth sets. Such sets have been seen in Proposition 7.4 to be tangentially regular. In fact the corresponding characterization below of hemi-subsmooth sets will allow us to prove more, in the sense that we even have the stronger property of tangential regularity of the distance function d_S. We will also be able to show that hemi-subsmoothness and metric hemi-subsmoothness are the same property.

Proposition 7.7. *Let S be a subset of a normed space X which is closed near $\overline{x} \in S$. Consider the following assertions.*

(a) *The set S is metrically hemi-subsmooth at $\overline{x}$.*
(b) *The set S is hemi-subsmooth at $\overline{x}$.*
(c) *For any $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $x \in B(\overline{x}, \delta)$ and all $u^* \in \partial_C d_S(\overline{x})$ one has*

$$\langle u^*, x - \overline{x} \rangle \le d_S(x) + \varepsilon \|x - \overline{x}\|.$$

(d) *The distance function d_S is Clarke-Fréchet regular at $\overline{x}$, in the sense that*

$$\partial_C d_S(\overline{x}) = \partial_F d_S(\overline{x}).$$

(e) *The distance function d_S is tangentially regular at $\overline{x}$, that is,*

$$d_S^o(\overline{x}; \cdot) = d_S'(\overline{x}; \cdot).$$

(f) $N^C(S; \overline{x}) = N^F(S; \overline{x})$ *and* $N^C(S; \overline{x}) \cap \mathbb{B} = \partial_C d_S(\overline{x})$.

Then (b) $\Rightarrow$ (a) $\Leftrightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e). *If X is a Banach space, the implications* (a) $\Leftrightarrow$ (b) *and* (d) $\Rightarrow$ (f) *also hold.*

Proof. The implications (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a) are obvious and (a) $\Rightarrow$ (c) is obtained like for (a) $\Rightarrow$ (b) in Proposition 7.5. The property (c) entails that any $x^* \in \partial_C d_S(\overline{x})$ is a Fréchet subgradient of d_S at $\overline{x}$, and hence the equality $\partial_C d_S(\overline{x}) = \partial_F d_S(\overline{x})$. This means that (c) $\Rightarrow$ (d) holds.

Suppose now that (d) is satisfied. It is not difficult to see that for any function f and $x, h \in X$ one has $\sup_{x^* \in \partial_F f(x)} \langle x^*, h\rangle \le f^B(x;h)$, and hence by (d)

$$(d_S)^o(\overline{x};h) = \sup_{x^* \in \partial_C d_S(\overline{x})} \langle x^*, h\rangle \le (d_S)^B(\overline{x};h).$$

The reverse inequality being always true, we obtain the directional regularity of d_S at $\overline{x}$, that is, (d) $\Rightarrow$ (e) is shown.

Let us prove (f) under (d) and the completeness of X. Since $\partial_F d_S(\overline{x}) = N^F(S;\overline{x}) \cap \mathbb{B}_{X^*}$ (see (2.8)), the equality in (d) assures us that

$$\partial_C d_S(\overline{x}) = N^F(S;\overline{x}) \cap \mathbb{B}_{X^*}. \tag{7.9}$$

Thanks to the weak-star closedness of the Clarke subdifferential of a function, the latter equality yields that $N^F(S;\overline{x}) \cap \mathbb{B}_{X^*}$ is weak-star closed and hence $N^F(S;\overline{x})$ is weak-star closed as well, according to the Krein-Šmulian theorem since $N^F(S;\overline{x})$ is a convex cone of X^*. This weak-star closedness property along with (7.9) gives

$$N^C(S;\overline{x}) = \mathrm{cl}_{w^*}[\mathbb{R}_+ \partial_C d_S(\overline{x})] = \mathrm{cl}_{w^*}[N^F(S;\overline{x})] = N^F(S;\overline{x}), \tag{7.10}$$

that is, $N^C(S;\overline{x}) = N^F(S;\overline{x})$. Combining the latter equality with

$$N^F(S;\overline{x}) \cap \mathbb{B}_{X^*} = \partial_F d_S(\overline{x}) = \partial_C d_S(\overline{x}),$$

we see that $N^C(S;\overline{x}) \cap \mathbb{B}_{X^*} = \partial_C d_S(\overline{x})$. So the implication (d)$\Rightarrow$(f) holds.

It remains to establish (a) $\Rightarrow$ (b) if X is a Banach space. Under (a) and the completeness of X we know by what precedes that (f) holds, and hence $\partial_C d_S(\overline{x}) = N^C(S;\overline{x}) \cap \mathbb{B}$. Consequently, the metric hemi-subsmoothness of S at $\overline{x}$ implies its hemi-subsmoothness at $\overline{x}$ as well. The proof is then complete. ☐

The next theorem provides in addition to Proposition 7.5 some other characterizations of metric subsmoothness in the context of Asplund space but its interest essentially rests on the important characterizations furnished by (*e*) and (*f*) when the space X is Hilbert or finite-dimensional.

Theorem 7.8. *Assume that X is an Asplund space and S is a subset of X which is closed near $\overline{x} \in S$. Then the following assertions are equivalent:*

(a) *The set S is metrically subsmooth at $\overline{x}$.*
(b) *The multimapping $\partial_F d_S$ is submonotone at $\overline{x}$ relative to the set S*
(c) *The multimapping $\partial_L d_S$ is submonotone at $\overline{x}$ relative to the set S.*
(d) *The multimapping $N^F(S;\cdot) \cap \mathbb{B}_{X^*}$ is submonotone at $\overline{x}$.*

If in addition X is a Hilbert space, the following assertions may be added to the list of equivalences:

(e) *For any $\varepsilon > 0$ there exists some $\delta > 0$ such that, for all $y \in X$, $x, u \in S \cap B(\overline{x}, \delta)$ with $u \in \mathrm{Proj}_S(y)$ one has*

$$\langle y - u, x - u\rangle \le \varepsilon \|y - u\| \, \|x - u\|.$$

(f) *The multimapping* $N^P(S;\cdot)\cap\mathbb{B}$ *is submonotone at* $\overline{x}$.

(g) *The multimapping* $\partial_P d_S$ *is submonotone at* $\overline{x}$ *relative to the set* S.

If X *is finite-dimensional, then anyone of all the above properties is equivalent to:*

(h) *The multimapping* $N^L(S;\cdot)\cap\mathbb{B}$ *is submonotone at* $\overline{x}$.

Proof. The equivalence between (a), (b), and (c) is a reformulation of Lemma 5.17. The equivalence between (b) and (d) follows from the equality $\partial_F d_S(x) = N^F(S;x)\cap\mathbb{B}$ for all $x\in S$ (see (2.8)).

Assume now that X is a Hilbert space. Without loss of generality we may suppose that S is closed. We know (see Section 2) that any proximal normal vector of the form $\|y-u\|^{-1}(y-u)$, for $u\in\mathrm{Proj}_S(y)$ with $d_S(y)>0$, is a unit Fréchet normal vector of S at u and this justifies the implication (d) $\Rightarrow$ (e). Taking into account the definition of the proximal normal cone it is easily seen that (e) entails that the multimapping $N^P(S;\cdot)\cap\mathbb{B}$ is submonotone at $\overline{x}$, which is exactly (e) $\Rightarrow$ (f). By (2.13) we know that $\partial_P d_S(x) = N^P(S;x)\cap\mathbb{B}$ for all $x\in S$. Therefore, the assertion (g) is just a reformulation of (f). Let us show that (g) implies (c). Fix any $\varepsilon>0$. By the assumption (g) there exists some $\delta>0$ such that $\langle w^*, y-w\rangle\le\varepsilon\|y-w\|$ for all $y,w\in S\cap B(\overline{x},\delta)$ and $w^*\in\partial_P d_S(w)$. Fix any $y,x\in S\cap B(\overline{x},\delta)$ and $x^*\in\partial_L d_S(x)$. By Proposition 2.7(d) there are a sequence $(x_n)_n$ in S converging to x and a sequence $(x_n^*)_n$ converging weakly to x^* with $x_n^*\in\partial_P d_S(x_n)$. For n large enough we have $\langle x_n^*, y-x_n\rangle\le\varepsilon\|y-x_n\|$, and hence $\langle x^*, y-x\rangle\le\varepsilon\|y-x\|$. So the equivalence of any assertion among (a) to (d) with anyone of (e) to (g) is established under the Hilbert assumption of X.

The implication (h)$\Rightarrow$(d) being obvious, suppose that (d) holds and X is finite-dimensional. Without loss of generality, we may suppose that the norm of X is a Euclidean norm and we may identify X^* with X through the Euclidean inner product. Fix any $\varepsilon>0$ and take $\delta>0$ such that $\langle w, x-y\rangle\le\frac{\varepsilon}{2}\|x-y\|$ for all $x,y\in S\cap B(\overline{x},\delta)$ and $w\in N^F(S;y)\cap\mathbb{B}$. Fix any $x,u\in S\cap B(\overline{x},\delta)$ and any $v\in N^L(S;u)$ with $\|v\|=1$. By definition of $N^L(S;u)$, there exist some sequence $(u_n)_n$ in $S\cap B(\overline{x},\delta)$ converging to u and some sequence $(v_n)_n$ converging to v with $v_n\in N^F(S;u_n)$. Then for any integer n sufficiently large we have

$$\langle\frac{1}{\|v_n\|}v_n, x-u_n\rangle\le(\varepsilon/2)\|x-u_n\|,$$

which yields

$$\langle v, x-u\rangle\le(\varepsilon/2)\|x-u\|\quad\text{for all } x,u\in S\cap B(\overline{x},\delta)\text{ and } v\in N^L(S;u)\cap\mathbb{S}.$$

This property is easily seen to entail the submonotonicity of the multimapping $N^L(S;\cdot)\cap\mathbb{B}$ at $\overline{x}$. So the equivalence between (h) and (d) is established provided that X is finite-dimensional. The proof is then complete. □

Remark 7.9. The property (e) in Theorem 7.8 has been used in $\mathbb{R}^n$ by A.S. Lewis, R.D. Luke and J. Malick [50]. Roughly speaking, as noticed in [50] it means that the angle between a unit proximal normal vector $x^*\in N^P(S;x)$ and $(y-x)/\|y-x\|$ is not much less than $\pi/2$ for $x,y\in S$ sufficiently close to $\overline{x}$.

We use Theorem 7.8(e) in the next two examples. The first one is an example in $\mathbb{R}^2$ of a metrically subsmooth set at a point $\overline{x}$ which fails to be subsmooth at $\overline{x}$.

Example 7.10 (Lewis-Luke-Malick example [50]). Let $f : [-1, 1] \to \mathbb{R}$ be the continuous even (f(-t)=f(t)) function defined by $f(1/2^n) = 1/4^n$ and f is affine on $[1/2^{n+1}, 1/2^n]$ for all $n \in \mathbb{N}$. Considering S as the graph of f, $S := \operatorname{gph} f$, we see by Theorem 7.8(e) that S is metrically subsmooth at $\overline{x} = (0, 0)$ since the angle between a unit proximal normal vector $x^* \in N^P(S; x)$ and $(y - x)/\|y - x\|$ is not much less than $\pi/2$ for $x, y \in S$ sufficiently close to $\overline{x}$, as illustrated in Figure 7.1.

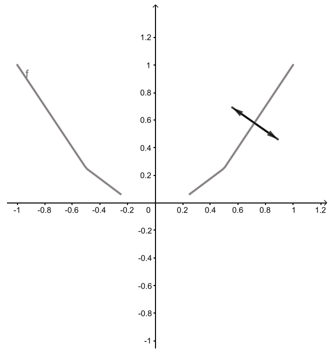

FIGURE 7.1. A metrically subsmooth set at $\overline{x}$ but not subsmooth at $\overline{x}$

On the other hand, setting $x_n := (1/2^n, 1, /4^n)$ we see that $N^C(S; x_n) = \mathbb{R}^2$ since $T^C(S; x_n) = \{(0, 0)\}$ (as illustrated in Figure 7.1). So, taking $u_n := (\overline{x} - x_n)/\|\overline{x} - x_n\|$ we have

$$u_n \in N^C(S; x_n), \quad \langle u_n, \frac{\overline{x} - x_n}{\|\overline{x} - x_n\|} \rangle = 1, \quad S \ni x_n \to \overline{x}.$$

Consequently, the set S is not subsmooth at $\overline{x}$.

The second example provides a set in $\mathbb{R}^2$ which is tangentially regular at a point $\overline{x}$ but not metrically subsmooth at $\overline{x}$.

Example 7.11 (Lewis-Luke-Malick example [50]). Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be the lower semicontinuous even function defined by $f(0) = 0$, $f(1) = 1/4$, $f(t) = +\infty$ if $t > 1$ and

$$f(t) = \frac{1}{2^n}(t - \frac{1}{2^n}) \quad \text{if } t \in [\frac{1}{2^n}, \frac{1}{2^{n-1}}[, \ n \in \mathbb{N}.$$

Define S as the epigraph of f, $S := \operatorname{epi} f$, and $\overline{x} = (0, 0)$. We claim that S is tangentially regular at $\overline{x}$ but it fails to be metrically subsmooth at $\overline{x}$.

First, it is clear that

$$T^B(S; \overline{x}) = \mathbb{R}^2_+ \quad \text{and} \quad \operatorname*{Lim\,inf}_{S' \ni x \to \overline{x}} T^B(S; x) = \mathbb{R}^2_+,$$

where $S' := S \backslash \{\overline{x}\}$, so $T^B(S; \overline{x}) = \operatorname*{Lim\,inf}_{S \ni x \to \overline{x}} T^B(S; x) = \mathbb{R}^2_+$. Since $\operatorname*{Lim\,inf}_{S \ni x \to \overline{x}} T^B(S; x) = T^C(S; \overline{x})$ (see (2.4)) , it ensues that S is tangentially regular at $\overline{x}$.

On the other hand, for $x_n := (1/2^n, 0)$ we have $u_n := (-1, 0) \in N^P(S; x_n)$, as seen in Figure 7.2. Further,

$$\langle u_n, \frac{x_{n+1} - x_n}{\|x_{n+1} - x_n\|} \rangle = 1 \quad \text{with} \quad S \ni x_n \to \overline{x}.$$

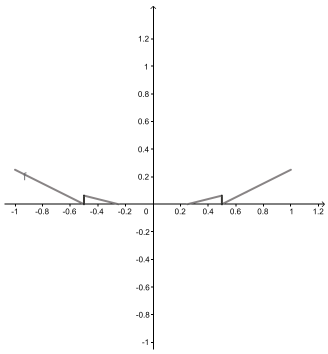

FIGURE 7.2. A tangentially regular set at $\overline{x}$ but not metrically subsmooth at $\overline{x}$

It follows by Theorem 7.8(e) that S is not metrically subsmooth at $\overline{x}$.

The next proposition compares the metric subsmoothness of a set with the Jensen-type inequality of the distance function involved in (a) of Proposition 4.21. The implications (a)$\Rightarrow$(b) and (c)$\Rightarrow$(a) in this proposition are taken from Theorem 10 and Corollary 11 respectively in the paper [58] by H.V. Ngai and J.-P. Penot; the proofs given here follow [58].

Proposition 7.12. *Let S be subset of a normed space X and $\overline{x} \in S$. Consider the assertions:*

(a) *For any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in B(\overline{x}, \delta)$, $x^* \in \partial_C d_S(x)$ and $y \in S \cap B(\overline{x}, \delta)$*
$$d_S(x) + \langle x^*, y - x\rangle \le \varepsilon \|y - x\|.$$
(b) *For every $\varepsilon > 0$ there exists $\delta > 0$ such that*
$$d_S(tx + (1-t)y) \le \varepsilon t(1-t)\|x - y\|$$
for all $t \in [0, 1]$ and all $x, y \in S \cap B(\overline{x}, \delta)$.

(c) *The set S is metrically subsmooth at $\overline{x}$.*

The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) *hold. If X is an Asplund space, these implications are equivalences.*

Proof. First, we note that the implication (b) $\Rightarrow$ (c) follows directly from Proposition 4.21. Now, assume that (a) holds and fix any $\varepsilon > 0$. By (a) there is $\delta > 0$ such that for all $x' \in B(\overline{x}, \delta)$, $x^* \in \partial_C d_S(x')$ and $y \in S \cap B(\overline{x}, \delta)$
$$d_S(x') + \langle x^*, y - x'\rangle \le (\varepsilon/2)\|y - x'\|.$$
Let any $x, y \in S \cap B(\overline{x}, \delta)$ with $x \ne y$ and any $t \in]0, 1[$, and set $u := tx + (1-t)y$. By Lebourg mean value equality (see Proposition 2.1(b)) there are $z \in [x, u]$, $z^* \in \partial_C d_S(z)$ such that with $\alpha := \|y - z\|^{-1}\|u - x\|$
$$d_S(u) = \langle z^*, u - x\rangle = \alpha\langle z^*, y - z\rangle,$$
which entails by the above inequality that
$$d_S(u) \le -\alpha d_S(z) + \frac{\varepsilon}{2}\alpha\|y - z\| = -\alpha d_S(z) + \frac{\varepsilon}{2}\|u - x\| = -\alpha d_S(z) + \frac{\varepsilon}{2}(1-t)\|y - x\|.$$

It results that

$$(7.11)\qquad d_S(u) \le \frac{\varepsilon}{2}(1-t)\|y-x\|.$$

Similarly, there are $\zeta \in [u,y]$ and $\zeta^* \in \partial_C d_S(\zeta)$ such that with $\beta := \|x-\zeta\|^{-1}\|u-y\|$

$$d_S(u) = \langle \zeta^*, u-y\rangle \le -\beta d_S(\zeta) + \frac{\varepsilon}{2}\beta\|y-\zeta\| = -\beta d_S(\zeta) + \frac{\varepsilon}{2}t\|y-x\|,$$

which implies that

$$(7.12)\qquad d_S(u) \le (\varepsilon/2)t\|y-x\|.$$

Multiplying (7.11) by t and (7.12) by $(1-t)$, and adding the new inequalities yield

$$d_S(u) \le \varepsilon t(1-t)\|y-x\|,$$

which justifies the implication (a) $\Rightarrow$ (b).

Now, assume that X is an Asplund space and that (c) holds. Let any $\varepsilon > 0$. There is $\delta > 0$ such that for any $z, z' \in S \cap B(\overline{x},\delta)$ and $z^* \in \partial_C d_S(z)$ one has $\langle z^*, z'-z\rangle \le (\varepsilon/4)\|z'-z\|$. Fix any $y \in S \cap B(\overline{x},\delta)$ and any $(x,x^*) \in \operatorname{gph}\partial_F d_S(x)$ with $x \in B(\overline{x},\delta)\backslash S$. By Proposition 2.5(e) there are $u_n \in S \cap B(\overline{x},\delta)$, $u_n^* \in \partial_F d_S(u_n)$ such that

$$\|x-u_n\| \to d_S(x) \quad \text{and} \quad \|x^*-u_n^*\| \to 0 \quad \text{as } n \to \infty.$$

Further, one has $\langle x^*, x-u_n\rangle \to d_S(x)$ according to Lemma 2.6. Write with $A_n := \langle x^*, x-u_n\rangle + \langle x^*-u_n^*, u_n-y\rangle$ that

$$\langle x^*, x-y\rangle = \langle x^*, x-u_n\rangle + \langle x^*, u_n-y\rangle = A_n + \langle u_n^*, u_n-y\rangle,$$

and note by what precedes that

$$\langle u_n^*, u_n-y\rangle \ge -(\varepsilon/4)\|u_n-y\| \ge -(\varepsilon/4)\|x-y\| - (\varepsilon/4)\|u_n-x\|.$$

Since $\|u_n-x\| \to d_S(x)$ and $d_S(x) > 0$, we have $(\varepsilon/4)\|u_n-x\| \le (\varepsilon/4)d_S(x) + (\varepsilon/4)d_S(x)$ for n large enough, say $n \ge N$. Noting also that $A_n \to d_S(x)$, we may choose an integer $k \ge N$ such that $A_k \ge d_S(x) - (\varepsilon/4)d_S(x)$. Taking $n = k$ in the preceding inequalities, it follows that

$$\begin{aligned}\langle x^*, x-y\rangle &\ge d_S(x) - \frac{\varepsilon}{4}d_S(x) - \frac{\varepsilon}{4}\|x-y\| - \frac{\varepsilon}{2}d_S(x)\\ &\ge d_S(x) - \varepsilon\|x-y\|.\end{aligned}$$

This ensures that $\langle x^*, x-y\rangle \ge d_S(x) - \varepsilon\|y-x\|$ for all $(x,x^*) \in \operatorname{gph}\partial_F d_S(x)$ with $x \in B(\overline{x},\delta)$, since the case $x \in S \cap B(\overline{x},\delta)$ is obvious by the choice of δ above. Remembering the definition of L-subdifferential, we deduce that the inequality holds true for all $(x,x^*) \in \operatorname{gph}\partial_L d_S(x)$ with $x \in B(\overline{x},\delta)$. Finally, the equality $\partial_C d_S(v) = \overline{\text{co}}^*\,\partial_L d_S(v)$ for every $v \in X$ (see Proposition 2.5(g)) assures us that $\langle x^*, x-y\rangle \ge d_S(x) - \varepsilon\|y-x\|$ for all $y \in S \cap B(\overline{x},\delta)$, $x \in B(\overline{x},\delta)$ and $x^* \in \partial_C d_S(x)$, and this finishes the proof of the proposition. □

Let us establish in addition to Proposition 7.7 two other characterizations of hemi-subsmooth sets. We also show that, in finite dimensions, the hemi-subsmoothness of S at $\overline{x}$ is equivalent to its tangential regularity at $\overline{x}$.

Proposition 7.13. *Let S a be set of a normed space X and let $\overline{x} \in S$. The following are equivalent:*

(a) *The set S is hemi-subsmooth at $\overline{x}$.*
(b) *For every $\overline{x}^* \in N^C(S;\overline{x})$, every $\beta > 0$, and every $\varepsilon > 0$ there exists $\delta > 0$ such that (5.5) holds for all $x \in S \cap B(\overline{x},\delta)$ and all $x^* \in N^C(S;\overline{x})$ with $\|x^* - \overline{x}^*\| \le \beta$.*
(c) *For every $\overline{x}^* \in N^C(S;\overline{x})$ there is $\beta > 0$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that (5.5) holds for all $x \in S \cap B(\overline{x},\delta)$ and $x^* \in N^C(S;\overline{x})$ with $\|x^* - \overline{x}^*\| \le \beta$.*
(d) *For every $\overline{x}^* \in N^C(S;\overline{x})$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that (5.5) holds for all $x \in S \cap B(\overline{x},\delta)$ and $x^* \in N^C(S;\overline{x})$ with $\|x^* - \overline{x}^*\| \le \varepsilon$.*

If the space X is finite-dimensional, then one may also add to the list of equivalences the property (e) below:

(e) *The set S is tangentially regular at $\overline{x}$.*

Proof. Suppose that (a) holds. Fix $\overline{x}^* \in N^C(S;\overline{x})$, $\beta > 0$ and $\varepsilon > 0$. By (a) choose some $\delta > 0$ such that
$$\langle x^*, x - \overline{x}\rangle \le \frac{\varepsilon}{\beta + \|\overline{x}^*\|}\|x - \overline{x}\|$$
for all $x \in S \cap B(\overline{x},\delta)$ and $x^* \in N^C(S;\overline{x}) \cap \mathbb{B}_{X^*}$. Then for any $x \in S \cap B(\overline{x},\delta)$ and any $x^* \in N^C(S;\overline{x})$ with $\|x^* - \overline{x}^*\| \le \beta$, we have $\|\rho^{-1}x^*\| \le 1$, for $\rho := \beta + \|\overline{x}^*\|$, and hence $\langle \rho^{-1}x^*, x - \overline{x}\rangle \le \rho^{-1}\varepsilon\|x - \overline{x}\|$, that is, $\langle x^*, x - \overline{x}\rangle \le \varepsilon\|x - \overline{x}\|$. This means that (b) holds.

The fact that (b) implies both (c) and (d) is obvious. To see that (c) (resp. (d)) entails (a), assume (c) (resp. (d)) and take some $\beta > 0$ corresponding to the choice $\overline{x}^* = 0$ in (c) (resp. and take the choice $\overline{x}^* = 0$ in (d)). Fix any $\varepsilon > 0$ and by (c) (resp. (d)) choose $\delta > 0$ such that $\langle x^*, x - \overline{x}\rangle \le \varepsilon\beta\|x - \overline{x}\|$ for any $x \in S \cap B(\overline{x},\delta)$ and $x^* \in N^C(S;\overline{x})$ with $\|x^*\| \le \beta$ (resp. $\langle x^*, x - \overline{x}\rangle \le \varepsilon^2\|x - \overline{x}\|$ for any $x \in S \cap B(\overline{x},\delta)$ and $x^* \in N^C(S;\overline{x})$ with $\|x^*\| \le \varepsilon$) . Anyone of the two latter properties is easily seen to give $\langle x^*, x - \overline{x}\rangle \le \varepsilon\|x - \overline{x}\|$ for all $x \in S \cap B(\overline{x},\delta)$ and $x^* \in N^C(S;\overline{x}) \cap \mathbb{B}_{X^*}$, that is, (a) is fulfilled.

Now assume that X is finite-dimensional. We note first that (a) $\Rightarrow$ (e) is true by Proposition 5.14(b). Conversely, suppose that (a) does not hold. Then, there exist a real $\varepsilon > 0$, a sequence $(x_n)_n$ in S converging to $\overline{x}$ and a sequence $(x_n^*)_n$ in $N^C(S;\overline{x}) \cap \mathbb{B}_{X^*}$ such that $\langle x_n^*, x_n - \overline{x}\rangle > \varepsilon\|x_n - \overline{x}\|$ for all $n \in \mathbb{N}$. Extracting subsequences, we may suppose that $x_n^* \to x^*$ and $\|x_n - \overline{x}\|^{-1}(x_n - \overline{x}) \to h$. Clearly, $x^* \in N^C(S;\overline{x})$ and $\langle x^*, h\rangle \ge \varepsilon$, hence $h \notin T^C(S;\overline{x})$. Further, the convergence $\|x_n - \overline{x}\|^{-1}(x_n - \overline{x}) \to h$ also ensures that $h \in T^B(S;\overline{x})$, so S is not tangentially regular at $\overline{x}$, which finishes the proof. □

Remark 7.14. Characterizations of subsmoothness of the set S similar to (a) and (b) in Proposition 7.13 also hold true with the same arguments.

8. Subsmoothness of a set and α-far property of the C-subdifferential of its distance function

For a closed S of a normed space X and for any $x \in X \setminus S$ we know (see (2.9)) that $\|x^*\| = 1$ for any $x^* \in \partial_F d_S(x)$, and also for $x^* \in \partial_L d_S(x)$ whenever X is finite-dimensional. So, given $\overline{x} \in \operatorname{bdry} S$ the following question arises: Is there a real $\delta > 0$ such that zero is kept far away from the C-subdifferential on $B(\overline{x}, \delta) \setminus S$, that is, $0 \notin \partial_C d_S(x)$ for all $x \in B(\overline{x}, \delta) \setminus S$? For some sets the answer is negative. Consider in $\mathbb{R}$ the closed set $S := \{0\} \cup \{1/n : n \in \mathbb{N}\}$ and $0 \in \operatorname{bdry} S$. Putting $\mu_n := \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right)$, we see that

$$d_S(x) = \frac{1}{n} - x \text{ if } x \in [\mu_n, \frac{1}{n}[\quad \text{and} \quad d_S(x) = x - \frac{1}{n+1} \text{ if } x \in]\frac{1}{n+1}, \mu_n].$$

The gradient representation of C-subdifferential (see Proposition 2.1(g)) enables us to say that $\partial_C d_S(\mu_n) = [-1, 1] \ni 0$. Consequently, there is no neighborhood U of $\overline{x} := 0$ such that the set S enjoys the above desirable positively far property of zero from the C-subdifferential of d_S on $U \setminus S$.

Suppose now that X is a finite-dimensional Euclidean space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$ and that S is a closed subset which is metrically subsmooth at $\overline{x} \in \operatorname{bdry} S$. Let any real $\varepsilon \in]0, 1[$. By metric subsmoothness choose a real $\delta > 0$ such that

$$\langle z_1^* - z_2^*, z_1 - z_2 \rangle \geq -\varepsilon \|z_1 - z_2\|$$

for all $z_i \in S \cap B(\overline{x}, 2\delta)$ and $z_i^* \in \partial_C d_S(z_i)$, i$= 1, 2$. Fix any $x \in B(\overline{x}, \delta) \setminus S$. We know (see [68, Example 8.53]) that

$$\partial_L d_S(x) = \frac{1}{d_S(x)}(x - \operatorname{Proj}_S x).$$

Take $x_1^*, x_2^* \in \partial_L d_S(x)$, so $x_i^* = (x - u_i)/d_S(x)$ with $u_i \in \operatorname{Proj}_S x$ for $i = 1, 2$. Then $x_i^* \in N^P(S; u_i) \cap \mathbb{B}_X = \partial_P d_S(u_i)$ (see (2.13)), which yields

$$\langle x_1^* - x_2^*, u_1 - u_2 \rangle \geq -\varepsilon \|u_1 - u_2\|$$

since $\|u_i - \overline{x}\| \leq 2\|x - \overline{x}\| < 2\delta$. Put $t := d_S(x)$ and note that

$$\begin{aligned} \langle x_1^* - x_2^*, u_1 - u_2 \rangle &= \langle x_1^* - x_2^*, x - tx_1^* - x + tx_2^* \rangle \\ &= -t\|x_1^* - x_2^*\|^2 = -2t + 2t\langle x_1^*, x_2^* \rangle, \end{aligned}$$

where the last equality is due to the fact that $\|x_i^*\| = 1$. Note also that $\|u_1 - u_2\| \leq \|u_1 - x\| + \|u_2 - x\| = 2t$. We derive that

$$-2t + 2t\langle x_1^*, x_2^* \rangle \geq -2t\varepsilon, \text{ or equivalently } \langle x_1^*, x_2^* \rangle \geq 1 - \varepsilon.$$

Since $\partial_C d_S(x) = \operatorname{co}(\partial_L d_S(x))$, we easily see that the latter inequality still holds for $x_i^* \in \partial_C d_S(x)$, that is, for every $x \in B(\overline{x}, \delta) \setminus S$

$$\langle x_1^*, x_2^* \rangle \geq 1 - \varepsilon \quad \text{for all } x_1^*, x_2^* \in \partial_C d_S(x), \tag{8.1}$$

which entails in particular

$$\|x^*\| \geq \sqrt{1 - \varepsilon} \quad \text{for all } x^* \in \partial_C d_S(x). \tag{8.2}$$

Our aim in this section is to extend in some sense both properties (8.1) and (8.2) to metrically subsmooth sets of Hilbert spaces.

Definition 8.1. Let S be a nonempty closed set of a normed space $(X, \|\cdot\|)$ with $S \neq X$ and let $\alpha > 0$. We say that the origin is kept α-far away from the C-subdifferential of d_S on a set $Q \subset X \setminus S$ if

$$\alpha \leq \inf_{x \in Q} \operatorname{dist}(0, \partial_C d_S(x)).$$

If the above inequality holds true for some real $\alpha > 0$, we say that the origin is kept positively far away from the C-subdifferential of d_S on Q. When Q is a singleton set $\{u\}$, we just say that the origin is kept α-far away from $\partial_C d_S(u)$.

Taking into account Proposition 2.4, let us start with the following lemma.

Lemma 8.2. *Let S be a nonempty closed set in a Hilbert space H and let $x \in H \setminus S$.*

(a) *Assume that there exist two reals $\alpha > 0$, $\eta_x > 0$ and a function $\theta_x :]0, \eta_x[\to \mathbb{R}$ with $\lim_{\eta \downarrow 0} \theta_x(\eta) = 0$ such that for each $\eta \in]0, \eta_x[$*

$$\langle x_1^*, x_2^* \rangle \geq \alpha^2 + \theta_x(\eta)$$

for all x_1^*, x_2^* in $\left(x - \operatorname{Proj}_{S,\eta} x\right)/d_S(x)$. Then the origin is kept α-far away from $\partial_C d_S(x)$, that is, $\operatorname{dist}(0, \partial_C d_S(x)) \geq \alpha$.

(b) *As a partial converse, if $\operatorname{dist}(0, \partial_C d_S(x)) \geq \alpha$, then one has*

$$\langle x_1^*, x_2^* \rangle \geq 2\alpha^2 - 1 \quad \text{for all } x_1^*, x_2^* \in \partial_C d_S(x).$$

Proof. (a) For each $\eta \in]0, \eta_x[$ it is clear that for each $x_1^* \in \left(x - \operatorname{Proj}_{S,\eta} x\right)/d_S(x)$ the inequality $\langle x_1^*, x_2^* \rangle \geq \alpha^2 + \theta_x(\eta)$ holds for all x_2^* in $\overline{\operatorname{co}}\left(\frac{1}{d_S(x)}(x - \operatorname{Proj}_{S,\eta} x)\right)$ since any set $\{u^* \in H : \langle x_1^*, u^* \rangle \geq t\}$ is closed and convex, hence by Proposition 2.4 the inequality holds for all $x_2^* \in \partial_C d_S(x)$. Fixing $x_2^* \in \partial_C d_S(x)$, the same argument gives that the above inequality holds for all x_1^*, x_2^* in $\partial_C d_S(x)$ according to Proposition 2.4 again. In particular, for each $x^* \in \partial_C d_S(x)$ we have $\|x^*\|^2 \geq \alpha^2 + \theta_x(\eta)$ for all $0 < \eta < \eta_x$, and hence $\|x^*\| \geq \alpha$.

(b) Under the assumption in (b), for any x_1^*, x_2^* in $\partial_C d_S(x)$ the inclusion $(x_1^* + x_2^*)/2 \in \partial_C d_S(x)$ due to the convexity of $\partial_C d_S(x)$ ensures that

$$\alpha^2 \leq \left\| \frac{x_1^* + x_2^*}{2} \right\|^2 = \frac{\|x_1^*\|^2}{4} + \frac{\|x_2^*\|^2}{4} + \frac{1}{2}\langle x_1^*, x_2^* \rangle \leq \frac{1}{2} + \frac{1}{2}\langle x_1^*, x_2^* \rangle,$$

which justifies the inequality in (b). □

Remark 8.3. As observed in [47] the inequality in (b) above is *sharp*. Indeed, for the closed set $S = \operatorname{epi}(-|\cdot|)$ in $\mathbb{R}^2$ it is easily seen that $d_S(r, s) = ||r| + s|$ for $(r, s) \notin S$. Consequently, for each $(r, s) \in \mathbb{R}^2 \setminus S$

$$\partial_C d_S(r, s) = \begin{cases} \{\frac{1}{\sqrt{2}}(1, -1)\} & \text{if } r < 0 \\ \{\frac{1}{\sqrt{2}}(1 - 2t, -1) : t \in [0, 1]\} & \text{if } r = 0 \\ \{\frac{1}{\sqrt{2}}(-1, -1)\} & \text{if } r > 0, \end{cases}$$

so $\inf_{(r,s) \notin S} \operatorname{dist}((0, 0), \partial_C d_S(r, s)) = 1/\sqrt{2}$. Setting $\alpha := 1/\sqrt{2}$ the origin is exactly α-far away from the C-subdifferential of d_S on $X \setminus S$. We note for $(0, s) \notin S$ that both $x_1^* := \frac{1}{\sqrt{2}}(-1, -1)$ and $x_2^* := \frac{1}{\sqrt{2}}(1, -1)$ are in $\partial_C d_S(0, s)$ and satisfy the equality

$\langle x_1^*, x_2^* \rangle = 0 = 2\alpha^2 - 1$. This confirms the sharpness of the inequality in (b) in Lemma 8.2.

Given $r > 0$ and a subset S of a normed space X, we define the open r-tube around S as the set

$$\mathrm{Tube}_r(S) := \{x \in X : 0 < d_S(x) < r\}.$$

The assertion (b) in the next theorem has been established by A. Jourani and E. Vilches [47]. The same arguments also yield the assertion (a).

Theorem 8.4 (Jourani-Vilches [47]). *Let S be a nonempty closed set in a Hilbert space H with $S \neq H$. The following hold.*

(a) *If S is metrically subsmooth at a point $\overline{x} \in \mathrm{bdry}\, S$, then for each $\varepsilon \in]0,1[$ there exists a real $\delta > 0$ such that $\sqrt{1-\varepsilon} \le \mathrm{dist}(0, \partial_C d_S(x))$ for all $x \in B(\overline{x}, \delta) \setminus S$.*

(b) *If S is uniformly subsmooth, then for each $\varepsilon \in]0,1[$ there exists a real $r > 0$ such that the origin is kept $\sqrt{1-\varepsilon}$-far away from the C-subdifferential of d_S on the open r-tube $\mathrm{Tube}_r(S)$, that is,*

$$\sqrt{1-\varepsilon} \le \inf_{x \in \mathrm{Tube}_r(S)} \mathrm{dist}(0, \partial_C d_S(x)).$$

Proof. (a) Fix $\varepsilon \in]0,1[$ and choose $0 < \delta_0 < 2$ such that $\langle z_1^* - z_2^*, z_1 - z_2 \rangle \ge -\varepsilon \|z_1 - z_2\|$ for all $z_i \in S \cap B(\overline{x}, \delta_0)$ and $z_i^* \in \partial_C d_S(z_i)$, for $i = 1, 2$. Put $\delta := \delta_0/4$ and fix any $x \in B(\overline{x}, \delta) \setminus S$. Put also $\overline{\eta} := \delta$ and fix any positive real $\eta < \overline{\eta}$. For $i = 1, 2$ take $x_i^* = (x - z_i)/d_S(x)$ with $z_i \in \mathrm{Proj}_{S,\eta} x$, so $\|x - z_i\| \le \|x - u\| + \eta$ for all $u \in S$. The Ekeland variational principle (see [31]) furnishes some $u_i \in S$ such that

(i) $\|z_i - u_i\| \le \sqrt{\eta}$, $\|u_i - x\| + \sqrt{\eta}\|z_i - u_i\| \le \|z_i - x\|$

(ii) $\|u_i - x\| \le \|u - x\| + \sqrt{\eta}\|u - u_i\|$ for all $u \in S$.

By Lemma 2.2(b) the point u_i is a global minimizer on H of the function $u \mapsto (1 + \sqrt{\eta}) d_S(u) + \|u - x\| + \sqrt{\eta}\|u - u_i\|$. Noticing that $u_i \neq x$ (because $u_i \in S$ and $x \notin S$) it ensues (see Proposition 2.1(d)) that

$$0 \in (1 + \sqrt{\eta}) \partial_C d_S(u_i) + \frac{u_i - x}{\|u_i - x\|} + \sqrt{\eta}\mathbb{B}_H,$$

hence there is $b_i \in \mathbb{B}_H$ such that

$$u_i^* := \frac{x - u_i}{\|x - u_i\|} + \sqrt{\eta} b_i \in (1 + \sqrt{\eta}) \partial_C d_S(u_i).$$

It results that

$$x_i^* - u_i^* = \frac{x - u_i}{d_S(x)} - \frac{x - u_i}{\|x - u_i\|} + \frac{u_i - z_i}{d_S(x)} - \sqrt{\eta} b_i,$$

which entails that

$$\begin{aligned} \|x_i^* - u_i^*\| &\le \left(\frac{1}{d_S(x)} - \frac{1}{\|x - u_i\|} \right) \|x - u_i\| + \frac{\|u_i - z_i\|}{d_S(x)} + \sqrt{\eta} \\ &= \frac{\|x - u_i\|}{d_S(x)} - 1 + \frac{\|u_i - z_i\|}{d_S(x)} + \sqrt{\eta}. \end{aligned}$$

This combined with the inequalities $\|u_i - x\| \le \|z_i - x\| \le d_S(x) + \eta$ and $\|z_i - u_i\| \le \sqrt{\eta}$ gives

$$\|x_i^* - u_i^*\| \le \frac{\eta}{d_S(x)} + \frac{\sqrt{\eta}}{d_S(x)} + \sqrt{\eta} \le \sqrt{\eta}\left(\frac{2}{d_S(x)} + 1\right).$$

Observing that

$$\|u_i - \overline{x}\| \le \|u_i - x\| + \|x - \overline{x}\| < d_S(x) + \eta + \delta < \eta + 2\delta < 4\delta,$$

we also have

$$\langle u_1^* - u_2^*, u_1 - u_2\rangle \ge -\varepsilon(1 + \sqrt{\eta})\|u_1 - u_2\|. \tag{8.3}$$

Further, from the definition of u_i^* we see that

$$\langle u_1^* - u_2^*, u_1 - u_2\rangle \le \left\langle \frac{x - u_1}{\|x - u_1\|} - \frac{x - u_2}{\|x - u_2\|}, u_1 - u_2 \right\rangle + 2\sqrt{\eta}\|u_1 - u_2\|,$$

so writing

$$\begin{aligned}
&\left\langle \frac{x - u_1}{\|x - u_1\|} - \frac{x - u_2}{\|x - u_2\|}, u_1 - u_2 \right\rangle \\
&= \left\langle \frac{x - u_1}{\|x - u_1\|} - \frac{x - u_2}{\|x - u_2\|}, (u_1 - x) - (u_2 - x) \right\rangle \\
&= -\|x - u_1\| - \|x - u_2\| + \left\langle \frac{x - u_1}{\|x - u_1\|}, x - u_2 \right\rangle + \left\langle \frac{x - u_2}{\|x - u_2\|}, x - u_1 \right\rangle \\
&= -\|x - u_1\| - \|x - u_2\| + (\|x - u_2\| + \|x - u_1\|)\left\langle \frac{x - u_1}{\|x - u_1\|}, \frac{x - u_2}{\|x - u_2\|} \right\rangle,
\end{aligned}$$

we deduce according to the equality $(x - u_i)/\|x - u_i\| = u_i^* - \sqrt{\eta}b_i$ that

$$\begin{aligned}
&\langle u_1^* - u_2^*, u_1 - u_2\rangle \\
&\le (\|x - u_1\| + \|x - u_2\|)[-1 + \langle u_1^* - \sqrt{\eta}b_1, u_2^* - \sqrt{\eta}b_2\rangle] + 2\sqrt{\eta}\|u_1 - u_2\| \\
&\le (-1 + 5\sqrt{\eta} + \langle u_1^*, u_2^*\rangle)(\|x - u_1\| + \|x - u_2\|).
\end{aligned}$$

Combining with (8.3) both the latter inequality and the inequality

$$-\varepsilon(1 + \sqrt{\eta})\|u_1 - u_2\| \ge -\varepsilon(1 + \sqrt{\eta})(\|x - u_1\| + \|x - u_2\|),$$

and dividing by $\|x - u_1\| + \|x - u_2\|$ it follows that

$$-1 + 5\sqrt{\eta} + \langle u_1^*, u_2^*\rangle \ge -\varepsilon(1 + \sqrt{\eta}),$$

or equivalently

$$\langle u_1^*, u_2^*\rangle \ge 1 - \varepsilon(1 + \sqrt{\eta}) - 5\sqrt{\eta}. \tag{8.4}$$

On the other hand, we have

$$\begin{aligned}
\langle u_1^*, u_2^*\rangle &= \langle u_1^* - x_1^*, u_2^* - x_2^*\rangle + \langle u_1^* - x_1^*, x_2^*\rangle + \langle x_1^*, u_2^* - x_2^*\rangle + \langle x_1^*, x_2^*\rangle \\
&\le \eta\left(\frac{2}{d_S(x)} + 1\right)^2 + 2\sqrt{\eta}\left(\frac{2}{d_S(x)} + 1\right)\left(\frac{\eta}{d_S(x)} + 1\right) + \langle x_1^*, x_2^*\rangle \\
&\le 3\sqrt{\eta}\left(\frac{2}{d_S(x)} + 1\right)^2 + \langle x_1^*, x_2^*\rangle.
\end{aligned}$$

From the latter inequality and (8.4) it results that

$$\langle x_1^*, x_2^* \rangle \geq 1 - \varepsilon(1 + \sqrt{\eta}) - 5\sqrt{\eta} - 3\sqrt{\eta}\left(\frac{2}{d_S(x)} + 1\right)^2$$
$$\geq 1 - \varepsilon - 9\sqrt{\eta}\left(\frac{2}{d_S(x)} + 1\right)^2,$$

thus Lemma 8.2(a) guarantees that the origin is kept $\sqrt{1-\varepsilon}$-far away from the C-subdifferential of d_S on $B(\overline{x}, \delta) \setminus S$.
(b) Fix any $\varepsilon \in]0, 1[$ and choose $0 < \delta_0 < 2$ such that $\langle z_1^* - z_2^*, z_1 - z_2 \rangle \geq -\varepsilon \|z_1 - z_2\|$ for all $z_i \in S$ with $\|z_1 - z_2\| < \delta_0$ and all $z_i^* \in \partial_C d_S(z_i)$, for $i = 1, 2$. Put $\delta := \delta_0/4$ and fix any $x \in \mathrm{Tube}_\delta(S)$. There is some $\overline{x} \in \mathrm{bdry}\, S$ such that $x \in B(\overline{x}, \delta) \setminus S$. Then the proof in (a) shows that $\sqrt{1-\varepsilon} \leq \mathrm{dist}(0, \partial_C d_S(x))$, which justifies (b). $\square$

9. Preservation of subsmoothness under operations

Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(0) = 0$ and

(9.1) $$g(x) := |x|^3(1 - \cos\frac{1}{x}) \quad \text{for all } x \in \mathbb{R} \setminus \{0\},$$

and consider also the linear mapping $A : \mathbb{R}^2 \to \mathbb{R}^2$ with $A(x, r) := (x, 0)$ for all $(x, r) \in \mathbb{R}^2$. The function g being (easily seen to be) of class C^1 on $\mathbb{R}$, it is locally Lipschitz and subsmooth on $\mathbb{R}$, so by Corollary 6.4 the set $S := \mathrm{epi}\, g$ is subsmooth (and hence metrically subsmooth). We observe that

$$A^{-1}(S) = \big(\{0\} \cup \{\pm\frac{1}{2k\pi} : k \in \mathbb{N}\}\big) \times \mathbb{R} =: Q,$$

and the latter set is not subsmooth at $(0, 0)$, since it is not even tangentially regular at $(0, 0)$ due the fact that $T^C(Q; (0, 0)) = \{0\} \times \mathbb{R}$ and $T^B(Q; (0, 0)) = \mathbb{R} \times \mathbb{R}$. This says that the subsmoothness (resp. metric subsmoothness) property is not preserved by inverse image with a continuous linear mapping.

On the other hand, considering the closed convex set $S' := \mathbb{R} \times \{0\}$ we also see that $S \cap S' = \big(\{0\} \cup \{\pm\frac{1}{2k\pi} : k \in \mathbb{N}\}\big) \times \{0\}$, and the latter set is not even tangentially regular at $(0, 0)$. This is a counterexample for the preservation of subsmoothness (resp. metric subsmoothness) under intersection.

Accordingly, such desired preservation properties require additional conditions.

Let $G : X \to Y$ be a mapping between two normed spaces and let S be a subset of Y. Suppose that G is of class C^1 near $\overline{x} \in G^{-1}(S)$. We say that the inverse image set-representative $G^{-1}(S)$ of the set S by G has the (local) *truncated C-normal cone inverse image property* near $\overline{x}$ with a real constant $\gamma > 0$ provided there is a neighborhood U of $\overline{x}$ such that

(9.2)
$$N^C(G^{-1}(S); x) \cap \mathbb{B}_{X^*} \subset DG(x)^* \big(N^C(S; G(x)) \cap (\gamma \mathbb{B}_{Y^*})\big) \quad \text{for all } x \in U \cap G^{-1}(S),$$

where $DG(x)^*$ denotes the adjoint of the derivative mapping $DG(x)$ of G at x. If in place of the C-normal cones the above inclusion holds true with L-normal cones in both members, one says that the truncated L-normal cone inverse image property is satisfied. When G is of class C^1 on an open set containing $G^{-1}(S)$ and the inclusion

(9.2) with C-normal (resp. L-normal, etc) cones holds with the same real $\gamma > 0$ for all $x \in G^{-1}(S)$, one says that the set-representative set has the *global truncated C-normal (resp. L-normal) cone inverse image property* with constant γ.

Similarly, subdifferential of distance function can be employed in place of normal cone. This corresponds to the following (local) *linear inclusion property of C-subdifferential of distance from inverse image* near $\overline{x}$ (relative to $G^{-1}(S)$) with a real constant $\gamma > 0$: There exists a neighborhood U of $\overline{x}$ such that

$$\partial_C d\left(\cdot, G^{-1}(S)\right)(x) \subset \gamma\, DG(x)^* \left(\partial_C d(\cdot, S)(G(x))\right) \tag{9.3}$$

for all $x \in U \cap G^{-1}(S)$. One defines analogously the linear inclusion of L-subdifferential of distance function from inverse image. If G is of class C^1 on an open set containing $G^{-1}(S)$ and if the inclusion (9.3) with C-subdifferential (resp. L-subdifferential, etc) holds with the same real $\gamma > 0$ for all $x \in G^{-1}(S)$, one says that the *global linear inclusion property is satisfied with constant γ for C-subdifferential (resp. L-subdifferential) of distance function to $G^{-1}(S)$*.

Note that anyone of properties (9.2) and (9.3) holds if and only if it holds for all $x \in U \cap \operatorname{bdry}\left(G^{-1}(S)\right)$.

The other important concept of metric subregularity related to the distance function from the set $G^{-1}(S)$ does not require the differentiability of the mapping G. Recall that a multimapping $M : X \rightrightarrows Y$ is *metrically subregular at a point $\overline{x} \in X$ for a point $\overline{y} \in M(\overline{x})$* provided there are a real $\gamma > 0$ and a neighborhood U of $\overline{x}$ such that (see, e.g., [68])

$$d(x, M^{-1}(\overline{y})) \le \gamma\, d(\overline{y}, M(x)) \quad \text{for all } x \in U. \tag{9.4}$$

The infimum of $\gamma > 0$ for which there exists a neighborhood U over which (9.4) holds is called the *rate (or modulus) of metric subregularity of M at $\overline{x}$ for $\overline{y}$*. It will be denoted $\operatorname{subreg}[M](\overline{x}, \overline{y})$, so M is metrically subregular at $\overline{x}$ for $\overline{y}$ if and only if $\operatorname{subreg}[M](\overline{x}, \overline{y}) < +\infty$. Considering the mapping $G : X \to Y$ and the set $S \subset Y$, it is clear that the multimapping $\Phi_{G,S} : X \rightrightarrows Y$, defined by $\Phi_{G,S}(x) := G(x) - S$, is metrically subregular at $\overline{x} \in G^{-1}(S)$ for $0 \in \Phi_{G,S}(\overline{x})$ means that there exist a real constant $\gamma > 0$ and a neighborhood U of $\overline{x}$ such that

$$d(x, G^{-1}(S)) \le \gamma\, d(G(x), S) \quad \text{for all } x \in U. \tag{9.5}$$

In such a case we will say that *the mapping G is metrically subregular at $\overline{x}$ with respect to the subset S of the image space Y*, and the corresponding rate $\operatorname{subreg}[\Phi_{G,S}](\overline{x}, 0)$ will instead be denoted by $\operatorname{subreg}[G]_{\cdot,S}(\overline{x})$. When the role of $G^{-1}(S)$ as a set-constraint needs to be emphasized one also says that *the set-representative $G^{-1}(S)$ or the inverse image of the set S by G is metrically subregular* at $\overline{x}$.

Recall also that the multimaping M satisfies the stronger property of *metric regularity at $\overline{x}$ for $\overline{y} \in M(\overline{x})$* whenever there exist a real $\gamma > 0$ and neighborhoods U and W of $\overline{x}$ and $\overline{y}$ respectively such that (see, e.g., [41, 52, 68])

$$d(x, M^{-1}(y)) \le \gamma\, d(y, M(x)) \quad \text{for all } x \in U, y \in W. \tag{9.6}$$

When M is a single-valued mapping, one just says that it is metrically regular at $\overline{x}$. The metric regularity of the previous multimapping $\Phi_{G,S}$ at $\overline{x}$ for $0 \in \Phi_{G,S}(\overline{x})$

is readily seen to hold if and only if there are a real $\gamma > 0$ and neighborhoods U of $\overline{x}$ and V of zero in Y such that

$$d(x, G^{-1}(S - v)) \le \gamma\, d(G(x), S - v) \quad \text{for all } x \in U, v \in V; \tag{9.7}$$

in such a case we will then say that *the mapping G is metrically regular at $\overline{x}$ with respect to the subset S of the image space Y.*

We start by establishing two lemmas related the above concepts.

Lemma 9.1. *Let $G : X \to Y$ be a mapping between normed spaces which is continuous at $\overline{x} \in X$.*

(a) *The mapping G is metrically regular at $\overline{x}$ if and only if it is metrically regular at $\overline{x}$ with respect to any subset S of Y containing $G(\overline{x})$.*
(b) *If X, Y are Banach spaces, G is C^1 near $\overline{x}$ and $DG(\overline{x})$ is surjective, then G is metrically regular at $\overline{x}$ with respect to any set S of Y containing $G(\overline{x})$.*

Proof. (a) The implication $\Leftarrow$ is obvious by taking S as the singleton $\{G(\overline{x})\}$. Suppose now that G is metrically regular at $\overline{x}$. There are two reals $\gamma > 0$ and $\varepsilon > 0$ such that $d(x, G^{-1}(y)) \le \gamma \, \|G(x) - y\|$ for all $x \in B(\overline{x}, \varepsilon)$ and $y \in B(\overline{y}, 3\varepsilon)$, where $\overline{y} := G(\overline{x})$. By continuity of G at $\overline{x}$ choose a real $\delta \in]0, \varepsilon[$ such that for every $x \in B(\overline{x}, \delta)$ one has $G(x) \in B(\overline{y}, \varepsilon/2)$. Fix any $x \in B(\overline{x}, \delta)$ and any $v \in B_Y(0, \varepsilon/2)$. For every $y \in S \cap B(\overline{y}, 2\varepsilon)$ we have

$$d(x, G^{-1}(S - v)) \le d(x, G^{-1}(y - v)) \le \gamma \, \|G(x) + v - y\|,$$

hence $d(x, G^{-1}(S - v)) \le \gamma \, d(G(x) + v, S \cap B(\overline{y}, 2\varepsilon))$. We deduce

$$d(x, G^{-1}(S - v)) \le \gamma \, d(G(x) + v, S) = \gamma \, d(G(x), S - v)$$

since $d(G(x) + v, S \cap B(\overline{y}, 2\varepsilon)) = d(G(x) + v, S)$ according to Lemma 2.2(a). This translates the desired implication $\Rightarrow$.
(b) Assume that X, Y are Banach spaces. If G is C^1 near $\overline{x}$ and $DG(\overline{x})$ is surjective, it is known that G is metrically regular at $\overline{x}$ (see, e.g, [52, Theorem 1.57]), so the assertion (b) follows from (a). $\square$

Lemma 9.2. *Let X, Y be normed spaces, Q and S be subsets of X and Y respectively, and $G : X \to Y$ be a mapping which is of class C^1 on an open set U of X. Let $\overline{x} \in U \cap G^{-1}(S)$ and $A_x :^= DG(x)$ for every $x \in U$.*

(a) *If for a real $\gamma > 0$ one has*

$$d_P(x) \le \gamma\big(d_Q(x) + d_S(G(x))\big) \quad \text{for all } x \in U, \tag{9.8}$$

where $P := Q \cap G^{-1}(S)$ is assumed to be nonempty, then for each $x \in P \cap U$

$$d^o_P(x; h) \le \gamma\big(d^o_Q(x; h) + d^o_S(G(x); A_x h)\big) \ \forall h \in X,$$

$$\partial_C d_P(x) \subset \gamma\big(\partial_C d_Q(x) + A^*_x(\partial_C d_S(x))\big);$$

further, the latter inclusion also holds with ∂_L in place of ∂_C in the three subdifferentials whenever X, Y are Asplund spaces.
(b) *If G is metrically subregular at $\overline{x}$ with respect to the set S of the image space Y in such a way that (9.5) holds on U with the real $\gamma > 0$, then for each $x \in U \cap G^{-1}(S)$*

$$d^o_{G^{-1}(S)}(x; h) \le \gamma \, d^o_S(G(x); A_x h) \ \forall h \in X, \quad \partial_C d_{G^{-1}(S)}(x) \subset \gamma \, A^*_x \left(\partial_C d_S(G(x))\right);$$

and the latter inclusion also holds with ∂_L in place of ∂_C in both subdifferentials whenever X, Y are Asplund spaces.

(c) *If $DG(\overline{x})$ is surjective and X, Y are Banach spaces, then there exists an open neighborhood $U_0 \subset U$ of $\overline{x}$ such that for each $x \in U_0 \cap G^{-1}(S)$*

$$N^C(G^{-1}(S); x) \subset A_x^* \left(N^C(S; G(x)) \right);$$

and the latter inclusion also holds with N^L in place of N^C whenever X, Y are Asplund spaces.

Proof. (a) For any $x \in P \cap U$ and $h \in X$ we have by Proposition 2.1(a)

$$\begin{aligned} d_P^o(x; h) &= \limsup_{P \ni u \to x, t \downarrow 0} t^{-1} d_P(u + th) \\ &\leq \gamma \left(\limsup_{P \ni u \to x, t \downarrow 0} t^{-1} d_Q(u + th) + \limsup_{P \ni u \to x, t \downarrow 0} t^{-1} d_S(G(u + th)) \right) \\ &= \gamma \left(\limsup_{P \ni u \to x, t \downarrow 0} t^{-1} d_Q(u + th) + \limsup_{P \ni u \to x, t \downarrow 0} t^{-1} d_S(G(u) + tA_x h) \right) \\ &\leq \gamma \left(d_Q^o(x; h) + d_S^o(G(x); A_x h) \right), \end{aligned}$$

where the second equality is due to the strict differentiability of G at x and the Lipschitz property of d_S. This justifies the inequality in (a) which in turn entails the inclusion in (a) since $d_Q^o(x; \cdot)$ and $\gamma\, d_S^o(G(x); A_x \cdot)$ are the support functions of $\partial_C d_Q(x)$ and $\gamma\, A_x^* \big(\partial_C d_S(G(x))$ respectively.

Suppose now that X, Y are Asplund spaces. Putting $\varphi(\cdot) := \gamma \big(d_Q(\cdot) + d_S(G(\cdot))$ and noting for $u \in P$ that $\varphi(u) = d_P(u) = 0$, we clearly see that $\partial_F d_P(u) \subset \partial_F \varphi(u)$ for all $u \in U \cap P$. This and Proposition 2.5(f) entails for any fixed $x \in U \cap P$ that $\partial_L d_P(x) \subset \partial_L \varphi(x)$. Then it remains to observe according to Proposition 2.5(c) that $\partial_L \varphi(x) \subset \gamma\, (\partial_L d_Q(x) + A_x^* (\partial_L d_S(G(x)))$.
(b) Taking $Q = X$, the assertion (b) follows directly from (a).
(c) By Lemma 5.10 and Lemma 9.1(b) there is an open neighborhood $U_0 \subset U$ of $\overline{x}$ such that $DG(x)$ is surjective for each $x \in U_0$ and such that (9.5) holds on U_0. Fix any $x \in U_0 \cap G^{-1}(S)$, put $y := G(x)$, $T_x := T^C(G^{-1}(S); x)$, $\mathcal{T}_y := T^C(S; y)$. By (b) and the second equality in (2.3) we have $A_x^{-1}(\mathcal{T}_y) \subset T_x$, and hence $\psi_{T_x} \leq \psi_{\mathcal{T}_y} \circ A_x$, which ensures that $\partial \psi_{T_x}(0) \subset \partial(\psi_{\mathcal{T}_y} \circ A_x)(0)$ since the convex functions ψ_{T_x} and $\psi_{\mathcal{T}_y} \circ A_x$ take on the same (null) value at zero. Further, the convex function $\psi_{\mathcal{T}_y}$ being proper and lower semicontinuous and A_x being surjective, we have (see, e.g., [12, Theorem 4.1.19])

$$\partial(\psi_{\mathcal{T}_y} \circ A_x)(0) = A_x^*(\partial \psi_{\mathcal{T}_y}(0)).$$

Recalling that $\partial \psi_{T_x}(0) = N^C(G^{-1}(S); x)$ it results that

$$N^C(G^{-1}(S); x) \subset A_x^* \big(N^C(S; G(x))\big),$$

which justifies the inclusion in the case of C-subdifferential.

If X, Y are asplund spaces, the desired ∂_L-inclusion follows directly from (b) since we know that $N^L(P; x) = \mathbb{R}_+ \partial_L d_P(x)$ by Proposition 2.5(d). □

Theorem 9.3 (**Subsmoothness of inverse image**). *Let $G : X \to Y$ be a mapping between normed spaces X and Y and let S be a subset of Y. Assume that G is of class C^1 near $\overline{x} \in G^{-1}(S)$. The following hold.*

(a) *If the set S is subsmooth at $G(\overline{x})$ and if the truncated C-normal cone inverse image property is satisfied for $G^{-1}(S)$ near $\overline{x}$, then $G^{-1}(S)$ is subsmooth at $\overline{x}$.*

(b) *If the set S is metrically subsmooth at $G(\overline{x})$ and if the linear inclusion property (9.3) for C-subdifferential of distance function to $G^{-1}(S)$ holds (which is the case in particular whenever the mapping G is metrically subregular at $\overline{x}$ with respect to the set S of the image space Y), then the set $G^{-1}(S)$ is metrically subsmooth at $\overline{x}$.*

Proof. (a) Fix any $\varepsilon > 0$. Take some neighborhood U' of $\overline{x}$ over which (9.2) holds for some constant real number $\gamma > 0$ and over which the mapping G is Lipschitz continuous with Lipschitz constant $\beta > 0$. By definition of subsmooth set, choose a neighborhood V of $G(\overline{x})$ such that for all $y, y' \in S \cap V$ and $y^* \in N^C(S; y) \cap \mathbb{B}_{Y^*}$ we have

$$\langle y^*, y' - y\rangle \le (2\beta\gamma)^{-1}\varepsilon\|y' - y\|. \tag{9.9}$$

Take a convex neighborhood $U \subset U'$ of $\overline{x}$ such that $G(U) \subset V$ and such that, by the continuity of the derivative mapping $DG(\cdot)$, we have $\|DG(x') - DG(x)\| \le (2\gamma)^{-1}\varepsilon$ for all $x, x' \in U$. Fix any $x, x' \in U \cap G^{-1}(S)$ and $x^* \in N^C(G^{-1}(S); x) \cap \mathbb{B}_{X^*}$. By (9.2) there exists some $y^* \in N^C(S; G(x)) \cap (\gamma\mathbb{B}_{Y^*})$ such that $x^* = y^* \circ DG(x)$. Write

$$G(x') - G(x) = DG(x)(x' - x) + \int_0^1 \big(DG(x + t(x' - x)) - DG(x)\big)(x' - x)\, dt.$$

Then we have

$$\begin{aligned}
&\langle x^*, x' - x\rangle \\
=\ &\langle y^*, DG(x)(x' - x)\rangle \\
=\ &\langle y^*, G(x') - G(x)\rangle - \textstyle\int_0^1 \langle y^*, (DG(x + t(x' - x)) - DG(x))\,(x' - x)\rangle\, dt \\
\le\ &\langle y^*, G(x') - G(x)\rangle + (2\gamma)^{-1}\varepsilon\|y^*\|\|x' - x\|.
\end{aligned}$$

Taking (9.9) and the inequality $\|y^*\| \le \gamma$ into account, we obtain

$$\langle x^*, x' - x\rangle \le (2\beta)^{-1}\varepsilon\|G(x') - G(x)\| + (\varepsilon/2)\|x' - x\|,$$

and hence according to the Lipschitz continuous behavior of G with Lipschitz constant β over U

$$\langle x^*, x' - x\rangle \le \varepsilon\|x' - x\|.$$

This means that the set $G^{-1}(S)$ is subsmooth at $\overline{x}$.

(b) Let $\varepsilon > 0$. As above, fix some open neighborhood U' of $\overline{x}$ for which (9.3) holds with some constant $\gamma > 0$ and over which G is Lipschitz continuous with some Lipschitz constant $\beta > 0$. According to the definition of metric subsmoothness of S take some neigborhood V of $G(\overline{x})$ such that for all $y, y' \in S \cap V$ and $y^* \in \partial_C d_S(G(x))$ we have

$$\langle y^*, y' - y\rangle \le (2\beta\gamma)^{-1}\varepsilon\|y' - y\|. \tag{9.10}$$

For $P := G^{-1}(S)$, by (9.3) we have for all $x \in U' \cap P$

$$\partial_C d_P(x) \subset \gamma\, DG(x)^*(\partial_C d_S(G(x))). \tag{9.11}$$

Take a convex neighborhood U of $\overline{x}$ with $U \subset U'$ and such that

$$\|DG(x') - DG(x)\| \le (2\gamma)^{-1}\varepsilon \quad \text{for all } x, x' \in U.$$

Of course, the inclusion (9.11) holds for all $x \in P \cap U$. Fix any $x, x' \in P \cap U$ and $x^* \in \partial_C d_P(x)$. By (9.11) there exists some $y^* \in \partial_C d_S(G(x))$ such that $x^* = \gamma(y^* \circ DG(x))$. As in (a) above, writing

$$G(x') - G(x) = DG(x)(x' - x) + \int_0^1 \big(DG(x + t(x' - x)) - DG(x)\big)\,(x' - x)\,dt$$

gives

$$\begin{aligned} &\langle x^*, x' - x\rangle \\ =\;& \gamma\langle y^*, G(x') - G(x)\rangle - \gamma \textstyle\int_0^1 \langle y^*, (DG(x + t(x' - x)) - DG(x))\,(x' - x)\rangle\,dt \\ \le\;& \gamma\langle y^*, G(x') - G(x)\rangle + (2\gamma)^{-1}\gamma\varepsilon\|y^*\|\|x' - x\|. \end{aligned}$$

Invoking (9.10) and the inequality $\|y^*\| \le 1$, we see that

$$\langle x^*, x' - x\rangle \le (2\beta)^{-1}\varepsilon\|G(x') - G(x)\| + (\varepsilon/2)\|x' - x\|,$$

and hence

$$\langle x^*, x' - x\rangle \le \varepsilon\|x' - x\|.$$

We then conclude that the set $P = G^{-1}(S)$ is metrically subsmooth at $\overline{x}$.

The case of metric subregularity of G follows from Lemma 9.2(b). $\square$

Let $S := \{x \in X : g_1(x) \le 0, \dots, g_m(x) \le 0\}$ and $\overline{x} \in S$, where X is a normed space and $g_1, \dots, g_m : X \to \mathbb{R}$ are functions which are of class $\mathcal{C}^1$ on an open neighborhood U of $\overline{x}$. Let $K := \{1, \dots, m\}$ and $K(x) := \{k \in K : g_k(x) = \max\limits_{j \in K} g_j(x)\}$ for each x. Assume that there is a real $\sigma > 0$ such that for each $x \in U$ there is $\overline{v} \in X$ (depending on x) for which $\langle Dg_k(x), \overline{v}\rangle \le -\sigma$ for every $k \in K(x)$. Defining $G : X \to \mathbb{R}^m$ by $G(x) := (g_1(x), \dots, g_m(x))$ and putting $S' := \mathbb{R}^m_+$ we see that $S = G^{-1}(S')$ and G is of class $\mathcal{C}^1$ on U. We claim that the truncated C-normal cone inverse image property (9.2) is satisfied. Indeed, let any $x \in U \cap G^{-1}(S')$. We may suppose that $x \in \operatorname{bdry}(G^{-1}(S'))$. Take any $x^* \in N^C(G^{-1}(S')) \cap \mathbb{B}_{X^*}$. We know (see, e.g., Corollary 2 of Theorem 2.4.7 in [18]) that there is $\lambda \in \mathbb{R}^m_+$ such that $x^* = \sum_{k=1}^m \lambda_k Dg_k(x) = DG(x)(\lambda)$ and such that $\lambda_k = 0$ for $k \notin K(x)$, that is, $\lambda \in N^C(\mathbb{R}^m_+; G(x))$. Endowing $\mathbb{R}^m$ with its natural Euclidean norm and considering the vector $\overline{v}$ given by the assumption we can write

$$\langle x^*, -\overline{v}\rangle = \sum_{k \in K(x)} \lambda_k \langle Dg_k(x), -\overline{v}\rangle \ge \sigma \sum_{k \in K(x)} \lambda_k \ge \sigma\|\lambda\|,$$

which ensures that $\|x^*\| \ge \sigma\|\lambda\|$, and hence $\|\lambda\| \le 1/\sigma$. It ensues with $\gamma := 1/\sigma > 0$ that for all $x \in U \cap G^{-1}(S')$) one has

$$N^C(G^{-1}(S'); x) \cap \mathbb{B}_{X^*} \subset DG(x)^*\big(N^C(S'; G(x)) \cap (\gamma\mathbb{B}_{R^m})\big),$$

which translates the desired truncated inverse image property. Theorem 9.3(a) then tells us that the set $S = G^{-1}(S')$ is subsmooth at $\overline{x}$. In fact, the next proposition says more and remove the $\mathcal{C}^1$ property of g_i. Its proof is an adaptation of the proof of S. Adly, F. Nacry and L. Thibault of Theorem 9.1 in [3]. The uniform subsmoothness of sublevel sets (even the uniform equi-subsmoothness of families of such sets) will be studied in Proposition 10.1.

Proposition 9.4. *Let $K := \{1, \ldots, m\}$ and $S = \{x \in X : g_k(x) \le 0,\ \forall k \in K\}$ be a subset of a normed space X, where the functions $g_1, \ldots, g_m : X \to \mathbb{R}$ are locally Lipschitz on a neighborhood U of a point $\overline{x} \in S$. Assume that the functions $g_1, \ldots, g_m$ are subsmooth at $\overline{x}$ and assume also that the following generalized Slater condition holds: there exists a real $\sigma > 0$ such that for each $x \in U \cap \operatorname{bdry} S$ there exists a vector $\overline{v} \in \mathbb{B}_X$ (depending on x) for which*

$$\langle x^*, \overline{v} \rangle \le -\sigma$$

for every $k \in K(x) := \{k \in K : g_k(x) = \max_{j \in K} g_j(x)\}$ and every $x^* \in \partial_C g_k(x)$. Then the set S is subsmooth at $\overline{x}$.

Proof. Define $g : X \to \mathbb{R}$ by $g(x) := \max_{k \in K} g_k(x)$ for all $x \in X$ and note that $S = \{x \in X : g(x) \le 0\}$. By Proposition 2.1(e) we have

$$\partial_C g(x) \subset \operatorname{co}\Big(\bigcup_{k \in K(x)} \partial_C g_k(x) \Big) \quad \text{for all } x \in U \cap \operatorname{bdry} S. \tag{9.12}$$

This inclusion and the assumption on $\overline{v}$ give us

$$0 \notin \partial_C g(x) \quad \text{for all } x \in U \cap \operatorname{bdry} S.$$

By Corollary 1 of Theorem 2.4.7 in [18] one has

$$N^C(S; x) \subset \mathbb{R}_+ \partial_C g(x) \quad \text{for all } x \in U \cap \operatorname{bdry} S. \tag{9.13}$$

Take any $\varepsilon > 0$ and put $\varepsilon' := \varepsilon\sigma$. The subsmoothness assumption allows us by Proposition 4.16 to choose $\delta > 0$ with $B(\overline{x}, \delta) \subset U$ such that for any $x, y \in S \cap B(\overline{x}, \delta)$, for any $k \in K$, for any $x^* \in \partial_C g_k(x)$, and for any $y^* \in \partial_C g_k(y)$

$$\langle x^* - y^*, x - y \rangle \ge -\varepsilon' \|x - y\| .$$

Fix any $x \in S \cap B(\overline{x}, \delta)$ with $x \in \operatorname{bdry} S$ and any $u^* \in N^C(S; x) \cap \mathbb{B}_{X^*}$. By (9.13) choose a real $\alpha \ge 0$ and $x^* \in \partial_C g(x)$ such that $u^* = \alpha x^*$. From (9.12) there are $x_k^* \in \partial g_k(x)$ and $\lambda_k \ge 0$ with $\lambda_k = 0$ if $k \notin K(x)$ and with $\sum_{k \in K} \lambda_k = 1$, such that $x^* = \sum_{k \in K} \lambda_k x_k^*$. Fix any $y \in S \cap B(\overline{x}, \delta)$. By Lebourg mean value equality (see Proposition 2.1(b)) choose for each $k \in K(x)$ some $z_k := x + t_k(y - x)$ with $t_k \in]0, 1[$ and some $z_k^* \in \partial_C g_k(z_k)$ such that $g_k(y) - g_k(x) = \langle z_k^*, y - x \rangle$. Then for each $k \in K(x)$ writing

$$\begin{aligned} 0 \ge g_k(y) - g_k(x) &= \langle z_k^*, y - x \rangle \\ &= \langle z_k^* - x_k^*, y - x \rangle + \langle x_k^*, y - x \rangle \\ &= \frac{1}{t_k} \langle z_k^* - x_k^*, z_k - x \rangle + \langle x^*, y - x \rangle \end{aligned}$$

we obtain that

$$0 \ge -\frac{1}{t_k} \varepsilon' \|z_k - x\| + \langle x_k^*, y - x \rangle = -\varepsilon' \|y - x\| + \langle x_k^*, y - x \rangle.$$

which means that $\langle x_k^*, y-x\rangle \le \varepsilon\sigma\|y-x\|$. Recalling that $\lambda_k = 0$ if $k \notin K(x)$ it ensues that $\langle x^*, y-x\rangle \le \varepsilon\sigma\|y-x\|$. On the other hand, using again the equality $\lambda_k = 0$ if $k \notin K(x)$ as well as the assumption on $\overline{v} \in \mathbb{B}_X$, we also have $\langle x^*, -\overline{v}\rangle \ge \sigma$. Therefore, the above equality $u^* = \alpha x^*$ gives

$$1 \ge \|u^*\| \ge \langle u^*, -\overline{v}\rangle = \alpha\langle x^*, -\overline{v}\rangle \ge \alpha\sigma,$$

so $\alpha \le 1/\sigma$. It follows that

$$\langle u^*, y-x\rangle = \alpha\langle x^*, y-x\rangle \le \varepsilon\|y-x\|,$$

and this inequality still holds when $x \in B(\overline{x}, \delta) \cap \operatorname{int} S$. This confirms that the set S is subsmooth at $\overline{x}$. $\square$

We already observed in Proposition 7.2 that the uniform subsmoothness and the uniform metric subsmoothness of a set in a Banach space coincide. The next theorem provides a result of preservation of such a property for inverse image $G^{-1}(S)$ under suitable conditions on an open enlargement of this set. Recall that the r-open enlargement (for $r \in]0, +\infty]$) of a set C in a normed space X is defined as

$$U_r(C) := \{x \in X : d_C(x) < r\}.$$

Theorem 9.5 (**Uniform subsmoothness of inverse image).** *Let $G : X \to Y$ be a mapping between normed spaces X and Y and let S be a uniformly subsmooth subset of Y. Assume that G is Lipschitz on $G^{-1}(S)$ and differentiable on an open enlargement of $G^{-1}(S)$ with DG uniformly continuous therein. If the global truncated C-normal cone inverse image property (resp. the global linear inclusion property for C-subdifferential of distance function to $G^{-1}(S)$) is satisfied with a same constant $\gamma > 0$, then the set $G^{-1}(S)$ is uniformly subsmooth (resp. uniformly metrically subsmooth).*

Proof. We make the proof under the global truncated C-normal cone inverse image property. Let $\beta > 0$ be a Lipschitz constant of G over $G^{-1}(S)$ and let $r > 0$ be such that on $U_r(G^{-1}(S)) := \{x \in X : d(x, G^{-1}(S)) < r\}$ the mapping G is differentiable with DG uniformly continuous therein. Fix any $\varepsilon > 0$. By definition of uniformly subsmooth set, choose $\eta > 0$ such that for any $y, y' \in S$ with $\|y'-y\| \le \eta$ and any $y^* \in N^C(S; y) \cap \mathbb{B}_{Y^*}$ we have

$$\langle y^*, y'-y\rangle \le (2\beta\gamma)^{-1}\varepsilon\|y'-y\|. \tag{9.14}$$

Choose $\delta \in]0, r[$ such that $\|G(x') - G(x)\| \le \eta$ for all $x, x' \in G^{-1}(S)$ with $\|x' - x\| \le \delta$ and such that $\|DG(x') - DG(x)\| \le (2\gamma)^{-1}\varepsilon$ for all $x, x' \in U_r(G^{-1}(S))$ with $\|x'-x\| \le \delta$. Then fix any $x, x' \in G^{-1}(S)$ with $\|x'-x\| \le \delta$ and any $x^* \in N^C(G^{-1}(S); x) \cap \mathbb{B}_{X^*}$. By (9.2) there exists some $y^* \in N(S; G(x)) \cap (\gamma\mathbb{B}_{Y^*})$ such that $x^* = y^* \circ DG(x)$. We continue like in the proof of Theorem 9.3(a). Write

$$G(x') - G(x) = DG(x)(x'-x) + \int_0^1 \big(DG(x + t(x'-x)) - DG(x)\big)(x'-x)\,dt,$$

and note for every $t \in [0,1]$ that $x + t(x'-x) \in U_r(G^{-1}(S))$ since

$$d_{G^{-1}(S)}(x + t(x'-x)) \le \|x + t(x'-x) - x\| = t\|x'-x\| \le \delta < r.$$

Then we have

$$\begin{array}{ll} & \langle x^*, x' - x\rangle \\ = & \langle y^*, DG(x)(x'-x)\rangle \\ = & \langle y^*, G(x') - G(x)\rangle - \int_0^1 \langle y^*, \left(DG(x+t(x'-x)) - DG(x)\right)(x'-x)\rangle\, dt \\ \leq & \langle y^*, G(x') - G(x)\rangle + (2\gamma)^{-1}\varepsilon\|y^*\|\|x'-x\|. \end{array}$$

Using (9.14) and the inequality $\|y^*\| \leq \gamma$, it ensues that

$$\langle x^*, x'-x\rangle \leq (2\beta)^{-1}\varepsilon\|G(x') - G(x)\| + (\varepsilon/2)\|x'-x\| \leq \varepsilon\|x'-x\|,$$

which confirms that the set $G^{-1}(S)$ is uniformly subsmooth. □

Under the surjectivity of $Dg(\overline{x})$ we then have the assertion (a) below of preservation of metric subsmoothness.

Proposition 9.6. *Let $G : X \to Y$ be a mapping between Banach spaces X and Y and let S be a subset of Y. Let also $(S_i)_{i\in I}$ be a family of subsets of Y and $(G_i)_{i\in I}$ be a family of mappings from X into Y such that $G_i^{-1}(S_i) \neq \emptyset$ for all $i \in I$.*

(a) *Assume that G is of class C^1 near $\overline{x} \in G^{-1}(S)$ with $DG(\overline{x})$ surjective and that the set S is subsmooth (resp. metrically subsmooth) at $G(\overline{x})$. Then the set $G^{-1}(S)$ is subsmooth (resp. metrically subsmooth) at $\overline{x}$.*

(b) *Assume that S is uniformly subsmooth, G is Lipschitz on $G^{-1}(S)$ and differentiable on an open enlargement $U_r(G^{-1}(S))$ with DG uniformly continuous on this enlargement. Assume also that there is a real $\rho > 0$ such that*

$$\rho\mathbb{B}_Y \subset DG(x)(\mathbb{B}_X) \quad \textit{for all}\, x \in G^{-1}(S).$$

Then the set $G^{-1}(S)$ is uniformly subsmooth.

(c) *Assume that the family of sets $(S)_{i\in I}$ is uniformly equi-subsmooth, that for each $i \in I$ the mapping G_i is Lipschitz on $G_i^{-1}(S_i)$ with a common Lipschitz constant $\gamma > 0$, and that there is $r \in]0, +\infty]$ such that each G_i is differentiable on the open enlargement $U_r(G_i^{-1}(S_i))$ with $(DG_i)_{i\in I}$ uniformly equi-continuous relative to the family of open sets $(U_r(G_i^{-1}(S_i))_{i\in I}$. Assume also that there is a real $\rho > 0$ such that for each $i \in I$*

$$\rho\mathbb{B}_Y \subset DG_i(x)(\mathbb{B}_X) \quad \textit{for all}\, x \in G_i^{-1}(S_i).$$

Then the family of sets $(G_i^{-1}(S_i))_{i\in I}$ is uniformly equi-subsmooth.

Proof. (a) By Lemma 9.1(b) the mapping G is metrically regular at $\overline{x}$ with respect to the subset S of Y. Thus, the metric subsmoothness of S at $G(\overline{x})$ entails the metric subsmoothness of $G^{-1}(S)$ at $\overline{x}$ according to Theorem 9.3(b).

Now suppose that S is subsmooth at $G(\overline{x})$. Choose by Lemma 5.10(b) a real $\gamma > 0$ and an open neighborhood U of $\overline{x}$ where G is C^1 and such that for each $x \in U$ the continuous linear mapping $DG(x)$ is surjective and $\|y^*\| \leq \gamma\|x^*\|$ for all $x^* \in X^*$ and $y^* \in Y^*$ satisfying the equality $x^* = y^* \circ DG(x)$. For each $x \in U \cap G^{-1}(S)$, it ensues by the surjectivity of $DG(x)$ and by Lemma 9.2(c) that

$$N^C(G^{-1}(S); x) \subset DG(x)^*\big(N^C(S; G(x))\big). \tag{9.15}$$

Fix any $x \in U \cap G^{-1}(S)$ and any $x^* \in N^C(G^{-1}(S);x) \cap \mathbb{B}_{X^*}$. Then there is an element $y^* \in N^C(S;G(x))$ such that $x^* = DG(x)^*(y^*) = y^* \circ DG(x)$. It results by the choice of U that $\|y^*\| \le \gamma$, which gives

$$N^C(G^{-1}(S);x) \cap \mathbb{B}_{X^*} \subset DG(x)^*\big(N^C(S;G(x)) \cap (\gamma \mathbb{B}_{X^*})\big).$$

The subsmoothness of $G^{-1}(S)$ at $\overline{x}$ then follows from Theorem 9.3(a).
(c) Fix any $\varepsilon > 0$ and put $\varepsilon' := \varepsilon\rho/(2+2\gamma))$. Choose a real $\delta > 0$ such that for each $i \in I$ we have $\langle y^*, y'-y\rangle \le \varepsilon'\|y'-y\|$ for all $y, y' \in S_i$ with $\|y'-y\| < \delta$ and all $y^* \in N^C(S_i;y) \cap \mathbb{B}_{Y^*}$. By the uniform equi-continuity of the family $(DG_i)_{i\in I}$ relative to $(U_r(G_i^{-1}(S_i)))_{i\in I}$ there is a positiive real $\delta' < \min\{r, \delta/\gamma\}$ such that for any $i \in I$ and any $x, x' \in U_r(G_i^{-1}(S_i))$ with $\|x'-x\| < \delta'$ one has $\|DG_i(x') - DG_i(x)\| < \varepsilon'$. Fix any $i \in I$, any $x \in G_i^{-1}(S_i)$ and any $x^* \in N^C(G_i^{-1}(S_i);x) \cap \mathbb{B}_{X^*}$. As in (a) the surjectivity of $DG_i(x)$ entails by Lemma 9.2(c) that

$$N^C(G_i^{-1}(S_i);x) \subset DG_i(x)^*\big(N^C(S_i;G(x))\big),$$

hence there exists some $y^* \in N^C(S_i;G_i(x))$ (depending on i, x) such that $x^* = y^* \circ DG_i(x)$. For any $b \in \mathbb{B}_Y$ taking by assumption some $u \in \mathbb{B}_X$ such that $\rho b = DG_i(x)(u)$, we obtain

$$\rho\langle y^*, b\rangle = \langle y^*, DG_i(x)(u)\rangle = \langle x^*, u\rangle \le 1,$$

thus $\|y^*\| \le 1/\rho$. Now take any $x' \in G_i^{-1}(S_i)$ with $\|x'-x\| < \delta'$, so putting $z_t := x + t(x'-x)$ we have for every $t \in [0,1]$ that $\|z_t - x\| < \delta'$ and $z_t \in U_r(G_i^{-1}(S_i))$ since

$$\operatorname{dist}\big(z_t, G_i^{-1}(S_i)\big) \le \|z_t - x\| = t\|x'-x\| < r.$$

Noticing that $\|G_i(x') - G_i(x)\| \le \gamma\|x'-x\| < \delta$ with $G_i(x'), G_i(x) \in S_i$, it ensues that

$$\begin{aligned}\langle x^*, x'-x\rangle &= \langle y^*, DG_i(x)(x'-x)\rangle \\ &= \langle y^*, G_i(x') - G_i(x)\rangle - \int_0^1 \langle y^*, \big(DG_i(z_t) - DG_i(x)\big)(x'-x)\rangle\, dt \\ &\le (\varepsilon'/\rho)\|G_i(x') - G_i(x)\| + (\varepsilon'/\rho)\|x'-x\|,\end{aligned}$$

which implies that

$$\langle x^*, x'-x\rangle \le \frac{\varepsilon'\gamma}{\rho}\|x'-x\| + \frac{\varepsilon'}{\rho}\|x'-x\| \le \varepsilon\|x'-x\|$$

according to the γ-Lipschitz assumption of G_i on $G_i^{-1}(S_i)$. The uniform equi-subsmoothness of the family of sets $(G_i^{-1}(S_i))_{i\in I}$ is established.
(b) The assertion (b) is clearly a particular case of (c). □

The next two corollaries apply the previous proposition to graphs of certain basic multimappings.

Corollary 9.7. *Let $G : X \to Y$ be a mapping between Banach spaces which is of class C^1 near $\overline{x} \in X$. Let S be a subset of Y which is closed near a point $\overline{y} \in S$ and subsmooth (resp. metrically subsmooth) at $\overline{y}$. Then the graph of the multimapping $x \mapsto G(x) - S$ is subsmooth (resp. metrically subsmooth) at $(\overline{x}, G(\overline{x}) - \overline{y})$.*

Proof. Denoting by Γ the graph of the multimapping in the corollary, we see that $\Gamma = g^{-1}(S)$, where $g : X \times Y \to Y$ denotes the mapping defined by $g(x, y) := G(x) - y$. It is clear that g of class C^1 near $(\overline{x}, \overline{y})$ with $Dg(\overline{x}, \overline{y})(u, v) = DG(\overline{x})(u) - v$, so $Dg(\overline{x}, \overline{y})$ is surjective. Assuming that the set S is subsmooth (resp. metrically subsmooth) at $\overline{y}$, Proposition 9.6(a) tells us that $\Gamma = g^{-1}(S)$ is submooth (resp. metrically subsmooth) at $(\overline{x}, g(\overline{x}, \overline{y}))$, that is, at $(\overline{x}, G(\overline{x}) - \overline{y})$. □

Corollary 9.8. *Let $g : X \to Y$ be a mapping between Banach spaces which is of class C^1 near a point $\overline{x} \in X$ with $Dg(\overline{x})$ surjective. Let Z be another Banach space and $M : Y \rightrightarrows Z$ be a multimapping from Y into Z whose graph is closed near $(\overline{y}, \overline{z})$ in $\operatorname{gph} M$ and subsmooth (resp. metrically subsmooth) at $(\overline{y}, \overline{z})$, where $\overline{y} := g(\overline{x})$. Then the graph of the multimapping $M \circ g$ is subsmooth (resp. metrically subsmooth) at $(\overline{x}, \overline{z})$.*

Proof. Since $z \in M(g(x)) \Leftrightarrow (g(x), z) \in \operatorname{gph} M$, we see that $\operatorname{gph}(M \circ g) = G^{-1}(\operatorname{gph} M)$, where $G : X \times Z \to Y \times Z$ is defined by $G(x, z) := (g(x), z)$. The mapping G is obviously of class C^1 near $(\overline{x}, \overline{z})$ with $DG(\overline{x}, \overline{z})(u, w) = (Dg(\overline{x})(u), w)$. We then see that $DG(\overline{x}, \overline{z})$ is surjective, thus Proposition 9.6(a) guarantees the desired subsmoothness (resp. metric subsmoothness) property of $\operatorname{gph}(M \circ g)$. □

The next proposition provides another example extending the one in Proposition 5.11(b). We prove first three lemmas. The first lemma uses the concept of core of a set. Given a subset S of a vector space X, recall that its core, denoted by $\operatorname{Core} S$, is defined as the set of $x \in S$ such that for every $y \in X$ there is some real $t > 0$ such that $[x, x + t(y - x)] \subset S$.

Lemma 9.9. *Let C and D be closed convex sets of Banach spaces X and Y respectively and let $A : X \to Y$ be a continuous linear mapping. Let $\overline{x} \in C$ and $\overline{y} \in D$ such that*

$$0 \in \operatorname{Core}\big(A(C - \overline{x}) - (D - \overline{y})\big).$$

Then there exist reals $s > 0, \delta > 0$ and open neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that, for all $u \in U \cap C$, $v \in V \cap D$ and for every continuous linear mapping $\Lambda : X \to Y$ with $\|\Lambda - A\| < \delta$, one has

$$s\mathbb{B}_Y \subset \Lambda((C - u) \cap \mathbb{B}_X) - (D - v).$$

Proof. For any continuous linear mapping $\Lambda : X \to Y$ and any $u \in C$ and $v \in D$, define $M^{\Lambda}_{u,v} : X \rightrightarrows Y$ by $M^{\Lambda}_{u,v}(x) := \Lambda(x) - (D - v)$ if $x \in C - u$ and $M^{\Lambda}_{u,v}(x) = \emptyset$ otherwise. Observe that $\operatorname{gph} M^{\Lambda}_{u,v}$ is closed convex and $0 \in M^{\Lambda}_{u,v}(0)$. Setting $M := M^{A}_{\overline{x},\overline{y}}$, we note in addition that $0 \in \operatorname{Core} M(X)$ by the Core-assumption of the statement. The Robinson-Ursescu theorem (see, e.g., [64, 79]) says that there is a real $r > 0$ such that $r\mathbb{B}_Y \subset M(\mathbb{B}_X)$ (keep in mind that $0 \in M(0)$), which means

$$r\mathbb{B}_Y \subset A((C - \overline{x}) \cap \mathbb{B}_X) - (D - \overline{y}).$$

Take any reals $\eta > 0$ and $0 < r' < r$. Fix any $v \in D$ with $\|v - \overline{y}\| < (r - r')/3$, any $u \in C$ with $\|u - \overline{x}\| < \eta$ and $\|A\| \, \|u - \overline{x}\| < (r - r')/3$, and any continuous linear mapping $\Lambda : X \to Y$ with $(1 + \eta)\|\Lambda - A\| < (r - r')/3$. For every $b \in \mathbb{B}_Y$ there are $d \in D$ and $c \in C$ with $\|c - \overline{x}\| \le 1$ such that $rb = A(c - \overline{x}) - (d - \overline{y})$, and hence

$$rb = \Lambda(c - u) + (A - \Lambda)(c - u) + A(u - \overline{x}) - (d - v) + (\overline{y} - v).$$

This and the inequality $\|(A-\Lambda)(c-u)\| \leq (1+\eta)\|A-\Lambda\| < (r-r')/3$ ensure that

$$r\mathbb{B}_Y \subset \Lambda\big((C-u)\cap(1+\eta)\mathbb{B}_X\big) - (D-v) + (r-r')\mathbb{B}_Y,$$

or equivalently

$$r'\mathbb{B}_Y + (r-r')\mathbb{B}_Y \subset M^{\Lambda}_{u,v}((1+\eta)\mathbb{B}_X) + (r-r')\mathbb{B}_Y.$$

Taking support functions yields with $\alpha := 1+\eta$

$$r'\mathbb{B}_Y \subset \mathrm{cl}_Y\, M^{\Lambda}_{u,v}(\alpha\mathbb{B}_X) = \alpha\,\mathrm{cl}_Y\big(\frac{1}{\alpha}M^{\Lambda}_{u,v}(\alpha\mathbb{B}_X)\big) \subset \alpha\,\mathrm{cl}_Y\, M^{\Lambda}_{u,v}(\mathbb{B}_X), \tag{9.16}$$

where the latter inclusion is due to the fact that $\frac{1}{\alpha}M^{\Lambda}_{u,v}(\mathbb{B}_X) \subset M^{\Lambda}_{u,v}(\mathbb{B}_X)$ since the graph of $M^{\Lambda}_{u,v}$ is convex with $0 \in M^{\Lambda}_{u,v}(0)$. On the other hand, from the Robinson-Ursescu theorem again it is easily seen that $M^{\Lambda}_{u,v}(\alpha\mathbb{B}_X)$ is a neighborhood of zero, hence its interior is nonempty. We then deduce from (9.16) and the convexity of $M^{\Lambda}_{u,v}(\mathbb{B}_X)$ that

$$r'\mathbb{U}_Y \subset \alpha\,\mathrm{int}\,\big(\mathrm{cl}_Y\, M^{\Lambda}_{u,v}(\mathbb{B}_X)\big) = \alpha\,\mathrm{int}\,\big(M^{\Lambda}_{u,v}(\mathbb{B}_X)\big)\,.$$

In conclusion, for any positive real $s < r'/(1+\eta)$ we get $s\mathbb{B}_Y \subset M^{\Lambda}_{u,v}(\mathbb{B}_X)$, that is,

$$s\mathbb{B}_Y \subset \Lambda((C-u)\cap\mathbb{B}_X) - (D-v),$$

which finishes the proof of the lemma. □

Lemma 9.10. *Let C and D be closed convex sets of Banach spaces X and Y respectively and let $u \in C$ and $v \in D$. Let $\Lambda : X \to Y$ be a continuous linear mapping for which there is a real $s > 0$ such that*

$$s\mathbb{U}_Y \subset \Lambda((C-u))\cap\mathbb{B}_X) - (D-v).$$

Then given $u^* \in N(C;u)$, $v^* \in N(D;v)$ and $x^* = v^*\circ\Lambda + u^*$, one has

$$\|v^*\| \leq s^{-1}\|x^*\| \quad \textit{and} \quad \|u^*\| \leq (1+s^{-1}\|\Lambda^*\|)\|x^*\|.$$

Proof. Take any $b \in \mathbb{U}_Y$ and choose $d \in D$ and $c \in C$ with $\|c-u\| \leq 1$ such that $sb = -\Lambda(c-u) + (d-v)$. We then have

$$\begin{aligned} s\langle v^*, b\rangle &= \langle v^*\circ\Lambda, -c+u\rangle + \langle v^*, d-v\rangle \\ &= \langle x^*, u-c\rangle + \langle u^*, c-u\rangle + \langle v^*, d-v\rangle \\ &\leq \langle x^*, u-c\rangle \leq \|x^*\|. \end{aligned}$$

This being true for any $b \in \mathbb{U}_Y$, it follows that $s\|v^*\| \leq \|x^*\|$. Further, using this in the equality $u^* = x^* - \Lambda^*(v^*)$ gives $\|u^*\| \leq (1+s^{-1}\|\Lambda^*\|)\|x^*\|$. □

Lemma 9.11. *Let C and D be closed convex sets of Banach spaces X and Y respectively and let $g : X \to Y$ be a mapping which is continuously differentiable near a point $\overline{x} \in C\cap g^{-1}(D)$. Assume that, for $\overline{y} := g(\overline{x})$ the Robinson qualification condition*

$$0 \in \mathrm{Core}\,\big(Dg(\overline{x})(C-\overline{x}) - (D-\overline{y})\big)$$

is satisfied. Then for some neighborhood U of $\overline{x}$ one has

$$N^C(C\cap g^{-1}(D);x) = N(C;x) + Dg(x)^*(N(D;g(x))) \quad \textit{for all}\, x \in U\cap C\cap g^{-1}(D).$$

Proof. Put $P := C \cap g^{-1}(D)$. We know (see, e.g., [65]) that there is some real $\gamma > 0$ such that $d_P(x) \le \gamma\,(d_C(x) + d_D(g(x)))$ for all x in a neighborhood of $\overline{x}$. By Lemma 9.2(a) and by Lemma 9.9 there is an open neighborhood U of $\overline{x}$ such that for every $x \in P \cap U$ one has with $A_x := Dg(x)$ and $\mathcal{N}(x) := N(C;x) + A_x^*(N(D;g(x)))$

$$\partial_C d_P(x) \subset \mathcal{N}(x) \quad \text{and} \quad 0 \in \operatorname{Core}\big(A_x(C-x) - (D - g(x))\big).$$

Fixing any $x \in S \cap U$ and putting $C_0 := C - x$, $D_0 := D - g(x)$, by the second inclusion above we can write (see, e.g., the equality in Theorem 2.8.3 under condition (vii) in [86])

$$\mathcal{N}(x) = N(C_0;0) + A_x^*(N(D_0;0)) = N(C \cap A_x^{-1}(D_0);0),$$

so the cone $\mathcal{N}(x)$ is w^*-closed. Using this and the first equality in (2.3) we obtain from the inclusion $\partial_C d_P(x) \subset \mathcal{N}(x)$ that $N^C(P;x) \subset \mathcal{N}(x)$. □

Proposition 9.12. *Let C and D be closed convex sets of Banach spaces X and Y respectively and let $g : X \to Y$ be mapping which is continuously differentiable near a point $\overline{x} \in C \cap g^{-1}(D)$. Assume that, for $\overline{y} := g(\overline{x})$ the Robinson qualification condition*

$$0 \in \operatorname{Core}\big(Dg(\overline{x})(C - \overline{x}) - (D - \overline{y})\big)$$

is satisfied. Then the set $C \cap g^{-1}(D)$ is subsmooth at $\overline{x}$.

Proof. Put $S := C \cap g^{-1}(D)$ and note that S is closed near $\overline{x}$. By Lemma 9.11, Lemma 9.9 and Lemma 9.10 there are a real $\gamma > 0$ and an open neighoborhood U of $\overline{x}$ over which g is of class C^1 and such for every $x \in U \cap S$ one has

$$N^C(S;x) \cap \mathbb{B}_{X^*} \subset Dg(x)^*\big(N(D;g(x)) \cap \gamma\mathbb{B}_{Y^*}\big) + N(C;x) \cap \gamma\mathbb{B}_{X^*}.$$

Define $G : X \to X \times Y$ by $G(x) = (x, g(x))$ for all $x \in X$, so $S = G^{-1}(Q)$, where $Q := C \times D$, and G is of class C^1 near $\overline{x}$ with $DG(x)(u) = (u, Dg(x)u)$ for all $u \in X$ and x in some open neighborhood U of $\overline{x}$ where g is C^1. Fix any $x \in U \cap S$ and note that $DG(x)^* : X^* \times Y^* \to X^*$ is given by $DG(x)^*(u^*, v^*) = u^* + Dg(x)^*(v^*)$ for all $(u^*, v^*) \in X^* \times Y^*$. Therefore, with $y := g(x)$, $L := Dg(x)$ and $T := DG(x)$ we have

$$\begin{aligned} T^*\big(N(Q;G(x)) \cap (\gamma(\mathbb{B}_{X^*} \times \mathbb{B}_{Y^*}))\big) &= T^*\big((N(C;x) \cap \gamma\mathbb{B}_{X^*}) \times (N(D;y) \cap \gamma\mathbb{B}_{Y^*})\big) \\ &= N(C;x) \cap \gamma\mathbb{B}_{X^*} + L^*\big(N(D;y) \cap \gamma\mathbb{B}_{Y^*}\big), \end{aligned}$$

which by what precedes yields

$$N^C(S;x) \cap \mathbb{B}_{X^*} \subset DG(x)^*\big(N(Q:G(x)) \cap (\gamma(\mathbb{B}_{X^*} \times \mathbb{B}_{Y^*}))\big).$$

Theorem 9.3(a) allows us to conclude that the set S is subsmooth at $\overline{x}$. □

Subsmoothness of more usual structured optimization constraint sets follows under the Mangasarian-Fromovitz qualification condition.

Corollary 9.13. *Let $g_1, \ldots, g_m$ be functions from a Banach space X into $\mathbb{R}$ and $G : X \to Y$ be a mapping from X into a Banach space Y, and let $S = \{x \in X : g_1(x) \le 0, \ldots, g_m(x) \le 0, G(x) = 0\}$. Assume that $g_1, \ldots, g_m$ and G are of class C^1 near a point $\overline{x} \in S$ and assume the following Mangasarian-Fromovitz qualification condition: the derivative $DG(\overline{x})$ is surjective and there exists a vector $\overline{v} \in X$ such* *that* $DG(\overline{x})\overline{v} = 0$ *and* $\langle Dg_k(\overline{x}), \overline{v}\rangle < 0$ *for all* $k \in K(\overline{x}) := \{k \in K : g_k(\overline{x}) = 0\}$, *where* $K := \{1, \dots, m\}$.

Proof. By the open mapping theorem choose a real $\rho > 0$ such that $\rho\mathbb{B}_Y \subset DG(\overline{x})(\mathbb{B}_X)$. Write $K(\overline{x}) := \{k_1, \dots, k_p\}$ with distinct k_i and define $g : X \to \mathbb{R}^p \times Y$ by $g(x) = (g_{k_1}(x), \dots, g_{k_p}(x), G(x))$ for all $x \in X$. Put $D := (-\mathbb{R}_+)^p \times \{0_Y\}$ and $S_0 := g^{-1}(D)$, so $U \cap S = U \cap S_0$ for some neighborhood U of $\overline{x}$. Choose a real $\sigma > 0$ such that

$$\eta_i := -\sigma\langle Dg_{k_i}(\overline{x}), \overline{v}\rangle - \|Dg_{k_i}(\overline{x})\| - \rho > 0 \quad \text{for any } i = 1, \dots, p.$$

Consider any $(\zeta, y) \in \mathbb{B}_{\mathbb{R}^p} \times \mathbb{B}_Y$. By the choice of ρ there is some $h \in \mathbb{B}_X$ such that $\rho y = DG(\overline{x})h$, hence $\rho y = DG(\overline{x})(h + \sigma\overline{v})$ since $DG(\overline{x})\overline{v} = 0_Y$ by assumption. For each $i \in \{1, \dots, p\}$ putting $\xi_i := -\sigma\langle Dg_{k_i}(\overline{x}), \overline{v}\rangle - Dg_{k_i}(\overline{x})h + \rho\zeta_i$ we notice that $\xi_i \geq \eta_i > 0$ and $\rho\zeta_i = \langle Dg_{k_i}(\overline{x}), h + \sigma\overline{v}\rangle + \xi_i$, hence

$$\rho(\zeta, y) = Dg(\overline{x})(h + \sigma\overline{v}) + (\xi, 0_Y).$$

Consequently, $\rho(\mathbb{B}_{\mathbb{R}^p} \times \mathbb{B}_Y) \subset Dg(\overline{x})(X) + \mathbb{R}^p_+ \times \{0_Y\}$, which means that the Robinson qualification condition in Proposition 9.12 is fulfilled with $\overline{x}$ and $g(\overline{x}) = (0_{\mathbb{R}^p}, 0_Y)$, hence the set S_0 is subsmooth at $\overline{x}$. It results that the set S is subsmooth at $\overline{x}$. □

A similar result for uniform equi-subsmoothness also holds true by adapting the above arguments of Proposition 9.12 and the arguments in Proposition 9.6(c).

Proposition 9.14. *Let* $(C_i)_{i\in I}$ *and* $(D_i)_{i\in I}$ *be two families of closed convex sets of Banach spaces* X *and* Y *respectively and let* $(g_i)_{i\in I}$ *be a family of mappings from* X *into* Y *such that every* g_i *is* γ*-Lipschitz on* $Q_i := C_i \cap g_i^{-1}(D_i)$ *with a common Lipschitz constant* $\gamma > 0$. *Assume that there is* $r \in]0, +\infty]$ *with* G_i *differentiable on the* r*-open enlargement* $U_r(Q_i)$ *for every* $i \in I$ *and such that the family* $(Dg_i)_{i\in I}$ *is uniformly equi-continuous relative to the family of open sets* $(U_r(Q_i))_{i\in I}$. *Assume also that there is a real* $\rho > 0$ *such that for every* $i \in I$ *the Robinson qualification condition*

$$\rho\mathbb{B}_Y \subset Dg_i(x)(C - x) - \big(D - g_i(x)\big) \quad \textit{for all } x \in Q_i,$$

is satisfied. Then the family of sets $(Q_i)_{i\in I}$ *is uniformly equi-subsmooth.*

Clearly, families $(Q_i)_{i\in I}$ of structured optimization constraint sets in the form

$$Q_i = \{x \in X : g_{1,i}(x) \leq 0, \dots, g_{m,i}(x) \leq 0, G_i(x) = 0\},$$

with functions $g_{k,i} : X \to \mathbb{R}$ and mappings $G_i : X \to Y$ are particular cases of the above proposition. The suitable statement based on the above proposition is left to the reader.

Concerning the intersection of finitely many sets, we need to translate the conditions in (9.2) and (9.5). Let $S_1, \dots, S_m$ be a finite system of sets of X and $\overline{x} \in \bigcap_{i=1}^{m} S_i$. We say that this system of sets satisfies the *truncated C-normal cone intersection property* near $\overline{x}$ if there are a positive real constant γ and a neighborhood U of $\overline{x}$ such that

$$N^C(\bigcap_{i=1}^{m} S_i; x) \cap \mathbb{B}_{X^*} \subset N^C(S_1; x) \cap (\gamma\mathbb{B}^*_X) + \cdots + N^C(S_m; x) \cap (\gamma\mathbb{B}^*_X) \tag{9.17}$$

for all $x \in U \cap S_1 \cap \cdots \cap S_m$. Obviously, the above property holds if and only if it holds for all $x \in U \cap S_1 \cap \cdots \cap S_m$ which lies in bdry $\left(\bigcap_{i=1}^{m} S_i \right)$.

We recall that $\overline{x}$ is a *metrically subregular point for the system of sets* $S_1, \ldots, S_m$ *relative to the intersection* if there exist a real $\gamma > 0$ and a neighborhood U of $\overline{x}$ such that

$$d(x, S_1 \cap \cdots \cap S_m) \leq \gamma[d(x, S_1) + \cdots + d(x, S_m)] \quad \text{for all } x \in U. \tag{9.18}$$

Corollary 9.15. *Let* $S_1, \ldots, S_m$ *be a finite system of sets of a normed space* X *and let* $\overline{x} \in \bigcap_{i=1}^{m} S_i$*. The following hold.*

(a) *If the sets* $S_1, \ldots, S_m$ *are subsmooth at* $\overline{x}$ *and if the truncated C-normal cone intersection property is satisfied for these sets near* $\overline{x}$*, then the intersection* $\bigcap_{i=1}^{m} S_i$ *is subsmooth at* $\overline{x}$*.*

(b) *If the sets* $S_1, \ldots, S_m$ *are metrically subsmooth at* $\overline{x}$ *and satisfy the metrically subregular intersection property* (9.18) *at* $\overline{x}$*, then the set* $\bigcap_{i=1}^{m} S_i$ *is metrically subsmooth at* $\overline{x}$*.*

Proof. Consider the normed space $Y := X \times \cdots \times X$ endowed with the sum norm (that is, $\|(x_1, \ldots, x_m)\| = \|x_1\| + \cdots + \|x_m\|$) and consider the subset $S := S_1 \times \cdots \times S_m$ of Y. Defining the continuous linear mapping $A : X \to Y$ by $A(x) := (x, \ldots, x)$ for all $x \in X$, we see that $S_1 \cap \cdots \cap S_m = A^{-1}(S)$.
(a) We know (see (2.1)) that

$$N^C(S; A(x)) = N^C(S_1; x) \times \cdots \times N^C(S_m; x). \tag{9.19}$$

Note that $\mathbb{B}_{Y^*} = \mathbb{B}_{X^*} \times \cdots \times \mathbb{B}_{X^*}$ since the dual norm in Y^* is the box norm related to the dual norm in X^* (the norm of Y being the sum norm). Fix a neighborhood U of $\overline{x}$ and a positive constant γ such that (9.17) holds. Observing that $A^*(x_1^*, \ldots, x_m^*) = x_1^* + \cdots + x_m^*$ for any $(x_1^*, \ldots, x_m^*) \in Y^*$, and using (9.19) we see that

$$N^C(A^{-1}(S); x) \cap \mathbb{B}_{X^*} \subset A^*(N^C(S; A(x)) \cap \gamma \mathbb{B}_{Y^*}) \quad \text{for all } x \in U \cap A^{-1}(S),$$

that is, the inverse image $A^{-1}(S)$ has the truncated normal cone inverse image property near $\overline{x}$. The set S being easily seen to inherit the subsmoothness property at $\overline{x}$ from the ones of S_i, $i = 1, \ldots, m$, it follows from Theorem 9.3 that the set $A^{-1}(S)$ is subsmooth at $\overline{x}$.
(b) Obviously, the definition of the sum norm yields that

$$d(y, S) = d(y_1, S_1) + \cdots + d(y_m, S_m) \quad \text{for all } y = (y_1, \ldots, y_m) \in Y.$$

The metrically subregular intersection property of the sets $S_1, \ldots, S_m$ with the constant $\gamma > 0$ and the neighborhood U of $\overline{x}$ may then be translated as

$$d(x, A^{-1}(S)) \leq \gamma \, d(A(x), S) \quad \text{for all } x \in U.$$

The property (b) of the corollary is then a consequence of (b) in Theorem 9.3, and this completes the proof. $\square$

Our next aim is to provide conditions for the metric subregularity in presence of subsmoothness. Let us prove first the following lemma. It is strongly in the line of Lemma 3.7 of Aussel, Daniilidis and Thibault [8] and Theorem 3.1 of Zheng and Ng [87]. Although it is similar to Proposition 2.5(e), it is different in two respects. On the one hand, the first assertion is concerned with any Banach space and not just the Asplund one and the proof works, for example, with the Clarke or Ioffe approximate subdifferential of the distance function instead of the Fréchet subdifferential. On the other hand, in the case of an Asplund space no element is required to be in the Fréchet subdifferential at the point outside the set.

Lemma 9.16. *Let S be a closed set of a Banach space X and $x \in X \setminus S$. Let also ∂ be a subdifferential on X such that $0 \in \partial f(\overline{x}) + \partial g(\overline{x})$ whenever $\overline{x}$ is a minimizer of $f + g$ and $f : X \to \mathbb{R}$ is locally Lipschitz and $g : X \to \mathbb{R}$ is convex continuous. Then for any $\varepsilon > 0$ there exist some $u \in S$ and $u^* \in \partial d_S(u)$ such that*

$$\|u - x\| \le (1 + \varepsilon(1 + \varepsilon))d_S(x) \quad \text{and} \quad \langle u^*, x - u\rangle \ge \frac{1-\varepsilon}{1+\varepsilon}\|x - u\|.$$

If X is an Asplund space, then for any $\varepsilon > 0$ there is $u \in S$ and $u^* \in N^F(S; u)$ with $\|u^*\| = 1$ such that the above inequalities hold.

Proof. Fix any positive $\varepsilon' < \min\{\varepsilon, \varepsilon d_S(x), \varepsilon\sqrt{d_S(x)}\}$. Choose some $x' \in S$ satisfying $\|x' - x\| \le d_S(x) + (\varepsilon')^2$, that is,

$$\|x' - x\| \le \|y - x\| + (\varepsilon')^2 \quad \forall y \in S.$$

According to the Ekeland variational principle applied to the function $y \mapsto \|y - x\|$ over the complete metric space S, there exists some $u \in S$ such that $\|u - x'\| \le \varepsilon'$ and

$$\|u - x\| \le \|y - x\| + \varepsilon'\|y - u\| \quad \forall y \in S.$$

Since the function $y \mapsto \|y-x\|+\varepsilon'\|y-u\|$ is Lipschitz continuous on X with $(1+\varepsilon')$ as Lipschitz constant, the latter inequality yields (see Lemma 2.2(b))

$$\|u - x\| \le \|y - x\| + \varepsilon'\|y - u\| + (1 + \varepsilon')d_S(y) \quad \forall y \in X,$$

that is, the point u is a minimizer on the whole space X of the function in y given by the second member of the inequality. The three functions involved in that second member being Lipschitz continuous, according to the assumption on the subdifferential ∂ we have

$$0 \in \partial\| \cdot -x\|(u) + \varepsilon'\mathbb{B} + (1 + \varepsilon')\partial d_S(u),$$

that is, $0 = v^* + \varepsilon' b^* + (1+\varepsilon')u^*$ for some $v^* \in \partial\|\cdot -x\|(u)$, $b^* \in \mathbb{B}$, and $u^* \in \partial d_S(u)$. Since $u \neq x$ we have $\langle v^*, u - x\rangle = \|u - x\|$ (and $\|v^*\| = 1$), which gives that

$$(1 + \varepsilon')\langle u^*, x - u\rangle + \varepsilon'\langle b^*, x - u\rangle = \|u - x\|,$$

and hence

$$(1 + \varepsilon')\langle u^*, x - u\rangle \ge (1 - \varepsilon')\|x - u\|,$$

which is the second inequality of the statement of the lemma.

Concerning the first one, it suffices to write

$$\begin{aligned}\|u-x\| \le \|u-x'\| + \|x'-x\| &\le \varepsilon' + d_S(x) + (\varepsilon')^2 \\ &\le \varepsilon d_S(x) + d_S(x) + \varepsilon^2 d_S(x).\end{aligned}$$

The proof of the first assertion is complete.

Assume now that X is an Asplund space. Clearly, we may suppose $\varepsilon \in]0,1[$. Fix any $\eta \in]0,\varepsilon[$. Let $u \in S$ and $u^* \in \partial_L d_S(u)$ satisfying the inequalities obtained above with η in place of ε, and let sequences $(u_n)_n$ in S converging to u and $(u_n^*)_n$ converging weakly* to u^* with $u_n^* \in \partial_F d_S(u_n)$. Since $d_S(x) > 0$, there is some $k \in \mathbb{N}$ with $d_S(u_k) > 0$ such that u_k and u_k^* satisfies the same inequalities with ε in place of η. Then, $u_k^* \neq 0$ and for $v := u_k$ and $v^* := u_k^*/\|u_k^*\|$ we have $v^* \in N^F(S; v)$ with $\|v^*\| = 1$ (since $\partial_F d_S(v) = N^F(S;v) \cap \mathbb{B}_{X^*}$) and

$$\langle v^*, x-v\rangle \ge \frac{1}{\|u_k^*\|}\frac{1-\varepsilon}{1+\varepsilon}\|x-v\| \ge \frac{1-\varepsilon}{1+\varepsilon}\|x-v\|,$$

where the latter inequality is due to the fact $\|u_k^*\| \le 1$. We conclude that the elements v and v^* satisfy the desired properties in the Asplund space setting. □

Theorem 9.17. *Let $G : X \to Y$ be a mapping from a Banach space X into a Banach space Y and let S be a subset of Y. Assume that G is of class C^1 near $\overline{x} \in G^{-1}(S)$ and that S is closed near $G(\overline{x})$. Then the following hold.*

(a) *If the metric subregularity condition (9.5) is satisfied with some real $\gamma > 0$ over some neighborhood U of $\overline{x}$, then the linear inclusion property of C-subdifferential of distance from inverse image (9.3) is fulfilled with the same constant γ over $U' \cap S$ for some neighborhood U' of $\overline{x}$.*

If in addition to the metric subregularity condition (9.5) both spaces X, Y are Asplund, then the linear inclusion property of L-subdifferential of distance from inverse image is satisfied near $\overline{x}$ (relative to $G^{-1}(S)$).

(b) *If S is metrically subsmooth at $G(\overline{x})$ and if (9.3) holds with a real constant $\gamma > 0$, then for any positive real number $\varepsilon < 1$ satisfying $1-\varepsilon > \varepsilon(1+\varepsilon)(1+\varepsilon(1+\varepsilon))$ there exists some neighborhood U' of $\overline{x}$ such that for all $x \in U'$*

$$d(x, G^{-1}(S)) \le \frac{\gamma(1+\varepsilon)}{1-\varepsilon-\varepsilon(1+\varepsilon)(1+\varepsilon(1+\varepsilon))} d(G(x), S).$$

(c) *If S is subsmooth at $G(\overline{x})$ with X, Y Asplund spaces and if for a real $\gamma > 0$ there exists a neighborhood U of $\overline{x}$ such that for all $x \in U \cap G^{-1}(S)$*

$$N^F\left(G^{-1}(S); x\right) \cap \mathbb{B}_{X^*} \subset \gamma DG(x)^*\left(N^C(S; G(x)) \cap \mathbb{B}_{Y^*}\right), \tag{9.20}$$

then the same conclusion in (b) holds.

Proof. The assertion (a) has been established in Lemma 9.2(b).

Let us prove (b) and (c). Without loss of generality, we may suppose that S is closed and G is continuous on X. Let γ, U, and ε be as in the statement of (b) (resp. (c)). We may suppose that G is Lipschitz on U with some Lipschitz constant $\beta > 0$ and that $\|DG(x_1) - DG(x_2)\| \le (2\gamma)^{-1}\varepsilon$ for all $x_1, x_2 \in U$. The set S being metrically subsmooth (resp. subsmooth) at $G(\overline{x})$, by (b) of Proposition 7.5 (resp. by (b) of Proposition 7.6) there exists some $\delta > 0$ such that $B(\overline{x}, \delta) \subset U$ and

$$\langle v^*, y - v\rangle \le d_S(y) + (2\beta\gamma)^{-1}\varepsilon\|y - v\| \tag{9.21}$$

for all $y \in B(G(\overline{x}), \delta)$, $v \in S \cap B(G(\overline{x}), \delta)$, and $v^* \in \partial_C d_S(v)$ (resp. $v^* \in N^C(S; v) \cap \mathbb{B}_{Y^*}$). Fix any positive $\eta < \delta$ such that

$$\eta(1 + \beta)(2 + \varepsilon(1 + \varepsilon)) < \delta \quad \text{and} \quad G(B(\overline{x}, \eta)) \subset B(G(\overline{x}), \delta).$$

Fix any $x \in B(\overline{x}, \eta) \setminus G^{-1}(S)$. By the above lemma there exist $u \in G^{-1}(S)$ and $u^* \in \partial_C d(\cdot, G^{-1}(S))(u)$ (resp. $u^* \in \partial_F d(\cdot, G^{-1}(S))(u)$) such that

$$\|u - x\| \le (1 + \varepsilon(1 + \varepsilon))d(x, G^{-1}(S)) \quad \text{and} \quad \langle u^*, x - u\rangle \ge \frac{1 - \varepsilon}{1 + \varepsilon}\|x - u\|. \tag{9.22}$$

Note that

$$\|G(x) - G(\overline{x})\| \le \beta\|x - \overline{x}\| < \eta\beta < \delta. \tag{9.23}$$

We also note that

$$\begin{aligned}\|u - \overline{x}\| &\le \|x - \overline{x}\| + (1 + \varepsilon(1 + \varepsilon))d(x, G^{-1}(D)) \\ &\le (2 + \varepsilon(1 + \varepsilon))\|x - \overline{x}\| \le \eta(2 + \varepsilon(1 + \varepsilon)),\end{aligned}$$

so $u \in B(\overline{x}, \delta)$. This ensures that

$$\|G(u) - G(\overline{x})\| \le \beta\|u - \overline{x}\| < \eta\beta(2 + \varepsilon(1 + \varepsilon)) < \delta. \tag{9.24}$$

Choose by the property (9.3) (resp. (9.20)) some $v^* \in \partial_C d_S(G(u))$ (resp. $v^* \in N^C(S; G(u)) \cap \mathbb{B}_{Y^*}$) such that we have $u^* = \gamma\, v^* \circ DG(u)$. By (9.23) and (9.24) and by the inclusion $G(u) \in S$ we may invoke (9.21) to write

$$\langle v^*, G(x) - G(u)\rangle \le d_S(G(x)) + (2\beta\gamma)^{-1}\varepsilon\|G(x) - G(u)\|,$$

and hence

$$\langle v^*, G(x) - G(u)\rangle \le d_S(G(x)) + (2\gamma)^{-1}\varepsilon\|x - u\|. \tag{9.25}$$

Then according to the second inequality of (9.22) and to the inequality $\operatorname{dist}(x, G^{-1}(S)) \le \|x - u\|$ (because $u \in G^{-1}(S)$) we have

$$\begin{aligned}&\frac{1 - \varepsilon}{1 + \varepsilon}\operatorname{dist}(x, G^{-1}(S)) \\ &\le \langle u^*, x - u\rangle = \langle \gamma\, v^* \circ DG(u), x - u\rangle \\ &= \gamma\langle v^*, G(x) - G(u)\rangle - \gamma\int_0^1 \langle v^* \circ [DG(u + t(x - u)) - DG(u)], x - u\rangle\, dt,\end{aligned}$$

which ensures by (9.25) and by the inequality $\|v^*\| \le 1$ that

$$\begin{aligned}\frac{1 - \varepsilon}{1 + \varepsilon}\operatorname{dist}(x, G^{-1}(S)) &\le \gamma\operatorname{dist}(G(x), S) + \varepsilon\|x - u\| \\ &\le \gamma\operatorname{dist}(G(x), S) + \varepsilon(1 + \varepsilon(1 + \varepsilon))\operatorname{dist}(x, G^{-1}(S)),\end{aligned}$$

the second inequality being due to the first inequality of (9.22). It ensues that

$$[\frac{1 - \varepsilon}{1 + \varepsilon} - \varepsilon(1 + \varepsilon(1 + \varepsilon))]\operatorname{dist}(x, G^{-1}(S)) \le \gamma\operatorname{dist}(G(x), S).$$

The latter inequality continuing to hold for $x \in G^{-1}(S)$, we conclude that for all $x \in B(\overline{x}, \eta)$

$$\operatorname{dist}(x, G^{-1}(S)) \leq \frac{\gamma(1+\varepsilon)}{1-\varepsilon-\varepsilon(1+\varepsilon)(1+\varepsilon(1+\varepsilon))} \operatorname{dist}(G(x), S).$$

□

Now consider the rate (or modulus) $\operatorname{subreg}_{\cdot,S}[G](\overline{x})$ of metric subregularity at $\overline{x}$ of the mapping G with respect to the subset S of the image space Y defined after (9.5). It can be directly written in terms of G and S as

$$\operatorname{subreg}_{\cdot,S}[G](\overline{x}) = \inf_{U \in \mathcal{N}(\overline{x})} \operatorname{subreg}_{\cdot,S}[G]_U,$$

where $\mathcal{N}(\overline{x})$ denotes the collection of all neighborhoods of $\overline{x}$ and

$$\operatorname{subreg}_{\cdot,S}[G]_U := \sup_{x \in U \setminus G^{-1}(S)} \operatorname{dist}(x, G^{-1}(S))/\operatorname{dist}(G(x), S).$$

The assertion (b) of Theorem 9.17 also leads us to consider in a similar way the following constant related to the property (9.3)

$$\operatorname{subdist}_{\cdot,S}[G](\overline{x}) := \inf_{U \in \mathcal{N}(\overline{x})} \operatorname{subdist}_{\cdot,S}[G]_U,$$

where $\operatorname{subdist}_{\cdot,S}[G]_U$ is the infimum of real numbers $\gamma > 0$ such that (9.3) is satisfied for all $x \in U \cap G^{-1}(S)$.

Theorem 9.17 then admits the following direct corollary.

Corollary 9.18. *Let X, Y be Banach spaces, S be a subset of Y and $G : X \to Y$ be a mapping which is of class C^1 near $\overline{x} \in G^{-1}(S)$ and such that S is closed near $G(\overline{x})$.*

(I) *Assume that S is metrically subsmooth at $G(\overline{x})$. Then the following hold.*
  (a) *The mapping G is metrically subregular at $\overline{x}$ with respect to the subset S of the image space Y if and only if the property (9.3) is fulfilled.*
  (b) *One has the equality between the above constants at $\overline{x}$, that is,*

$$\operatorname{subreg}_{\cdot,S}[G](\overline{x}) = \operatorname{subdist}_{\cdot,S}[G](\overline{x}).$$

(II) *Assume now that both spaces X, Y are Asplund and S is subsmooth at $G(\overline{x})$. Then the mapping G is metrically subregular at $\overline{x}$ with respect to the subset S of the image space Y if and only if* (9.20) *holds for some real constant $\gamma > 0$; in fact,* $\operatorname{subreg}_{\cdot,S}[G](\overline{x})$ *coincides with the infimum of all reals $\gamma > 0$ satisfying condition* (9.20).

In order to state the next proposition, let us recall that the infimum of $\gamma > 0$ for which there exists a neighborhood U of $\overline{x}$ over which the inequality (9.18) holds is called the *rate (or modulus) of metric subregularity at $\overline{x}$ for the system of sets $S_1, \ldots, S_m$ relative to the intersection*; it will be denoted by $\operatorname{subreg}_{\cap}[S_1, \ldots, S_m](\overline{x})$ and its finiteness translates the existence of $\gamma \in]0, +\infty[$ such that (9.18) holds over some neighborhood U of $\overline{x}$. Note that this constant clearly coincides with the infimum over all neighborhoods U of $\overline{x}$ of

$$\sup_{x \in U \setminus \cap_{i=1}^m S_i} \frac{\operatorname{dist}(x, S_1 \cap \cdots \cap S_m)}{\operatorname{dist}(x, S_1) + \cdots + \operatorname{dist}(x, S_m)}.$$

Consider also the other constant

$$\mathrm{subdist}_{\cap}[S_1, \ldots, S_m](\overline{x}) := \inf_{U \in \mathcal{N}(u_0)} \mathrm{subdist}_{\cap}[S_1, \ldots, S_m]_U,$$

where $\mathrm{subdist}_{\cap}[S_1, \ldots, S_m]_U$ is the infimum of real numbers $\gamma > 0$ such that the inclusion

$$\partial_C \mathrm{dist}\,(\cdot, \cap_{i=1}^m S_i)(x) \subset \gamma[\partial_C \mathrm{dist}\,(\cdot, S_1)(x) + \cdots + \partial_C \mathrm{dist}\,(\cdot, S_m)(x)] \tag{9.26}$$

is satisfied for all $x \in U \cap S_1 \cap \cdots \cap S_m$.

Proceeding like in the proof of Corollary 9.15 one obtains the following result.

Proposition 9.19. *Let $S_1, \ldots, S_m$ be finitely many sets of a Banach space X which are closed near $\overline{x} \in \cap_{i=1}^m S_i$.*

(I) *Assume that each set S_i is metrically subsmooth at $\overline{x}$. Then the following hold.*
  (a) *The point $\overline{x}$ is metrically subregular for the system of sets $S_1, \ldots, S_m$ relative to the intersection if and only if the property (9.26) is fulfilled with some real $\gamma > 0$ and some neighborhood U of $\overline{x}$.*
  (b) *One has the equality between the foregoing constants at $\overline{x}$, that is,*

$$\mathrm{subreg}_{\cap}[S_1, \ldots, S_m](\overline{x}) = \mathrm{subdist}_{\cap}[S_1, \ldots, S_m](\overline{x}).$$

(II) *Assume that X is an Asplund space and each set S_i is subsmooth at $\overline{x}$. Then the point $\overline{x}$ is metrically subregular for the system of sets $S_1, \ldots, S_m$ relative to the intersection if and only if there are a real $\gamma > 0$ and a neighborhood U of $\overline{x}$ such that*

$$N^F(S; x) \cap \mathbb{B}_{X^*} \subset N^C(S_1; x) \cap \gamma\mathbb{B}_{X^*} + \cdots + N^C(S_m; x) \cap \gamma\mathbb{B}_{X^*}$$

for all $x \in U \cap \mathrm{bdry}\, S$.

Similar conditions with the coderivative can be used to study the subregularity of multimappings with subsmooth graphs. Let $M : X \rightrightarrows Y$ be a multimapping between normed spaces and, for any $(x, y) \in \mathrm{gph}\, M$, let $D_C^* M(x, y) : Y^* \rightrightarrows X^*$ be its C-coderivative at (x, y) defined by

$$D_C^* M(x, y)(y^*) := \{x^* \in X^* : (x^*, -y^*) \in N^C(\mathrm{gph}\, M; (x, y))\} \quad \text{for all } y^* \in Y^*.$$

We say that the multimapping M satisfies the *truncated C-coderivative condition* at a point $\overline{x} \in \mathrm{Dom}\, M$ for $\overline{y} \in M(\overline{x})$ provided there exist a real $\gamma > 0$ and neighborhoods U and V of $\overline{x}$ and $\overline{y}$ respectively such that

$$N^C\left(M^{-1}(y); x\right) \cap \mathbb{B}_{X^*} \subset \gamma\, D_C^* M(x, y)(\mathbb{B}_{Y^*})$$

for all $x \in U \cap \mathrm{Dom}\, M$ and $y \in M(x) \cap V$.

Before giving the result with this C-coderivative condition, let us establish the metric subregularity under a similar condition with the C-subdifferential of the distance from the graph of M. We need first to recall that, in addition to Definition (9.4) of metric subgularity, a multimapping $M : X \rightrightarrows Y$ is metrically subregular at a point $\overline{x}$ for $\overline{y} \in M(\overline{x})$ if and only if it is graphically metrically subregular at

$\overline{x}$ for $\overline{y}$, that is, there exist a real $\gamma > 0$ and a neighborhood U of $\overline{x}$ such that (see [74, 75])

$$d(x, M^{-1}(\overline{y})) \le \gamma\, d((x,\overline{y}), \operatorname{gph} M) \quad \text{for all } x \in U. \tag{9.27}$$

Proposition 9.20. *Let $M : X \rightrightarrows Y$ be a multimapping between Banach spaces whose graph is metrically subsmooth at $(\overline{x},\overline{y}) \in \operatorname{gph} M$ and closed near $(\overline{x},\overline{y})$. Then M is metrically subsmooth at $\overline{x}$ for $\overline{y}$ if and only if there exist a real $\gamma > 0$ and a neighborhood U of $\overline{x}$ in X such for any $x \in U \cap M^{-1}(\overline{y})$*

$$\partial_C d(\cdot, M^{-1}(\overline{y}))(x) \subset \gamma\, \Pi_{X^*}\big(\partial_C d(\cdot, \operatorname{gph} M)(x,\overline{y})\big),$$

where $\Pi_{X^*} : X^* \times Y^* \to X^*$ is the projector defined by $\Pi_{X^*}(x^*, y^*) := x^*$.

Proof. Suppose first that M is metrically subregular at $\overline{x}$ for $\overline{y}$. By the characterization (9.27) there are a real $\gamma > 0$ and an open neighborhood U of $\overline{x}$ such that

$$d(x, M^{-1}(\overline{y})) \le \gamma d((x,\overline{y}), \operatorname{gph} M) \quad \text{for all } x \in U.$$

Putting $g(x) := \gamma\, d((x,\overline{y}), \operatorname{gph} M)$ and using Proposition 2.1 as in the proof of Lemma 9.2(a), it is not difficult to see that $\partial_C d_{M^{-1}(\overline{y})}(x) \subset \partial_C g(x)$ for all $x \in U \cap M^{-1}(\overline{y})$. On the other hand, for the continuous linear mapping $A : X \to X \times Y$ defined by $A(u) := (u, 0)$ for all $u \in X$ we have $\partial_C g(x) \subset \gamma\, A^*(\partial_C d_{\operatorname{gph} M}(x,\overline{y}))$. Since A^* coincides with Π_{X^*}, we deduce that for all $x \in U \cap M^{-1}(\overline{y}))$

$$\partial_C d(\cdot, M^{-1}(\overline{y}))(x) \subset \gamma\, \Pi_{X^*}\big(\partial_C d(\cdot, \operatorname{gph} M)(x,\overline{y})\big).$$

Conversely, suppose that the latter property holds and set $S_1 := \operatorname{gph} M$ and $S_2 := X \times \{\overline{y}\}$, so $S := S_1 \cap S_2 = M^{-1}(\overline{y}) \times \{\overline{y}\}$ and $d_S(x,y) = d_{M^{-1}(\overline{y})}(x) + \|y - \overline{y}\|$, where $X \times Y$ is equipped with the sum norm. For every $(x,y) \in (U \times Y) \cap S$ we see that $y = \overline{y}$ and $x \in M^{-1}(\overline{y})$, hence

$$\partial_C d_S(x,y) \subset \partial_C d_{M^{-1}(\overline{y})}(x) \times \mathbb{B}_{Y^*}.$$

Fix any $(x,y) \in (U \times Y) \cap S$ and any $(x^*, y^*) \in \partial_C d_S(x,y)$. The second inclusion means by the latter equality that $y^* \in \mathbb{B}_{Y^*}$ and $x^* \in \partial_C d_{M^{-1}(\overline{y})}(x)$. Thus, by assumption there exists $v^* \in Y^*$ such that $(x^*, v^*) \in \gamma \partial_C d_{S_1}(x,\overline{y})$. Since

$$(x^*, y^*) = (x^*, v^*) + (0, y^* - v^*) \in \gamma \partial_C d_{S_1}(x,\overline{y}) + (1+\gamma)\partial_C d_{S_2}(x,\overline{y}),$$

we derive that $\partial_C d_S(x,y) \subset (1+\gamma)[\partial_C d_{S_1}(x,y) + \partial_{S_2}(x,y)]$ for all $(x,y) \in (U \times Y) \cap S$. Taking any real $\gamma' > 1 + \gamma$ it results by Proposition 9.19(b) that there exist a real $\gamma' > 0$ and neighborhoods U' and V' of $\overline{x}$ and $\overline{y}$ such that for all $x \in U'$ and $y \in V'$

$$d((x,y), S_1 \cap S_2) \le \gamma'[d((x,y), \operatorname{gph} M) + d((x,y), X \times \{\overline{y}\})].$$

This gives with $y = \overline{y}$ that for all $x \in U'$

$$d(x, M^{-1}(\overline{y})) \le \gamma' d((x,\overline{y}), \operatorname{gph} M) \le \gamma' d(\overline{y}, M(x)),$$

which translates the metric subregularity of M at $\overline{x}$ for $\overline{y}$. $\square$

Proposition 9.21. *Let $M : X \rightrightarrows Y$ be a multimapping between Asplund spaces whose graph is subsmooth at $(\overline{x}, \overline{y}) \in \operatorname{gph} M$ and closed near $(\overline{x}, \overline{y})$. Then M is metrically subregular at $\overline{x}$ for $\overline{y}$ if and only if there exist a real $\gamma > 0$ and a neighborhood U of $\overline{x}$ such that*

$$N^F\left(M^{-1}(\overline{y}); x\right) \cap \mathbb{B}_{X^*} \subset \gamma\, D^*_C M(x, \overline{y})^*(\mathbb{B}_{Y^*})$$

for all $x \in U \cap \operatorname{bdry}\left(M^{-1}(\overline{y})\right)$.

Proof. Endow $X \times Y$ with the sum norm and note that $\mathbb{B}_{X^* \times Y^*} = \mathbb{B}_{X^*} \times \mathbb{B}_{Y^*}$. Then for $x \in M^{-1}(\overline{y})$ the inclusion

$$\Pi_{X^*}\left(\partial_C d(\cdot, \operatorname{gph} M)(x, \overline{y})\right) \subset \Pi_{X^*}\left(N^C(\operatorname{gph} M; (x, \overline{y})) \cap (\mathbb{B}_{X^*} \times \mathbb{B}_{Y^*}\right)$$

holds. According to this inclusion and the equality $N^F(M^{-1}(\overline{y}); x) \cap \mathbb{B}_{X^*} = \partial_F d(\cdot, M^{-1}(\overline{x}))(\overline{y})$, Proposition 9.20 clearly shows that the metric subregularity of M at $\overline{x}$ for $\overline{y}$ implies the condition of the proposition.

Let us show the converse implication. As in the proof of Proposition 9.20 set $S_1 := \operatorname{gph} M$ and $S_2 := X \times \{\overline{y}\}$, so $S := S_1 \cap S_2 = M^{-1}(\overline{y}) \times \{\overline{y}\}$. For every $(x, y) \in (U \times Y) \cap S$ we see that $y = \overline{y}$ and $x \in M^{-1}(\overline{y})$, hence

$$N^F(S; (x, y)) \cap \mathbb{B}_{X^* \times Y^*} = \left(N^F(M^{-1}(\overline{y}); x) \cap \mathbb{B}_{X^*}\right) \times \mathbb{B}_{Y^*}.$$

Fix any $(x, y) \in (U \times Y) \cap S$ and any $(x^*, y^*) \in N^F(S; (x, y)) \cap \mathbb{B}_{X^* \times Y^*}$. The second inclusion can be rewritten by the latter equality as $y^* \in \mathbb{B}_{Y^*}$ and $x^* \in N^F\left(M^{-1}(\overline{y}); x\right) \cap \mathbb{B}_{X^*}$. Thus, by assumption there exists $b^* \in \mathbb{B}_{Y^*}$ such that $(x^*, -\gamma b^*) \in N^C(\operatorname{gph} M; (x, \overline{y}))$. Writing $(x^*, y^*) = (x^*, -\gamma b^*) + (0, y^* + \gamma b^*)$, it ensues with $\gamma' := 1 + \gamma$ that

$$(x^*, y^*) \in N^C(S_1; (x, y)) \cap \gamma' \mathbb{B}_{X^* \times Y^*} + N^C(S_2; (x, y)) \cap \gamma' \mathbb{B}_{X^* \times Y^*}.$$

Taking any real $\gamma'' > \gamma'$ and applying Proposition 9.19(c) we obtain as in the end of the proof of Proposition 9.20 above that M is metrically subregular at $\overline{x}$ for $\overline{y}$ with constant γ''. □

10. Convergence

This section is concerned with closedness and convergence of normal cones of subsmooth sets.

In addition to examples in Propositions 5.4, 9.6 and 9.14, another example of uniform equi-subsmoothness for sets is furnished by suitable families of sublevel sets.

Proposition 10.1. *Let I be a nonempty set and $m \in \mathbb{N}$. For each $i \in I$ let $g_{1,i}, \ldots, g_{m,i}$ be m locally Lipschitz functions from a normed space X into $\mathbb{R}$ and let*

$$S_i := \{x \in X : g_{1,i}(x) \le 0, \ldots, g_{m,i}(x) \le 0\}$$

that we assume to be a nonempty set. Assume that there exists $r \in]0, +\infty]$ such that the families of functions $(g_{1,i})_{i \in I}, \ldots, (g_{m,i})_{i \in I}$ are uniformly equi-subsmooth relative to the family of open sets $(U_r(S_i))_{i \in I}$ (which holds in particular when for each $k \in \{1, \ldots, m\}$ every function $g_{k,i}$ is differentiable on $U_r(S_i)$ with $(Dg_{k,i})_{i \in I}$ uniformly equi-continuous relative to $(U_r(S_i))_{i \in I}$). Assume also that the following generalized Slater condition holds: there exists a real $\sigma > 0$ such that for for each

$i \in I$ and each $x \in \operatorname{bdry} S_i$ there exists a vector $\overline{v} \in \mathbb{B}_X$ (depending on i, x) for which

$$\langle x^*, \overline{v} \rangle \geq \sigma$$

for every $k \in K(x) := \{k \in K : g_{k,i}(x) = \max_{j \in K} g_{j,i}(x)\}$ and every $x^* \in \partial_C g_{k,i}(x)$, where $K := \{1, \ldots, m\}$.

Then the family of sets $(S)_{i \in I}$ is uniformly equi-subsmooth.

Proof. (I) For each $i \in I$ define the function $g_i : X \to \mathbb{R}$ by $g_i(x) := \max_{k \in K} g_{k,i}(x)$ for all $x \in X$ and observe that $S_i = \{x \in X : g_i(x) \leq 0\}$. By Proposition 2.1(e) we have for each $i \in I$

$$\partial_C g_i(x) \subset \operatorname{co}\Big(\bigcup_{k \in K(x)} \partial_C g_{k,i}(x) \Big) \quad \text{for all } x \in \operatorname{bdry} S_i. \tag{10.1}$$

This inclusion and the assumption on $\overline{v}$ give us for each $i \in I$

$$0 \notin \partial_C g_i(x) \quad \text{for all } x \in \operatorname{bdry} S_i.$$

By Corollary 1 of Theorem 2.4.7 in [18] one deduces that for each $i \in I$

$$N^C(S_i; x) \subset \mathbb{R}_+ \partial_C g_i(x) \quad \text{for all } x \in \operatorname{bdry} S_i. \tag{10.2}$$

Take any $\varepsilon > 0$ and set $\varepsilon' := \varepsilon\sigma$. The uniform equi-subsmoothness assumption for the families of functions allows us by Proposition 4.16 to choose $\delta \in]0, r[$ such that for any $i \in I$, for any $x, y \in U_r(S_i)$, for any $k \in K$, for any $x^* \in \partial_C g_{k,i}(x)$, and for any $y^* \in \partial_C g_{k,i}(y)$

$$\langle x^* - y^*, x - y \rangle \geq -\varepsilon' \|x - y\|.$$

Fix any $i \in I$, any $x \in \operatorname{bdry} S_i$ and any $u^* \in N^C(S_i; x) \cap \mathbb{B}_{X^*}$. By (10.2) choose a real $\alpha \geq 0$ and $x^* \in \partial_C g(x)$ (both depending on i) such that $u^* = \alpha x^*$. From (10.1) there are $x_k^* \in \partial g_{k,i}(x)$ and $\lambda_k \geq 0$ with $\lambda_k = 0$ if $k \notin K(x)$ and with $\sum_{k \in K} \lambda_k = 1$, such that $x^* = \sum_{k \in K} \lambda_k x_k^*$.

(II) Fix any $y \in S_i$ with $0 < \|y - x\| < \delta$. Fix for a moment $k \in K(x)$ and define the locally Lipschitz function $\varphi_k : \mathbb{R} \to \mathbb{R}$ by $\varphi_k(t) := g_{k,i}(x + t(y - x))$, and note that it is Lipschitz on $[0, 1]$. Denote by N a Lebesgue negligible subset of $[0, 1]$ such that at each $t \in [0, 1] \setminus N$ the function φ_k is derivable at t, so $\varphi_k'(t) \in \partial_C(g_{k,i} \circ G)(t)$ with $G(t) = x + t(y - x) =: z(t)$. Then for each $t \in [0, 1] \setminus N$ there exists some $z_k^*(t) \in \partial_C g_{k,i}(z(t))$ such that $\varphi_k'(t) = \langle z_k^*(t), y - x \rangle$. Note also for each $t \in [0, 1]$ that $z(t) \in U_r(S_i)$ since $d_{S_i}(z(t)) \leq \|z(t) - x\| = t\|y - x\| < r$. It results that

$$\begin{aligned}
0 \geq g_{k,i}(y) - g_{k,i}(x) &= \int_0^1 \langle z_k^*(t), y - x \rangle \, dt \\
&= \int_0^1 \langle z_k^*(t) - x_k^*, y - x \rangle \, dt + \langle x_k^*, y - x \rangle \\
&= \int_0^1 \frac{1}{t} \langle z_k^*(t) - x_k^*, z(t) - x \rangle \, dt + \langle x_k^*, y - x \rangle,
\end{aligned}$$

which yields

$$0 \geq -\int_0^1 \frac{1}{t}\varepsilon' \|z_(t) - x\|\, dt + \langle x_k^*, y - x\rangle = -\varepsilon'\|y - x\| + \langle x_k^*, y - x\rangle. \tag{10.3}$$

Recalling that $\lambda_k = 0$ if $k \notin K(x)$, we deduce that $\langle x^*, y - x\rangle \leq \varepsilon\sigma\|y - x\|$. Using again the equality $\lambda_k = 0$ if $k \notin K(x)$ and using the assumption on $\overline{v} \in \mathbb{B}_X$, we also have $\langle x^*, \overline{v}\rangle \geq \sigma$. Therefore, the above equality $u^* = \alpha x^*$ gives

$$1 \geq \|u^*\| \geq \langle u^*, \overline{v}\rangle = \alpha\langle x^*, \overline{v}\rangle \geq \alpha\sigma,$$

so $\alpha \leq 1/\sigma$. It follows that

$$\langle u^*, y - x\rangle = \alpha\langle x^*, y - x\rangle \leq \varepsilon\|y - x\|,$$

and this inequality still holds when $x \in \operatorname{int} S_i$. This confirms that the familiy of sets $(S_i)_{i\in I}$ is uniformly equi-subsmooth. □

Remark 10.2. Let us provide another way to see (10.3) above. Keep the part (I) in the proof of Proposition 10.1 and let us modify the beginning of the part (II) as follows. Fix any $y \in S_i$ with $0 < \|y - x\| < \delta$. For each $k \in K(x)$ choose by Lebourg mean value equality (see Proposition 2.1(b)) some $z_k := x + t_k(y - x)$ with $t_k \in]0, 1[$ and some $z_k^* \in \partial_C g_{k,i}(x)$ (both depending on i) such that $g_{k,i}(y) - g_{k,i}(x) = \langle z_k^*, y - x\rangle$. For each $k \in K(x)$, observing that $z_k \in U_r(S_i)$ since $d_{S_i}(z_k) \leq \|z_k - x\| = t_k\|y - x\| < r$, we can write

$$\begin{aligned} 0 \geq g_{k,i}(y) - g_{k,i}(x) &= \langle z_k^*, y - x\rangle \\ &= \langle z_k^* - x_k^*, y - x\rangle + \langle x_k^*, y - x\rangle \\ &= \frac{1}{t_k}\langle z_k^* - x_k^*, z_k - x\rangle + \langle x_k^*, y - x\rangle. \end{aligned}$$

This entails that

$$0 \geq -\frac{1}{t_k}\varepsilon'\|z_k - x\| + \langle x_k^*, y - x\rangle = -\varepsilon'\|y - x\| + \langle x_k^*, y - x\rangle,$$

which is (10.3), so the proof of Proposition 10.3 can be continued as above.

The next proposition allows us (as altreeady done for the local subsmoothness of a set in Proposition 7.5(c)) to take into account points x outside the sets S_i when working with uniformly equi-subsmooth families of sets.

Proposition 10.3. *Let $(S_i)_{i\in I}$ be a family of nonemty closed sets of an Asplund space X. This family is uniformly equi-subsmooth if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $i \in I$, any $y \in S_i$, any $x \in X$ with $\|x - y\| < \delta$ and any $x^* \in \partial_C d_{S_i}(x)$, one has*

$$\langle x^*, y - x\rangle \leq d_{S_i}(x) + \varepsilon\|y - x\|.$$

Proof. Suppose first that the family is uniformly equi-subsmooth. Fix any real $\varepsilon > 0$ and take some $\varepsilon' > 0$ such that $2\varepsilon' + \varepsilon'(2 + \varepsilon') < \varepsilon$. By definition of uniform equi-subsmoothness, choose some $\delta > 0$ such that for any $i \in I$, for any $u, v \in S_i$ with $\|u - v\| < 3\delta$ and $u^* \in N^C(S_i; u) \cap \mathbb{B}_{X^*}$ one has

$$\langle u^*, v - u\rangle \leq \varepsilon'\|v - u\|. \tag{10.4}$$

Fix any $i \in I$. Take any pair (x, y) such that $x \in (X \setminus S_i) \cap \operatorname{Dom} \partial_F d_{S_i}$, $y \in S_i$ and $\|x - y\| < \delta$. Let any $x^* \in \partial_F d_{S_i}(x)$. Following the proof of (a)⇒(c) in Proposition 7.5, let us take a positive real $\varepsilon_i < \min\{\delta, \varepsilon', \varepsilon' d_{S_i}(x)\}$ (depending on i). We use Proposition 2.5(e) with ε_i in place of ε to get some $v \in S_i$ and $v^* \in \partial_F d_{S_i}(v)$ (both depending on i) such that

$$\|v - x\| < \varepsilon_i + d_{S_i}(x) < (1 + \varepsilon')d_{S_i}(x) \quad \text{and} \quad \|v^* - x^*\| < \varepsilon'. \tag{10.5}$$

From the first inequality in (10.5) we notice that

$$\|v - y\| \le \|v - x\| + \|x - y\| < \varepsilon_i + d_{S_i}(x) + \|x - y\|,$$

and hence by the inclusion $y \in S_i$

$$\|v - y\| < \varepsilon_i + 2\|x - y\| < 3\delta. \tag{10.6}$$

It results that

$$\begin{aligned}\langle x^*, y - x\rangle &\le \langle v^*, y - x\rangle + \varepsilon'\|y - x\| \\ &= \langle v^*, y - v\rangle + \langle v^*, v - x\rangle + \varepsilon'\|y - x\| \\ &\le \varepsilon'\|y - v\| + \|v - x\| + \varepsilon'\|y - x\|,\end{aligned}$$

the first inequality being due to the last inequality in (10.5) and the second one being due to (10.4) and (10.6) and to the fact that $\|v^*\| \le 1$. From the second inequality in the first part of (10.5) it ensues that

$$\begin{aligned}\langle x^*, y - x\rangle &\le 2\varepsilon'\|y - x\| + (1 + \varepsilon')\|v - x\| \\ &\le 2\varepsilon'\|y - x\| + d_{S_i}(x) + \varepsilon'(2 + \varepsilon')d_{S_i}(x),\end{aligned}$$

which gives (since $y \in S_i$)

$$\begin{aligned}\langle x^*, y - x\rangle &\le 2\varepsilon'\|y - x\| + d_{S_i}(x) + \varepsilon'(2 + \varepsilon')\|y - x\| \\ &= (2\varepsilon' + \varepsilon'(2 + \varepsilon'))\|y - x\| + d_{S_i}(x).\end{aligned}$$

The choice of ε' yields

$$\langle x^*, y - x\rangle \le d_S(x) + \varepsilon\|y - x\|. \tag{10.7}$$

Clearly, this inequality is still satisfied for any pair (x, y) such that $x \in \operatorname{Dom} \partial_F d_{S_i}$, $y \in S_i$ and $\|x - y\| < \delta$ since the case when $x \in S_i$ with $\|x - y\| < \delta$ follows from (10.4). Now take any pair $(x, y) \in X \times S_i$ with $\|x - y\| < \delta$. Any element in $\partial_L d_{S_i}(x)$ being the weak* limit of some sequence of Fréchet subgradients at points x_n converging strongly to x, we see that (10.7) continues to hold for any $x^* \in \partial_L d_S(x)$, and hence also for any $x^* \in \overline{\mathrm{co}}^*(\partial_L d_{S_i}(x)) = \partial_C d_{S_i}(x)$ (see Proposition 2.5(f) for the equality). This justifies the implication ⇒.

Conversely, suppose that the property in the proposition holds. For each $i \in I$, this implies in particular that S_i is metrically subsmooth. This ensures for each $x \in S_i$ that $\partial_C d_{S_i}(x) = N^C(S_i; x) \cap \mathbb{B}_{X^*}$ according to the implication (a)⇒(f) in Proposition 7.7. This combined with the property of the proposition yields that the family of sets $(S_i)_{i \in I}$ is uniformly equi-subsmooth. □

We start now with two closedness properties.

Proposition 10.4. *Let E be a metric space and let $(S(q))_{q\in E}$ be a family of nonempty closed sets of an Asplund space X which is uniformly equi-subsmooth and let $\eta \in [0, +\infty[$. Let $Q \subset E$, $q_0 \in \mathrm{cl}Q$ and $x \in S(q_0)$. Then, for any net $(q_j)_{j\in J}$ in Q converging to q_0 with $d_{S(q_j)}(x) \underset{j\in J}{\to} 0$, for any net $(x_j)_{j\in J}$ converging to x in $(X, \|\cdot\|)$, and for any net $(x_j^*)_{j\in J}$ converging weakly* to x^* in X^* with $x_j^* \in \eta\partial_C d_{S(q_j)}(x_j)$, one has $x^* \in \eta\partial_C d_{S(q_0)}(x)$.*

Proof. We may suppose $\eta > 0$. Take any real $\varepsilon > 0$. By Proposition 10.3 above choose a real $\delta > 0[$ such that for all $q \in E$, $v \in S(q)$, $u \in X$ with $\|u - v\| < \delta$ and all $u^* \in \partial_C d_{S(q)}(u)$

$$\langle u^*, v - u\rangle \le d_{S(q)}(u) + \varepsilon\|v - u\|. \tag{10.8}$$

Fix any net $(q_j)_{j\in J}$ in Q converging to q_0 in E with $d_{S(q_j)}(x) \underset{j\in J}{\to} 0$, and any net $(x_j)_{j\in J}$ in X converging strongly to x, where $(J, \preccurlyeq)$ is a directed preordered set. Fix also any net $(x_j^*)_{j\in J}$ converging weakly* in X^* to x^* such that $x_j^* \in \eta\partial_C d_{S(q_j)}(x_j)$. Fix $y \in B(x, \frac{\delta}{2}) \cap S(q_0)$. For each $n \in \mathbb{N}$ and each $j \in J$, choose some $y_{j,n} \in S(q_j)$ such that

$$\|y_{j,n} - y\| \le d_{S(q_j)}(y) + \frac{1}{n}.$$

Endowing $J \times \mathbb{N}$ with the product preorder which is obviously directed, the family $(y_{j,n})_{(j,n)\in J\times\mathbb{N}}$ is a net in X. Since

$$d_{S(q_j)}(y) + \frac{1}{n} \underset{(j,n)\in J\times\mathbb{N}}{\longrightarrow} 0,$$

we have $\|y_{j,n} - y\| \underset{(j,n)\in J\times\mathbb{N}}{\longrightarrow} 0$, that is, $y_{j,n} \underset{(j,n)\in J\times\mathbb{N}}{\longrightarrow} y$ strongly in H, and hence there exists $j_0 \in J$ and $n_0 \in \mathbb{N}$ such that for all $(j, n) \in J \times \mathbb{N}$ with $j \succcurlyeq j_0$ and $n \ge n_0$ we have $y_{j,n} \in B(x, \frac{\delta}{2})$. Put $x_{j,n} := x_j$ for all $(j, n) \in J \times \mathbb{N}$. Obviously $x_{j,n} \underset{(j,n)\in J\times\mathbb{N}}{\longrightarrow} x$ strongly in X (because $x_j \underset{j\in J}{\to} x$). So, we may also suppose that $x_{j,n} \in B(x, \frac{\delta}{2})$ for all $(j, n) \in J \times \mathbb{N}$, with $j \succcurlyeq j_0$ and $n \ge n_0$. Thus, for all $(j, n) \in J \times \mathbb{N}$ with $j \succcurlyeq j_0$ and $n \ge n_0$ we have

$$\|y_{j,n} - x\| < \frac{\delta}{2} \text{ and } \|x_{j,n} - x\| < \frac{\delta}{2}.$$

Set $x_{j,n}^* := x_j^*$ and $q_{j,n} := q_j$ for all $(j, n) \in J \times \mathbb{N}$. The net $(q_{j,n})_{(j,n)\in J\times\mathbb{N}}$ converges to q_0 and the net $(x_{j,n}^*)_{(j,n)\in J\times\mathbb{N}}$ converges weakly* to x^* in X^*. Thanks to the latter inequalities above, for all $(j, n) \in J \times \mathbb{N}$ with $j \succcurlyeq j_0$ and $n \ge n_0$ we have $\|y_{j,n} - x_{j,n}\| < \delta$ with $y_{j,n} \in S(q_j)$, and hence according to (10.8))

$$\begin{aligned}\langle \eta^{-1}x_{j,n}^*, y_{j,n} - x_{j,n}\rangle &\le d_{S(q_j)}(x_{j,n}) + \varepsilon\|y_{j,n} - x_{j,n}\| \\ &\le d_{S(q_j)}(x) + \|x_{j,n} - x\| + \varepsilon\|y_{j,n} - x_{j,n}\|.\end{aligned}$$

Since the net $(\eta^{-1}x_{j,n}^*)_{(j,n)\in J\times\mathbb{N}}$ is bounded (by the real number 1), we may pass to the limit to obtain

$$\langle \eta^{-1}x^*, y - x\rangle \le \varepsilon\|y - x\|$$

for all $y \in B(x, \frac{\delta}{2}) \cap S(q_0)$. This entails that $\eta^{-1}x^* \in N^F(S(q_0); x)$. Further, $\eta^{-1}x^*_{j,n} \in \mathbb{B}$ for all $(j,n) \in J \times \mathbb{N}$ and this ensures $\eta^{-1}x^* \in \mathbb{B}$. Thus, $\eta^{-1}x^* \in N^F_{S(q_0)}(x) \cap \mathbb{B}$, so $\eta^{-1}x^* \in \partial_F d_{S(q_0)}(x) \subset \partial_C d_{S(q_0)}(x)$. The proof is then complete. □

The second proposition provides a partial upper semicontinuity property.

Proposition 10.5. *Let E be a metric space and let $(S(q))_{q\in E}$ be a family of nonempty closed sets of a normed space X which is uniformly equi-subsmooth. Let $Q \subset E$, $q_0 \in \mathrm{cl}Q$ and $x \in S(q_0)$. Then for any net $(q_j)_{j\in J}$ in Q converging to q_0 with $d_{S(q_j)}(x) \underset{j\in J}{\to} 0$, for any net $(x_j)_{j\in J}$ converging to x in $(X, \|\cdot\|)$, one has for every $h \in X$*

$$\limsup_{j\in J} \sigma\big(h, \partial_C d_{S(q_j)}(x_j)\big) \le \sigma\big(h, \partial_C d_{S(q_0)}(x)\big).$$

Proof. Fix any $h \in X$. Let $(q_j)_j$ and $(x_j)_j$ be as in the statement. Extracting a subnet if necessary, we may suppose that

$$\limsup_{j\in J} \sigma\big(h, \partial_C d_{S(q_j)}(x_j)\big) = \lim_{j\in J} \sigma\big(h, \partial_C d_{S(q_j)}(x_j)\big).$$

For each j, the weak* compactness of the convex set $\partial_C d_{Q(q_j)}(x_j)$ ensures the existence of some $x^*_j \in \partial d_{S(q_j)}(x_j)$ such that

$$\langle x^*_j, h\rangle = \sigma\big(h, \eta\partial_C d_{S(q_j)}(x_j)\big).$$

Since $\|x^*_j\| \le 1$, a subnet of $(x^*_j)_j$ (that we do not relabel) converges weakly* to some x^* in X^*. It results that

$$\langle x^*, h\rangle = \limsup_{j\in J} \sigma\big(h, \partial_C d_{S(q_j)}(x_j)\big). \tag{10.9}$$

On the other hand, Proposition 10.4 tells us that $x^* \in \partial_C d_{S(q_0)}(x)$. The latter inclusion combined with (10.9) yields

$$\limsup_{j\in J} \sigma\big(h, \partial_C d_{S(q_j)}(x_j)\big) \le \sigma\big(h, \partial_C d_{S(q_0)}(x)\big),$$

which completes the proof. □

The following corollary is a direct consequence of the previous proposition. It is often involved in the study of existence of solution for sweeping processes (see, e.g., [1, 3, 16, 33, 53, 54, 55, 76] and references therein). Before stating the corollary, recall that for an extended real $\rho \in]0, +\infty]$ and two subsets S, S' of a normed space X, the pseudo ρ-excess of S over S' is defined by

$$\widehat{\mathrm{exc}}_\rho(S, S') := \sup_{u\in S\cap\rho\mathbb{B}_X} d(u, S'),$$

where we employ the usual convention that the latter supremum is zero whenever $S \cap \rho\mathbb{B}_X = \emptyset$.

Corollary 10.6. *Let (Q, d) be a metric space and $(S(q))_{q\in Q}$ be a family of nonempty closed sets of an Asplund space X which are uniformly equi-subsmooth. Let W be a subset of $Q \times Q$ containing the diagonal set. Assume that there are an extended* *real* $\rho \in]0, +\infty]$ *and a function* $\vartheta : W \to [0, +\infty[$ *satisfying* $\vartheta(q_0, q) \to 0$ *as* $q \to q_0$ *with* $(q_0, q) \in W$ *and such that*

$$\widehat{\mathrm{exc}}_\rho(S(q), S(q')) \le \vartheta(q, q')$$

for every $(q, q') \in W$. *Then for any sequence* $(q_n)_n$ *in* Q *converging to* q *with* $(q, q_n) \in W$, *any sequence* $(x_n)_n$ *in* X *converging to* $x \in S(q) \cap \rho\mathbb{B}_X$, *and any* $h \in X$, *we have*

$$\limsup_{n\to\infty} \sigma\big(h, \partial_C d_{S(q_n)}(x_n)\big) \le \sigma\big(h, \partial_C d_{S(q)}(x)\big).$$

Now let us turn to convergence of normals. In Section 2 the limit inferior of a multimapping has been recalled. Now we need to recall first the Painlevé-Kuratowski convergence and the Mosco convergence of sequences of sets (see, e.g. [7, 9, 68]). Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of subsets of the normed space X. Given a topology τ on X, one defines the *sequential limit inferior* ${}^\tau \operatorname*{Lim\,inf}_{n\to\infty} S_n$ of the sequence $(S_n)_{n\in\mathbb{N}}$ with respect to the topology τ as the set of all τ-limits of sequences $(x_n)_n$ with $x_n \in S_n$ for all $n \in \mathbb{N}$ large enough. The *sequential limit superior* ${}^\tau \operatorname*{Lim\,sup}_{n\to\infty} S_n$ with respect to τ is defined as the set of all τ-limits of sequences $(x_n)_n$ with $x_n \in S_n$ for infinitely many $n \in \mathbb{N}$. Equivalently, $x \in {}^\tau \operatorname*{Lim\,sup}_{n\to\infty} S_n$ provided there are an increasing sequence $(k(n))_{n\in\mathbb{N}}$ in $\mathbb{N}$ and a sequence $(x_n)_{n\in\mathbb{N}}$ converging to x with $x_n \in S_{k(n)}$ for all $n \in \mathbb{N}$. Clearly,

$${}^\tau \operatorname*{Lim\,inf}_{n\to\infty} S_n \subset {}^\tau \operatorname*{Lim\,sup}_{n\to\infty} S_n.$$

One then says that the sequence $(S_n)_{n\in\mathbb{N}}$ τ-sequentially *Painlevé-Kuratowski converges* to a subset S of X whenever

$$S = {}^\tau \operatorname*{Lim\,inf}_{n\to\infty} S_n = {}^\tau \operatorname*{Lim\,sup}_{n\to\infty} S_n.$$

When τ is the topology associated with the norm of X, one just says that the sequence Painlevé-Kuratowski converges to S. When the sequence $(S_n)_{n\in\mathbb{N}}$ (sequentially) Painlevé-Kuratowski converges to S with respect to both the norm topology and the weak topology, one says that it *converges in the sense of Mosco* to S. It is easily seen that this is equivalent to

$$S = {}^{\|\,\|} \operatorname*{Lim\,inf}_{n\to\infty} S_n = {}^w \operatorname*{Lim\,sup}_{n\to\infty} S_n,$$

where w stands here for the weak topology $w(X, X^*)$ of X. Note that, in this case, the subset S is weakly sequentially closed in the sense that the limit of any weakly convergent sequence of S belongs to S. Indeed, suppose without loss of generality that every S_n is nonempty, and take any sequence $(x_m)_{m\in\mathbb{N}}$ of S converging weakly to $x \in X$. For each $m \in \mathbb{N}$, from the equality $S = {}^{\|\,\|} \operatorname*{Lim\,inf}_{n\to\infty} S_n$ there is a sequence $(x_{m,n})_{n\in\mathbb{N}}$ converging strongly to x_m with $x_{m,n} \in S_n$ for all $n \in \mathbb{N}$. We can then choose an increasing sequence $(k(m))_{m\in\mathbb{N}}$ in $\mathbb{N}$ such that $\|x_{m,k(m)} - x_m\| < 1/m$. So, for $x'_m := x_{m,k(m)}$, the sequence $(x'_m)_{m\in\mathbb{N}}$ converges weakly to x as $m \to \infty$ and $x'_m \in S_{k(m)}$ for all $m \in \mathbb{N}$. This and the equality $S = {}^w \operatorname*{Lim\,sup}_{m\to\infty} S_m$ justify the inclusion $x \in S$.

Denote by $\mathcal{N}_S = \mathcal{N}(S;\cdot)$ either the C-normal cone or the L-normal cone or the F-normal cone of S. Let ${}^{\|\,\|}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} \mathcal{N}_{S_n}$ denote the limit superior (with respect to the norm topology in $X \times X^*$) of the sequence $(\operatorname{gph} \mathcal{N}_{S_n})_{n\in\mathbb{N}}$ of the graphs of $\mathcal{N}_{S_n}$, that is, the set of all (x, x^*) in $X \times X^*$ for which there exists a sequence $(x_n, x_n^*)_{n\in\mathbb{N}}$ in $X \times X^*$ and an increasing sequence $(k(n))_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$x_n \in S_{k(n)} \quad \text{and} \quad x_n^* \in \mathcal{N}_{S_{k(n)}}(x_n) \quad \text{for } n \in \mathbb{N} \text{ large enough},$$

and such that $(x_n)_{n\in\mathbb{N}}$ and $(x_n^*)_{n\in\mathbb{N}}$ converge to x and x^* with respect to the norm topology of X and X^* respectively. Similarly, we denote by ${}^{\|\,\|,*}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} \mathcal{N}_{S_n}$ the sequential limit superior of $(\operatorname{gph} \mathcal{N}_{S_n})_{n\in\mathbb{N}}$ with respect to the $\|\,\| \times w(X^*, X)$ topology of $X \times X^*$, that is, the set of all (x, x^*) in $X \times X^*$ for which there exist a sequence $(x_n, x_n^*)_{n\in\mathbb{N}}$ in $X \times X^*$ and an increasing sequence $(k(n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$x_n \in S_{k(n)} \quad \text{and} \quad x_n^* \in \mathcal{N}_{S_k(n)}(x_n) \quad \text{for all } n \in \mathbb{N},$$

and such that $(x_n)_{n\in\mathbb{N}}$ and $(x_n^*)_{n\in\mathbb{N}}$ converge to x and x^* with respect to the norm topology of X and the weak* topology of X^* respectively. It is evident that

$${}^{\|\,\|}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} \mathcal{N}_{S_n} \subset {}^{\|\,\|,*}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} \mathcal{N}_{S_n}.$$

The sequence of sets $(S_n)_{n\in\mathbb{N}}$ is said to be *eventually subsmooth at a point* $\overline{x}$ *with respect to the nomal cone* $\mathcal{N}$ (where $\mathcal{N}$ is N^F, N^L or N^C) if for each real $\varepsilon > 0$ there exist an integer $N \in \mathbb{N}$ and a real $\delta > 0$ such that for each integer $n \geq N$ one has

$$\langle x^*, y - x\rangle \leq \varepsilon \|y - x\|$$

for all $x, y \in S_n \cap B(\overline{x}, \delta)$ and $x^* \in \mathcal{N}_{S_n}(x) \cap \mathbb{B}_{X^*}$. In [78] the sequence is rather called subsmooth at $\overline{x}$ with compatible indexation. Clearly, the sequence $(S_n)_{n\in\mathbb{N}}$ is eventually subsmooth whenever it is uniformly equi-subsmooth.

Using the above concepts and [78, Corollary 3.5(c)] we can now restate Theorem 3.15 of L. Thibault and T. Zakaryan [78] as follows.

Theorem 10.7. *Let X be a reflexive Banach space and let $(S_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of X converging in the sense of Mosco to a nonempty closed subset S of X. Assume that the sequence $(S_n)_{n\in\mathbb{N}}$ is eventually subsmooth with respect to the Fréchet normal cone at any point of S. Then one has*

$$\operatorname{gph} N_S^L = \operatorname{gph} N_S^F = {}^{\|\,\|}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N_{S_n}^F = {}^{\|\,\|,*}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N_{S_n}^F.$$

The next theorem reproduces Theorem 4.10 in [78] in an equivalent form.

Theorem 10.8. *Let X be a reflexive Banach space and let $(S_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of X Attouch-Wets convergent to a nonempty closed subset S of X. Assume that the sequence $(S_n)_{n\in\mathbb{N}}$ is eventually subsmooth with respect to the Fréchet normal cone at any point of S. Then one has*

$$\operatorname{gph} N_S^L = \operatorname{gph} N_S^F = {}^{\|\,\|}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N_{S_n}^F = {}^{\|\,\|,*}\operatorname*{Lim\,sup}_{n\to\infty} \operatorname{gph} N_{S_n}^F.$$

Corollary 10.9. *Let X be a reflexive Banach space and let $(S_n)_{n\in\mathbb{N}}$ be a sequence of closed subsmooth subsets of X converging to a nonempty closed subset S of X either in the sense of Mosco or in the sense of Attouch-Wets. Assume that the sequence $(S_n)_{n\in\mathbb{N}}$ is eventually subsmooth with respect to the Fréchet normal cone at any point of S (resp. $(S_n)_{n\in\mathbb{N}}$ is uniformly equi-subsmooth). Then the set S is subsmooth (resp. uniformly subsmmoth).*

Proof. Fix any $\overline{x} \in S$ and any $\varepsilon > 0$. By assumption there exist $\delta > 0$ and $N \in \mathbb{N}$ such that $\langle x^*, y - x\rangle \le \varepsilon\|y - x\|$ for all $n \ge N$, $x, y \in S_n \cap B(\overline{x}, \delta)$ and $x^* \in N^F(S_n; x)$. Take any $u, v \in S \cap B(\overline{x}, \delta)$ and any $u^* \in N^F(S; u) \cap \mathbb{B}$. By Theorem 10.7 (resp. Theorem 10.8) there exist an increasing sequence $(k(n))_{n\in\mathbb{N}}$ in $\mathbb{N}$, sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ converging to u and v respectively, a sequence $(u_n^*)_{n\in\mathbb{N}}$ converging strongly to u^* such that for all $n \in \mathbb{N}$ one has

$$u_n^* \in N^F(S_{k(n)}; u_n),\ u_n \in S_{k(n)},\ v_n \in S_{k(n)}.$$

Consider any real $\eta > 0$. There is some integer $N_0 \ge N$ such that for every $n \ge N_0$ we have $\|u_n^*\| \le 1 + \eta$ and both u_n and v_n are in $B(\overline{x}, \delta)$ (since this open ball contains u and v). Consequently, we see that for all $n \ge N_0$ we have

$$\langle \frac{u_n^*}{1+\eta}, v_n - u_n\rangle \le \varepsilon\|v_n - u_n\|,$$

which entails $\langle u^*, v - u\rangle \le \varepsilon(1+\eta)\|v - u\|$. This being true for every $\eta > 0$ it results that $\langle u^*, v - u\rangle \le \varepsilon\|v - u\|$, which justifies that S is submooth at any $\overline{x} \in S$.

The arguments for the uniform subsmoothness of S under the appropriate assumption in the corollary are quite similar. □

Before turning to convergence of subdifferentials, let us emphasize another way to define the subsmoothness of sets. We already noticed that the subsmoothness of a set S at $\overline{x} \in S$ does not mean that the multimapping $N^C(S; \cdot)$ is submonotone. The following question then arises: Which submonotone-like property does characterize the subsmoothness of S at $\overline{x}$? If S is subsmooth at $\overline{x}$, for any real $\varepsilon > 0$ there exists a real $\delta > 0$ such that for any $x, y \in S \cap B(\overline{x}, \delta)$ and any $u^* \in N^C(S; x) \cap \mathbb{B}_{X^*}$ we have $\langle u^*, y - x\rangle \le \varepsilon\|y - x\|$, so for any $x^* \in N^C(S; x)$ we see that $\langle x^*, y - x\rangle \le \varepsilon(1 + \|x^*\|)\|y - x\|$. The converse being evident, we obtain that S is subsmooth at $\overline{x}$ if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in S \cap B(\overline{x}, \delta)$ and $x^* \in N^C(S; x)$ one has

$$\langle x^*, y - x\rangle \le \varepsilon(1 + \|x^*\|)\|y - x\|.$$

Further, using the inclusion $0 \in N^C(S; y)$ it is clear that the latter property holds if and only if the following submonotone-like property at $\overline{x}$ is satisfied: for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in S \cap B(\overline{x}, \delta)$, $x^* \in N^C(S; x)$ and $y^* \in N^C(S; y)$

$$\langle x^* - y^*, x - y\rangle \ge -\varepsilon(1 + \|x^*\| + \|y^*\|)\|y - x\|. \tag{10.10}$$

According to the above analysis, a proper function $f : X \to \mathbb{R} \cup \{+\infty\}$ lower semicontinuous at $\overline{x} \in X$ will be called *subsmooth at $\overline{x}$ along ∂-subgradients* (where

∂ is ∂_F, ∂_L or ∂_C) provided that for any real $\varepsilon > 0$ there exists a real $\delta > 0$ such that for all $y \in B(\overline{x}, \delta)$, $x \in B(\overline{x}, \delta) \cap \operatorname{Dom} \partial f$ and $x^* \in \partial f(x)$

$$\langle x^*, y - x \rangle \le f(y) - f(x) + \varepsilon(1 + \|x^*\|)\|y - x\|. \tag{10.11}$$

Similarly, a family $(f_t)_{t \in T}$ of proper lower semicontinuous functions from X into $\mathbb{R} \cup \{+\infty\}$ is *equi-subsmooth at $\overline{x}$ along subgradients* whenever for any real $\varepsilon > 0$ there exists a real $\delta > 0$ such that for each $t \in T$

$$\langle x^*, y - x \rangle \le f_t(y) - f_t(x) + \varepsilon(1 + \|x^*\|)\|y - x\|$$

for all $y \in B(\overline{x}, \delta)$, $x \in B(\overline{x}, \delta) \cap \operatorname{Dom} \partial_C f_t$ and $x^* \in \partial_C f_t(x)$.

In the case of a sequence of functions the latter property can be weakened as follows. We say that a sequence $(f_n)_{n \in \mathbb{N}}$ of proper lower semicontinuous functions from X into $\mathbb{R} \cup \{+\infty\}$ is *eventually subsmooth at $\overline{x}$ along ∂-subgradients* if for any real $\varepsilon > 0$ there are some $N \in \mathbb{N}$ and some real $\delta > 0$ such that for each $n \ge N$

$$\langle x^*, y - x \rangle \le f_n(y) - f_n(x) + \varepsilon(1 + \|x^*\|)\|y - x\|$$

for all $y \in B(\overline{x}, \delta)$, $x \in B(\overline{x}, \delta) \cap \operatorname{Dom} \partial f_n$ and $x^* \in \partial f_n(x)$. We must also say that in [78] it is rather used the terminology that the sequence of functions is subsmooth at $\overline{x}$ with compatible indexation.

It will be convenient to denote by ${}^{\|\ \|}\operatorname*{Lim\,sup}_{n \to \infty, f_n} \operatorname{gph} \partial_F f_n$ (resp. ${}^{\|\ \|, *}\operatorname*{Lim\,sup}_{n \to \infty, f_n} \operatorname{gph} \partial_F f_n$) the set of all pairs (x, x^*) in $X \times X^*$ for which there exist an increasing sequence $(k(n))_{n \in \mathbb{N}}$ in $\mathbb{N}$ and a sequence $(x_n, x_n^*)_{n \in \mathbb{N}}$ with $(x_n, x_n^*) \in \operatorname{gph} \partial_F f_{k(n)}$ and such that $(x_n)_{n \in \mathbb{N}}$ converges to x with $f_{k(n)}(x_n) \to f(x)$ and $(x_n^*)_{n \in \mathbb{N}}$ converges to x^* (resp. $(x_n^*)_{n \in \mathbb{N}}$ converges weakly* to x^*).

Theorem 5.10 of L. Thibault and T. Zakaryan [78] according to the proof of Theorem 5.1 in [78] can be stated in the following form.

Theorem 10.10. *Let X be a reflexive Banach space and let $(f_n)_{n \in \mathbb{N}}$ be a sequence of proper lower semicontinuous functions from X into $\mathbb{R} \cup \{+\infty\}$ which converges in the sense of Mosco to a proper function f. Assume that at each point of* $\operatorname{dom} f$ *the sequence $(f_n)_{n \in \mathbb{N}}$ is eventually subsmooth along F-subgradients. Then, one has*

$$\operatorname{gph} \partial_L f = \operatorname{gph} \partial_F f = {}^{\|\ \|}\operatorname*{Lim\,sup}_{n \to \infty,\, f_n} \operatorname{gph} \partial_F f_n = {}^{\|\ \|, *}\operatorname*{Lim\,sup}_{n \to \infty,\, f_n} \operatorname{gph} \partial_F f_n.$$

11. Lower $\omega(\cdot)$-regular functions

Submooth functions along subgradients have been seen in the previous section to be a suitable framework for convergence of subgradients of sequences of functions. The present section is devoted to a fundamental subclass of subsmooth functions along subgradients.

Throughout the section, $\omega : [0, +\infty[\to [0, +\infty[$ is a (continuous) increasing convex function such that $\omega(0) = \omega'_+(0) = 0$; such $\omega(\cdot)$ is a particular convex modulus function. We note by convexity of ω that for each real $t > 0$ we have for all $s > 0$

$$\frac{\omega(t)}{t} = \frac{\omega(t - t) - \omega(t)}{-t} \le \frac{\omega(t + s) - \omega(t)}{s},$$

which gives $t^{-1}\omega(t) \le \omega'_+(t)$. Therefore, we have

$$\omega(t) \le t\,\omega'_+(t) \quad \text{for all } t \in [0,+\infty[.$$

Further, the convexity of ω also ensures that the function ω'_+ is nondecreasing on $[0,+\infty[$, hence in particular $t\,\omega'_+(t) \to 0$ as $t \downarrow 0$.

Definition 11.1. Let $f : U \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function on a nonempty open set U of a normed space $(X, \|\cdot\|)$. We say that f is lower $\omega(\cdot)$-regular at a point $\overline{x} \in U$ if there exist a real $c \ge 0$ and a real $\delta > 0$ such that for all $y \in B(\overline{x}, \delta)$, $x \in B(\overline{x}, \delta) \cap \mathrm{Dom}\,\partial_C f$ and $x^* \in \partial_C f(x)$

$$\langle x^*, y - x\rangle \le f(y) - f(x) + c(1 + \|x^*\|)\,\omega(\|y - x\|).$$

Clearly, any such function is subsmooth at $\overline{x}$ along subgradients. It is also readily seen with $\delta > 0$ as above that

$$\partial_C f(x) = \partial_F f(x) \quad \text{for all } x \in B(\overline{x}, \delta).$$

Further, if X is an Asplund space and f is lower semicontinuous and satisfies the property in Definition 11.1 with the F-subdifferential in place of the C-subdifferential, using the equality $\partial_C f(x) = \overline{\mathrm{co}}^*\big(\partial_L f(x) + \partial_L^\infty(x)\big)$ (see Proposition 2.5(g)) one can show that $\partial_C f(x) = \partial_F f(x)$ for all $x \in B(\overline{x}, \delta)$, so f is lower $\omega(\cdot)$-regular at $\overline{x}$.

Lemma 11.2. *Let $(X, \|\cdot\|)$ be a normed space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function with $f(\overline{x}) < +\infty$. Let r be a positive number such that f is bounded from below over $B[\overline{x}, r]$ by some real α. Let ω be as above and let a real $\theta \ge 0$. For each real $c \ge 0$, let*

$$F_c(x^*, x, y) := f(y) + \langle x^*, x - y\rangle + c(1 + \|x^*\|)\,\omega(\|x - y\|),$$

for all $x, y \in X$ and $x^* \in X^*$. Let any real $\underline{c}$ such that

$$\underline{c}\big(\omega(r/2) - \omega(r/4)\big) > \max\{r, f(\overline{x}) - \alpha + \theta\}.$$

Then, for any real $c \ge \underline{c}$, for any $x^* \in X^*$ and for any $x \in B[\overline{x}, \frac{r}{4}]$, every point $u \in B[\overline{x}, r]$ such that

$$F_c(x^*, x, u) \le \inf_{y \in B[\overline{x}, r]} F_{\beta,c}(x^*, x, y) + \theta$$

must belong to $B(\overline{x}, \frac{3r}{4})$.

Proof. Let any real $c \ge \underline{c}$. Fix $x \in B[\overline{x}, \frac{r}{4}]$ and $x^* \in X^*$. Take any $y \in B[\overline{x}, r]$ with $\|y - \overline{x}\| \ge \frac{3r}{4}$. Since

$$\|x - y\| \ge \|\overline{x} - y\| - \|x - \overline{x}\| \ge \frac{r}{2},$$

we have

$$\omega(\|x - y\|) - \omega(\|x - \overline{x}\|) \ge \omega(r/2) - \omega(r/4) =: \xi(r) > 0.$$

It ensues that, for $F(y) := F_c(x^*, x, y)$

$$\begin{aligned}
&F(y) - F(\overline{x}) - \theta \\
&\ge f(y) - f(\overline{x}) - \theta + \langle x^*, \overline{x} - y\rangle + c(1 + \|x^*\|)(\omega\big(\|x - y\|) - \omega(\|x - \overline{x}\|)\big) \\
&\ge \alpha - f(\overline{x}) - \theta - r\|x^*\| + c(1 + \|x^*\|)\xi(r) \\
&= \big(\alpha - f(\overline{x}) - \theta + c\xi(r)\big) + \|x^*\|\big(c\xi(r) - r\big),
\end{aligned}$$

hence with $\beta := \alpha - f(\overline{x}) - \theta + c\xi(r) > 0$ we obtain $F(y) - \beta \geq F(\overline{x}) + \theta$. This justifies the lemma. □

Theorem 11.3 (Subdifferential characterization of lower $\omega(\cdot)$-regularity). *Let ω be as above and derivable on $]0, +\infty[$. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function on a Banach space X which is finite at $\overline{x} \in X$ and lower semicontinuous near $\overline{x}$. The following are equivalent:*

(a) *The function f is lower $\omega(\cdot)$-regular at $\overline{x}$.*

(b) *There exist reals $\delta > 0$ and $c \geq 0$ such that for all $x_i^* \in \partial_C f(x_i)$ with $\|x_i - \overline{x}\| < \delta$, $i = 1, 2$, one has*

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -c(1 + \|x_1^*\| + \|x_2^*\|)\, \omega(\|x_1 - x_2\|).$$

If in addition X is an Asplund space, then the following assertion is also equivalent to the lower $\omega(\cdot)$-regularity of the function f at the point $\overline{x}$:

(c) *The inequality in (b) is fulfilled with $\partial_L f$ or ∂_F in place of $\partial_C f$.*

Proof. To prove $(a) \Rightarrow (b)$, suppose that f satisfies the lower $\omega(\cdot)$-regularity property for $\overline{x}$ over some ball $B(\overline{x}, \delta)$ with some coefficient $c \geq 0$. Then, for $x_i \in X$ with $\|x_i - \overline{x}\| < \delta$ and $x_i^* \in \partial_C f\,(x_i)$, $i = 1, 2$, we have by definition

$$\begin{array}{rcl} f(x_1) & \geq & f(x_2) + \langle x_2^*, x_1 - x_2 \rangle + c(1 + \|x_2^*\|)\, \omega(\|x_1 - x_2\|) \\ f(x_2) & \geq & f(x_1) + \langle x_1^*, x_2 - x_1 \rangle + c(1 + \|x_1^*\|)\, \omega(\|x_1 - x_2\|), \end{array}$$

and adding these inequalities gives according to the finiteness of $f(x_1)$ and $f(x_2)$

$$\begin{array}{rcl} \langle x_1^* - x_2^*, x_1 - x_2 \rangle & \geq & -c(2 + \|x_1^*\| + \|x_2^*\|)\, \omega(\|x_1 - x_2\|) \\ & \geq & -2c(1 + \|x_1^*\| + \|x_2^*\|)\, \omega(\|x_1 - x_2\|). \end{array}$$

This confirms that the implication $(a) \Rightarrow (b)$ holds true with $\partial_C f$, and hence also (a) implies (c).

Now let us prove the converse implication. Denote by ∂f anyone of the subdifferentials involved in (b) and (c) with the appropriate space. Let $\delta > 0, c \geq 0$ be such that the assertion (b) is fulfilled and f is lower semicontinuous on $B(\overline{x}, \delta)$. Let $0 < \varepsilon < \delta$ be such that $\varepsilon\omega'(\varepsilon) < 1/c$ and $\alpha := \inf\limits_{B[\overline{x},\varepsilon]} f$ is finite (according to the lower semicontinuity property of f). We fix a real $\underline{c}$ such that

$$\underline{c}\big(\omega(\varepsilon/2) - \omega(\varepsilon/4)\big) > \max\{\varepsilon, f(\overline{x}) - \alpha + 1\}.$$

and a real $c_0 > \max\left\{\underline{c}, \frac{2c}{1 - c\varepsilon\omega'(\varepsilon)}\right\}$. Let $u \in \operatorname{Dom} \partial f \cap B(\overline{x}, \frac{\varepsilon}{4})$ and $u^* \in \partial f(u)$. We define

$$\varphi(x) := f(x) + \langle u^*, u - x \rangle + c_0\, (1 + \|u^*\|)\, \omega(\|x - u\|) \quad \text{for all } x \in X$$

and

$$\overline{\varphi}\,(x) := \left\{ \begin{array}{ll} \varphi(x) & \text{if} \quad x \in B[\overline{x}, \varepsilon] \\ +\infty & \text{if} \quad x \in X \setminus B[\overline{x}, \varepsilon], \end{array} \right. \tag{11.1}$$

so clearly $\overline{\varphi}$ is lower semicontinuous on X (since f is lower semicontinuous on $B[\overline{x}, \varepsilon]$).

Let (ε_n) be a sequence of real numbers which converges to 0 with $0 < \varepsilon_n < \min\left\{\frac{1}{4}, (\frac{\varepsilon}{4})^2\right\}$. For every $n \in \mathbb{N}$, choose $u_n \in X$ such that

$$\overline{\varphi}(u_n) < \inf_X \overline{\varphi} + \varepsilon_n.$$

The above lemma with $\theta = 1$ entails that $u_n \in B(\overline{x}, \frac{3\varepsilon}{4})$ for all $n \in \mathbb{N}$. By the Ekeland variational principle, for each $n \in \mathbb{N}$, there exists $x_n \in X$ such that

$$\|x_n - u_n\| \leq \sqrt{\varepsilon_n},\ \overline{\varphi}(x_n) < \inf_X \overline{\varphi} + \varepsilon_n,\ \overline{\varphi}(x_n) = \inf_{x \in X} \left\{\overline{\varphi}(x) + \sqrt{\varepsilon_n}\|x - x_n\|\right\},$$

then

$$\|x_n - \overline{x}\| < \varepsilon \text{ and } 0 \in \partial(\overline{\varphi} + \sqrt{\varepsilon_n}\| \cdot - x_n\|)(x_n).$$

Since $\varphi = \overline{\varphi}$ on $B[\overline{x}, \varepsilon]$ and $x_n \in B(\overline{x}, \varepsilon)$, we deduce $0 \in \partial(\varphi + \sqrt{\varepsilon_n}\| \cdot - x_n\|)(x_n)$ hence by the fuzzy sum rule for the subdifferential ∂ and the $\sqrt{\varepsilon_n}$-Lipschitz property of the function $\sqrt{\varepsilon_n}\| \cdot \|$, we derive that there are x'_n, x''_n with $\|x'_n - x_n\| < \sqrt{\varepsilon_n}$, $|f(x'_n) - f(x_n)| < \sqrt{\varepsilon_n}$ and $\|x''_n - x_n\| < \sqrt{\varepsilon_n}$ such that

$$0 \in \partial f(x'_n) - u^* + c_0(1 + \|u^*\|)\partial_C(\omega \circ \| \cdot \|)(x''_n - u) + 2\sqrt{\varepsilon_n}\mathbb{B}_{X^*},$$

which furnishes some $x^*_n \in \partial f(x'_n)$ and $y^*_n \in -u^* + c_0(1 + \|u^*\|)\partial_C(\omega \circ \| \cdot \|)(x''_n - u)$ with

$$\|x^*_n + y^*_n\| \leq 2\sqrt{\varepsilon_n}. \tag{11.2}$$

Set $z^*_n := \dfrac{y^*_n + u^*}{c_0(1 + \|u^*\|)} \in \partial_C(\omega \circ \| \cdot \|)(x''_n - u)$ and note that (as easily seen through the subdifferential of the convex function $\omega \circ \| \cdot \|$)

$$\langle z^*_n, x''_n - u\rangle = \omega'_+(\|x''_n - u\|)\|x''_n - u\| \text{ and } \|z^*_n\| \leq \omega'_+(\|x''_n - u\|)\|x''_n - u\|. \tag{11.3}$$

Note also that

$$\|x''_n - u\| \leq \|x''_n - x_n\| + \|x_n - u_n\| + \|u_n - \overline{x}\| + \|\overline{x} - u\| < 2\sqrt{\varepsilon_n} + \frac{3\varepsilon}{4} + \|\overline{x} - u\|$$

and $\frac{3\varepsilon}{4} + \|\overline{x} - u\| < \varepsilon$, hence there exists some integer n_0 such that, for all $n \geq n_0$, $\|x''_n - u\| < \varepsilon$ and $\|z^*_n\| \leq \varepsilon\omega'_+(\varepsilon)$ (keep in mind that ω'_+ is nondecreasing). Fix any $n \geq n_0$. From the equality $y^*_n = -u^* + c_0(1 + \|u^*\|)z^*_n$ we see that

$$\|y^*_n\| \leq \|u^*\| + c_0\varepsilon\omega'(\varepsilon)(1 + \|u^*\|),$$

and from the inequality

$$\|x^*_n\| \leq \|x^*_n + y^*_n\| + \|y^*_n\|$$

and (11.2) we also see that

$$\|x^*_n\| \leq 2\sqrt{\varepsilon_n} + \|u^*\| + c_0\varepsilon\omega'(\varepsilon)(1 + \|u^*\|). \tag{11.4}$$

Further, the assertion (b), with $x^*_1 = x^*_n$ and $x^*_2 = u^*$, ensures that

$$\left\langle u^* - x^*_n, u - x'_n\right\rangle \geq -c\left(1 + \|u^*\| + \|x^*_n\|\right)\omega(\|u - x'_n\|).$$

Putting $\mu := c_0\varepsilon\omega'(\varepsilon)(1 + \|u^*\|)$ and writing by (11.3) and (11.2)

$$\begin{aligned}
&\langle u^* - x^*_n, u - x'_n\rangle \\
&= \langle c_0(1 + \|u^*\|)z^*_n - y^*_n - x^*_n, u - x''_n\rangle + \langle c_0(1 + \|u^*\|)z^*_n - y^*_n - x^*_n, x''_n - x'_n\rangle \\
&\leq -c_0(1 + \|u^*\|)\omega'_+(\|x''_n - u\|)\|x''_n - u\| + 2\sqrt{\varepsilon_n}\|x''_n - u\| + (\mu + 2\sqrt{\varepsilon_n})\|x''_n - x'_n\|,
\end{aligned}$$

it results that

$$\begin{aligned}&-c_0(1+\|u^*\|)\omega(\|x_n''-u\|)+2\sqrt{\varepsilon_n}\|u-x_n''\|+(\mu+2\sqrt{\varepsilon_n})\|x_n''-x_n'\|\\&\geq -c_0(1+\|u^*\|)\omega_+'(\|x_n''-u\|)\|x_n''-u\|+2\sqrt{\varepsilon_n}\|u-x_n''\|+(\mu+2\sqrt{\varepsilon_n})\|x_n''-x_n'\|\\&\geq -c(1+\|u^*\|+\|x_n^*\|)\omega(\|u-x_n'\|).\end{aligned}$$

Noticing that there is some real λ_n between $\|x_n''-u\|$ and $\|x_n'-u\|$ such that

$$\begin{aligned}\big|\,\omega(\|x_n'-u\|)-\omega(\|x_n''-u\|)\big| &= \big|\omega_+'(\lambda_n)(\,\|x_n'-u\|-\|x_n''-u\|)\big|\\&\leq \omega_+'(\lambda_n)\|x_n'-x_n''\|,\end{aligned}$$

with $\gamma_n := 2\sqrt{\varepsilon_n}\|u-x_n''\|+\big(\mu+2\sqrt{\varepsilon_n}+c_0(1+\|u^*\|)\omega_+'(\lambda_n)\big)\|x_n'-x_n''\|$ we obtain

$$\big(c_0(1+\|u^*\|)-c(1+\|u^*\|+\|x_n^*\|)\big)\omega(\|u-x_n'\|)\leq \gamma_n \tag{11.5}$$

along with $\gamma_n\to 0$ as $n\to\infty$ (since $(\lambda_n)_n$ is bounded and $\|x_n'-x_n''\|\leq 2\sqrt{\varepsilon_n}$). Further, the inequality (11.4) implies

$$\begin{aligned}&c_0(1+\|u^*\|)-c(1+\|u^*\|+\|x_n^*\|)\\&\geq c_0(1+\|u^*\|)-c(1+\|u^*\|)-c\big(2\sqrt{\varepsilon_n}+\|u^*\|+c_0(1+\|u^*\|)\varepsilon\omega_+'(\varepsilon)\big)\\&> c_0(1+\|u^*\|)-c(1+\|u^*\|)-c\big(1+\|u^*\|+c_0(1+\|u^*\|)\varepsilon\omega_+'(\varepsilon)\big)\\&=(1+\|u^*\|)\big(c_0-2c-cc_0\varepsilon\omega_+'(\varepsilon)\big).\end{aligned}$$

So by (11.5) we get

$$(1+\|u^*\|)\big(c_0-2c-cc_0\varepsilon\omega_+'(\varepsilon)\big)\omega(\|u-x_n'\|)\leq \gamma_n. \tag{11.6}$$

By the choice of c_0 we have

$$c_0>\frac{2c}{1-c\varepsilon\omega_+'(\varepsilon)}\ \text{ or equivalently }\ c_0-2c-cc_0\varepsilon\omega_+'(\varepsilon)>0,$$

then it follows from (11.6) that

$$\lim_{n\to\infty}x_n'=u,\quad\text{hence}\quad\lim_{n\to\infty}u_n=u.$$

Further, we know that $\varphi(u_n)\leq \inf_{x\in B[\overline{x},\varepsilon']}\varphi(x)+\varepsilon_n$, or equivalently

$$\begin{aligned}&f(u_n)+\langle u^*,u-u_n\rangle+c_0(1+\|u^*\|)\omega(\|u_n-u\|)\\&\leq \inf_{x\in B[\overline{x},\varepsilon]}\{f(x)+\langle u^*,u-x\rangle+c_0(1+\|u^*\|)\omega(\|x-u\|)\}+\varepsilon_n.\end{aligned}$$

Since f is lower semicontinuous and $\lim_{n\to\infty}u_n=u$, the latter inequality ensures that

$$f(u)\leq\liminf_{n\to\infty}f(u_n)\leq \inf_{x\in B[\overline{x},\varepsilon]}\big\{f(x)+\langle u^*,u-x\rangle+c_0(1+\|u^*\|)\omega(\|x-u\|)\big\},$$

and so

$$f(u)\leq f(x)+\langle u^*,u-x\rangle+c_0(1+\|u^*\|)\omega(\|x-u\|),\ \forall x\in B(\overline{x},\frac{\varepsilon}{4}).$$

We then conclude that f is lower $\omega(\cdot)$-regular at $\overline{x}$ by definition in the case ∂ is ∂_C and by the feature preceding the statement of Lemma 11.2 for the other cases ∂_F and ∂_L. □

12. Comments

As said in the introduction, submonotone multimappings have been introduced by J.E. Spingarn in his 1981 paper [72] under the name of "strictly submonotone" multimappings; "submonotone" multimappings in [72] correspond to multimappings called one-sided submonotone in the paper. Spingarn showed in [72] that the Clarke subdifferential of a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is submonotone on $\mathbb{R}^n$ (in the sense of the paper) if and only if it is lower-C^1, that is, for each point $\overline{x} \in \mathbb{R}^n$ there exist a compact topological space T, an open neighborhood V of $\overline{x}$ and a continuous function $\varphi : V \times T \to \mathbb{R}$ such that $D_1\varphi(\cdot,\cdot)$ exists and is continuous on $V \times T$, and such that $f(x) = \max_{t\in T} \varphi(x,t)$ for all $x \in V$. This result also yields that, for the locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, the Clarke subdifferential $\partial_C f$ is submonotone if and only if f is subsmooth on $\mathbb{R}^n$ in the sense of the paper. Independently, H.V. Ngai, D.T. Luc and M. Théra introduced in their 2000 paper [56] the class of approximate convex functions on a normed space as functions satisfying the condition (3.4). Such functions coincide with subsmooth ones by Proposition 3.11. Various important and significative results, in particular the subdifferential determination property, are proved by the authors in [56]. We used the term of "subsmooth functions" mainly because of Proposition 3.1, but also due to the fact that "approximately convex functions" were previously defined as another concept by D.H. Hyers and S.M. Ulam in their 1952 paper [39]. In [36] D.H. Hyers investigated the following question of S.M. Ulam: Given a function f which satisfies the linear functional equation $f(x+y) = f(x) + f(y)$ only approximately, does there exist a linear function g which approximates f? Of course, linear has to be understood as additive therein. Hyers showed in his 1941 paper [36] the following: Given a real $\varepsilon > 0$ and an ε-linear (in fact ε-additive) mapping $f : X \to X'$ between Banach spaces X and X', in the sense $\|f(x+y)-f(x)-f(y)\| \le \varepsilon$, then there exists a linear (in fact additive) mapping $g : X \to X'$ which approximates f with amount ε, that is, $\|f(x) - g(x)\| \le \varepsilon$ for all $x \in X$; further the mapping g is unique and it is continuous on X whenever f is continuous at some point. Similar questions have been also investigated, in the 1945 and 1947 papers [37, 38] of D.H. Hyers and S.M. Ulam and in the 1946 paper of D.G. Bourgin [14], for approximate ε-isometries, that is, mappings T between two metric spaces with $|d(T(x),T(y)) - d(x,y)| < \varepsilon$. All those papers naturally led D.H. Hyers and S.M. Ulam to study the similar problem when additivity or isometry is replaced by convexity. In the paper [39] published in 1952, they declared a function $f : C \to \mathbb{R}$ (defined on a convex set C) to be approximately convex with amount $\varepsilon > 0$ (or ε-convex) whenever $f(tx+(1-t)y) \le tf(x) + (1-t)f(y) + \varepsilon$ for all $x, y \in C$ and $t \in [0,1]$. Hyers and Ulam proved in [39] that for any function $f : U \to \mathbb{R}$ from an open convex set U of $\mathbb{R}^n$ which is approximately convex with amount ε there exists a convex function $g : U \to \mathbb{R}$ such that

$$|f(x) - g(x)| \le \kappa_n \varepsilon \quad \text{for all } x \in U, \tag{12.1}$$

where κ_n is a universal constant depending only on n (an exact value is even given). Other important results on approximately convex functions can be found in the papers by J.W. Green [32] and by S.J. Dilworth, R. Howard and J.W. Roberts [28]. The paper [28] also investigated approximately convex sets, which are defined therein as closed sets S whose distance functions d_S are approximately convex within amount $\varepsilon = 1$. According to both definitions of Hyers and Ulam and of Ngai, Luc and Théra, the concept of (e, δ)-convex functions has been considered by Z. Páles [59] as functions f such that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon t(1-t)\|x-y\| + \delta.$$

Many notions of approximate convexity are presented by J. Makó and Z. Páles [51]. For the use of approximate convex functions on the unit ball in the theory of geometry of Banach spaces we refer to the paper [15] by F. Cabello Sánchez, J.M.F. Castillo and P.L. Papini and to references therein.

Subsmooth sets in Banach spaces have been introduced and largely studied by D. Aussel, A. Daniilidis and L. Thibault in the 2005 paper [8]. The concept has been motivated by the necessity of a first-order viewpoint of the study realized by R.A. Poliquin, R.T. Rockafellar and L. Thibault [63] concerning the (second-order) hypomonotonicity property of the truncated (Clarke) normal cone $N^C(S;\cdot) \cap \mathbb{B}$ of a prox-regular set (see also [21, 42, 68, 20, 82]). The idea in [63] was the development of properties of sets S whose indicator functions are prox-regular at $\overline{x}$ in the sense introduced by R.A. Poliquin and R.T. Rockafellar [62]. Metrically subsmooth sets have been considered by A. Daniilidis and L. Thibault [26].

Complete comments concerning almost all results in Section 2 of preliminaries can be found in [18, 19, 52, 68]. For H. Berens' Proposition 2.4 we follow the proof of J.-B. Hiriart-Urruty [35] for a similar result concerning the farthest distance function.

Proposition 3.1 is due to L. Vesely and L. Zajíček [81] and the proof given in the survey follows the main arguments in [81, Proposition 3.7]. Proposition 3.8 corresponds to the result by L. Zajíček in [85, Corollary 3.3]. Proposition 3.9 and its proof are due to Zajíček [85, Proposition 3.4]. The equivalence in Proposition 3.11 for the subsmoothness property has been observed for locally Lipschitz functions by Zajíček [85, Lemma 3.2]. Proposition 3.14 has been established by H.V. Ngai, D.T. Luc and M. Théra [56] (with a slightly different proof) and Corollary 3.15 has also been observed in [56, Proposition 3.2].

Proposition 4.1 and Theorem 4.5 have been first established by H.V. Ngai, D.T. Luc and M. Théra in [56], when f is in addition lower semicontinuous, with different approaches based on Theorem 3.4 in [56] proving that for such a function f and for each $\varepsilon > 0$ there is some real $\delta > 0$ such that for each $y \in B(\overline{x}, \delta)$ there exists a proper lower semicontinuous convex function $\varphi_y : X \to \mathbb{R} \cup \{+\infty\}$ such that

$$|f(x) - \varphi_y(x)| \leq \varepsilon\|x-y| \quad \text{for all } x \in B(\overline{x}, \delta) \cap \operatorname{dom} f.$$

The proofs in the survey for Proposition 4.1 and Theorem 4.5 follow the approach utilized by L. Thibault and D. Zagrodny [77, Proposition 3.4].

Proposition 4.8 and Lemma 4.10 are probably new. The implication (c)⇒(a) in Theorem 4.11 seems to have been first obtained by L. Zajíček in [85, Lemma 5.4]; the other implications in the theorem are new. When the continuous function f is convex (instead of subsmooth) the equivalences in the theorem have been established for the first time in the 1968 paper [5] by E. Asplund and R.T. Rockafellar. Such a study for the differentiability of a norm (as well as other particular functions) has been previously realized by V.S. Šmulian in his 1939 paper [71] and others.

Example 4.14 and the development therein are due to J.E. Spingarn [72]. As said in Subsection 4.2, Theorem 4.15 has been independently proved by by A. Daniilidis, F. Jules and M. Lassonde [25] and by H.V. Ngai and J. P. Penot [57]: it was previously established for locally Lipschitz functions by A. Daniilidis and P. Georgiev [24]. Proposition 4.19 in its statement is new. The implication (a)$\Rightarrow$(b) in Proposition 4.19 corrects the statement and proof of the implication (i)$\Rightarrow$(ii) in [8, Lemma 4.4] where a gap occurred. The equivalence (iii)$\Leftrightarrow$(iv) in [8, Theorem 4.5] must then be replaced by the equivalence (b)$\Leftrightarrow$(c) in Proposition 4.20. Accordingly, Corollary 4.18 in [8] has also to be replaced by Proposition 6.6 in this survey. The equivalence (d)$\Leftrightarrow$(e) in Proposition 4.20 is due to J.E. Spingarn [72]: the proof here uses ideas in [8].

One-sided submonotone multimappings are called semi-submonotone in [8]: The name "one-sided submonotone" seems to be more appropriate in the sense that it translates the property of the definition. Lemmas 4.23 and 4.24 and Example 4.25 are taken from J.E. Spingarn [72]. Proposition 4.26 and its proof are taken from the paper [8] by D. Aussel, A. Daniliidis and L. Thibault. Proposition 4.31 is due to Spingarn [72]: the proof given here follows the approach in [8].

Except Proposition 5.12 which is new, all the results in Section 5 devoted to subsmooth sets are taken from D. Aussel, A. Daniilidis and L. Thibault [8]. The idea of the use of Krein-Šmulian theorem for the equality $N^C(S;\overline{x}) = N^F(S;\overline{x})$ in Lemma 5.15 is taken from the paper [87] by X.Y. Zheng and K.F. Ng. The subsmooth version of Theorem 6.3 as well as its proof are taken from the paper [8] of D. Aussel, A. Daniilidis and L. Thibault.

Property (e) in Theorem 7.8 has been considered by A.S. Lewis, D.R. Luke and J. Malick in their 2009 paper [50] under the name of *super regularity.* Example 7.10 and Example 7.11 are also due to Lewis, Luke and Malick [50]. Property (c) in Proposition 7.13 with the L-normal cone $N^L(S;\overline{x})$ in place of $N^C(S;\overline{x})$ is used, for an Asplund space X, by A. Jourani under the name of *weak-regularity* of S at $\overline{x}$ in his 2006 paper [45] (see Corollary 4.2 therein; diverse other results related to the weak regularity of sets can be found there). The implication (a)$\Rightarrow$(b) in Proposition 7.12 and its proof are due to H.V. Ngai and J.P. Penot who proved that result with the implication (e)$\Rightarrow$(a) in [58, Theorem 4.10]. The implication (c)$\Rightarrow$(a) in Proposition 7.12 appeared in the implication (c)$\Rightarrow$(e) of [58, Corollary 11]. When for any $\varepsilon > 0$ the distance function d_S satisfies for $\overline{x} \in S$ the property (3.4) in Proposition 3.11 on some ball $B(\overline{x},\delta)$ (resp. the relative similar property (b) in Proposition 7.12 on $S \cap B(\overline{x},\delta)$ for some $\delta > 0$), Ngai and Penot [58] defined the set S as approximately convex (resp. intrinsically approximately convex) at $\overline{x}$. The name of approximately convex set has been utilized much earlier in 1967 in the study of Chebyshev sets by L. Vlasov [83] to refer to a set S for which $\mathrm{Proj}_S x$ is a nonempty convex set for every $x \in X$; note that it was also used with another meaning in [28].

Lemma 8.2 and Theorem 8.4 as well as their proofs are due to A. Jourani and E. Vilches [47].

The truncated normal cone inverse image property in (9.2) as well as the linear inclusion property of subdifferential of distance from inverse image in (9.3) near a point have been introduced by A. Daniilidis and L. Thibault [26]. These concepts are used in [26] to prove the results in Theorem 9.3 and Corollary 9.15. Functions of type (9.1) are used in [4] to illustrate the non-preservation of prox-regularity of sets under usual operations. Lemma 9.16 is in the spirit of Lemma 3.7 of Aussel, Daniilidis and Thibault [8] and of Theorem 3.1 of X.Y. Zheng and K.F. Ng [87]; its proof combines the main ideas of these papers. Theorem 9.17 and Proposition 9.19 are taken from [26]. Under the Asplund property of the Banach space X, it was also proved by Zheng and Ng in [87, Theorem 4.2] that a point $\overline{x}$ is metrically subregular for a system of closed sets $S_1, \ldots, S_m$ of X with $\overline{x} \in S := \bigcap_{i=1}^m S_i$ if and only if there are reals $\gamma > 0$ and $\delta > 0$ such that for all $x \in S \cap B(\overline{x}, \delta)$

$$N^F(S; x) \cap \mathbb{B}_{X^*} \subset N^L(S_1; x) \cap \gamma\mathbb{B}_{X^*} + \cdots + N^L(S_m; x) \cap \gamma\mathbb{B}_{X^*}; \tag{12.2}$$

the equality between the infimum of such constants $\gamma > 0$ and $\mathrm{subreg}_\cap[S_1, \ldots, S_m](\overline{x})$ is also shown in the same theorem in [87]. The condition (12.2) is slightly different from the one in the assertion (II) in Proposition 9.19. Proposition 9.20 is a new result providing a necessary and sufficient condition for the metric subregularity of multimappings with metric subsmooth graphs. In the same framework of Asplund property of X and Y, the sufficiency of the condition in Proposition 9.21 for the metric subregularity of a mulimapping M has been established by Zheng and Ng in [88, Theorem 4.8] under a weakened partial subsmoothness property of $\mathrm{gph}\, M$. In [88] one can also find other sufficient (resp. necessary) conditions for the subregularity of a multimapping with various types of subsmoothness of its graph.

The proof of Proposition 10.1 utilizes ideas in the proof of [3, Theorem 9.1] by S. Adly, F. Nacry and L. Thibault. Proposition 10.3 is new while Propositions 10.4 and 10.5 improve similar results by T. Haddad, J. Noel and L. Thibault [33]. Theorems 10.7 and 10.8 as well as Theorem 10.10 are taken from [78]. Theorem 11.3 is new; its proof follows an approach by F. Bernard and L. Thibault [11] and by I. Kecis and L. Thibault [48] (we also refer to R.A. Poliquin [61] and M. Ivanov and N. Zlateva [43] for the case of primal lower nice functions). A large analysis (including sudifferential determination or integration, subdifferential characterization etc) of subsmooth functions along subgradients in the sense of (10.11) will appear in a forthcoming paper.

Concerning the genericity of differentiability we refer to L. Zajíček [85, page 10] where on can find the following result:

Theorem 12.1 (Zajíček). *Let $f : U \to \mathbb{R}$ be a continuous subsmooth function on a nonempty open set U of an Asplund space X. Then the set of points in U at which f is not Fréchet differentiable is a first category set.*

Several other smallness properties of non-differentiability points of continuous subsmooth functions are also established in the paper [85] for diverse suitable types of Banach spaces.

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