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# ON THE INVERSE OF CONVOLUTION INTEGRAL OPERATORS

HSIANG LIU, SY-MING GUU*, AND CHIN-TZONG PANG

Abstract. For an integrable function $a$ of which the Fourier transform never vanishes, the integral operator $\mathcal{K}_a u := a * u$ is injective and has dense range so that its inverse, denoted by $\mathcal{D}_a$, exists as a densely defined closed operator. In this paper, we study the conditions for $\mathcal{D}_a$ to generate $C_0$-semigroups. By using a theorem of Paley and Weiner, we find the spectra of $\mathcal{K}_a$ and $\mathcal{D}_a$. We establish conditions on the spectrum of $\mathcal{D}_a$ which is sufficient for $\mathcal{D}_a$ to be the generator of a strongly continuous semigroup on the space of almost periodic functions. As an application, we show that for each $n \in \mathbb{N}$, either $(I+\mathcal{D})^n$ or $-(I+\mathcal{D})^n$ generates a $C_0$-semigroup, where $\mathcal{D}$ is the operator of differentiation.

## 1. Introduction

Let $\mathbf{E}$ be a Banach space and $C_{ub}(\mathbb{R}, \mathbf{E})$ be the set of all bounded and uniformly continuous $\mathbf{E}$-valued functions on $\mathbb{R}$. Let $a$ be an integrable scalar-valued function on $\mathbb{R}$ such that its Fourier transform never vanishes. It is known that the integral operator $\mathcal{K}_a$ defined in $C_{ub}(\mathbb{R}, \mathbf{E})$ by $\mathcal{K}_a f := a * f$ is injective and hence has an inverse $\mathcal{D}_a$, where $a * f$ is the convolution integral:

$$(a * f)(t) = \int_{-\infty}^{\infty} a(t-s) f(s) ds.$$

It turns out that under fairly reasonable conditions $\mathcal{D}_a$ generates a strongly continuous semigroups of bounded linear operators on $AP(\mathbf{E})$, the space of all $\mathbf{E}$-valued almost periodic functions on $\mathbb{R}$.

The problem of finding sufficient conditions for the inverse of the generator of a $C_0$-semigroup to be a generator was first posed by DeLaubenfels [5], where the author proved that if $A$ generates a bounded analytic semigroup and $A^{-1}$ exists as a closed operator, then $A^{-1}$ also generates a bounded analytic semigroup. In the same paper, the author proposed open questions: Suppose $A$ is one-to-one with dense range and generates a bounded $C_0$-semigroup. Does $A^{-1}$ generate a bounded

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$C_0$-semigroup? Does $A^{-1}$ generate a $C_0$-semigroup? It was pointed out by Gomilko, Zwart and Tomilov [11] that, in 1966, Komatsu [14] gave a bounded operator

$$C = -I + S, \quad S\{\xi_1, \xi_2, \ldots\} = \{0, \xi_1, \xi_2, \ldots\},$$

generating a contraction $C_0$-semigroup $\left\{e^{tC}\right\}_{t\geq 0}$ in the Banach space $c_0$ with the standard sup-norm. While $C^{-1}$, which exists as a closed operator in $c_0$, does not generate a $C_0$-semigroup. This provides a negative answer for the DeLaubenfels' question. In [11], the authors also presented counterexamples for the Delaubenfels' question in $l^p$ spaces for $p \in (1,2)\cup(2,\infty)$ and in Hilbert spaces. However, in case $A$ is the generator of a bounded $C_0$-semigroup and $A$ is one-to-one with dense range, Gomilko [10] has found sufficient conditions on the resolvent of $A$ for the inverse $A^{-1}$ to generate a bounded $C_0$-semigroup.

In the present article, we first use the vector-valued version of a theorem from Payley and Weiner to find the spectra of $\mathcal{K}_a$ and $\mathcal{D}_a$. Then based on a generalized version of Trotter-Kato approximation theorem, we propose a sufficient condition on the spectrum of $\mathcal{D}_a$ to generate $C_0$-semigroups on the space of all $\mathbf{E}$-valued almost periodic functions on $\mathbb{R}$.

In the year of 1989, Delaubenfels proved a result concerning conditions for polynomials of generators of strongly continuous groups to generate semigroups, which are stated as follows:

**Theorem 1.1.** ( [6], Theorem 11) *Suppose $A$ generates a strongly continuous group, and $p(t) = t^{2n} + q(t)$, where $q$ is a polynomial of degree less than $2n$ . Then $-p(A)$ generates a $C_0$ holomorphic semigroup of angle $\pi/2$ .*

Note that this theorem involves only polynomials with even degrees. In the current paper, we apply our main theorem (Theorem 3.9) to show that for each $n \in \mathbb{N}$ either $(I+\mathcal{D})^n$ or $-(I+\mathcal{D})^n$ generate a $C_0$-semigroup, which extends DeLaubenfels' result for the translation group in $AP(\mathbf{E})$.

This paper is organized as follows. We give an introduction to the problem studied in this paper in Section 1 and collect some known notions and results which will be used throughout the paper in Section 2. The main results are presented in Section 3, where we study the spectra of the convolution operator $\mathcal{K}_a := a * f$ and its inverse $\mathcal{D}_a$. We also establish spectral conditions for $\mathcal{D}_a$ to generate $C_0$-semigroups and bounded analytic semigroups. As an application, it is proved that for each $n \in \mathbb{N}$ either $(I + \mathcal{D})^n$ or $-(I + \mathcal{D})^n$ generate a $C_0$-semigroup in $AP(\mathbf{E})$.

## 2. Notations and preliminaries

Throughout this paper, $\mathbf{E}$ is used to denote a complex Banach space. For a linear operator $T$ on $\mathbf{E}$, $D(T)$ and $R(T)$ stand for its domain and range, respectively, and as usual, $\sigma(T)$, $\rho(T)$, and $R(\lambda, T)$ denote the spectrum, resolvent set and resolvent operator of $T$, respectively. $B(\mathbf{E})$ is the space of all bounded linear operators on $\mathbf{E}$. The notation $C_{ub}(\mathbb{R}, \mathbf{E})$ will stand for the space of all bounded and uniformly continuous $\mathbf{E}$-valued functions on $\mathbb{R}$. The translation group on $C_{ub}(\mathbb{R}, \mathbf{E})$ will be denoted by $\{S(t)\}_{t\geq 0}$, i.e., $S(t)f(s) := f(t + s)$, for all $t, s \in \mathbb{R}$ and $f \in C_{ub}(\mathbb{R}, \mathbf{E})$. The infinitesimal generator $\mathcal{D} := d/dt$ of $\{S(t)\}_{t\geq 0}$ is defined on $D(\mathcal{D}) := C^1_{ub}(\mathbb{R}, \mathbf{E})$.

By $BV(\mathbb{R}, B(\mathbf{E}))$ and $BV(\mathbb{R}, B(C_{ub}(\mathbb{R}, \mathbf{E})))$, we denote the spaces of $B(\mathbf{E})$-valued and $B(C_{ub}(\mathbb{R}, \mathbf{E}))$-valued functions of bounded variation on $\mathbb{R}$, respectively.

$L^1(\mathbb{R})$ is the space of integrable scalar-valued functions on $\mathbb{R}$, and $|a|_1$ is the $L^1$-norm of $a$ whenever $a \in L^1(\mathbb{R})$. In this paper, we will always let $a \in L^1(\mathbb{R})$. $L^\infty(\mathbb{R}, \mathbf{E})$ is the space of essentially bounded $\mathbf{E}$-valued functions on $\mathbb{R}$ and $|f|_\infty$ is the $L^\infty$-norm of $f$ whenever $f \in L^\infty(\mathbb{R})$. If $g \in BV(\mathbb{R})$, the space of functions of bounded variation on $\mathbb{R}$, then $dg$ will stand for the Lebesgue-Stieltjes measure associated with $g$. In particular, we use $\delta_0$ to denote the Lebesgue-Stieltjes measure associated with the Heaviside function:

$$H(t) := \begin{cases} 0, & t < 0 \\ 1, & t \geq 0, \end{cases}$$

which is usually called the Dirac measure at 0.

If $f \in C_{ub}(\mathbb{R}, \mathbf{E})$, then the *Fourier-Carleman transform* $\hat{f}$ of $f$ is defined by

$$\hat{f}(\lambda) = \begin{cases} \displaystyle\int_0^\infty e^{-\lambda t} f(t)dt, & \text{Re}\lambda > 0, \\ \displaystyle -\int_{-\infty}^0 e^{-\lambda t} f(t)dt, & \text{Re}\lambda < 0. \end{cases}$$

Obviously, $\hat{f}(\lambda)$ is holomorphic in $\mathbb{C} \setminus i\mathbb{R}$. $\rho \in \mathbb{R}$ is called a regular point of $f$ if $\hat{f}$ can be analytically extended to some neighborhood $B(i\rho)$ of $i\rho$. The complement in $\mathbb{R}$ of the set of regular points of $f$ is called the *Carleman spectrum* of $f$ and is denoted by $\text{sp}(f)$. It coincides with the set (*Beurling spectrum*)

$$\{\xi \in \mathbb{R} : \forall \epsilon > 0 \; \exists g \in L^1(\mathbb{R}) \text{ such that } \text{supp}\tilde{g} \subset (\xi - \epsilon, \xi + \epsilon), f * g \neq 0\}$$

where

$$\tilde{g}(\rho) := \int_{-\infty}^\infty e^{-i\rho t} g(t)dt.$$

Moreover, the Carleman spectrum $\text{sp}(f)$ of a uniformly continuous and bounded function $f$ coincides with its Arveson spectrum, the set (*Arveson spectrum*) $\sigma(\mathcal{D}^{\mathcal{M}_f})$, where $\mathcal{M}_f \subset C_{ub}(\mathbb{R}, \mathbf{E})$ is the closure of the subspace spanned by all translations of $f$. The readers are referred to [1] for a concise introduction of these notions of spectrum.

For a subset $\Lambda$ of $\mathbb{R}$, we define

$$\Lambda(\mathbf{E}) := \{f \in C_{ub}(\mathbb{R}, \mathbf{E}) : \text{sp}(f) \subset \Lambda\},$$

and in particular, for each $n \in \mathbb{N}$, we let

$$\Lambda_n(\mathbf{E}) := \{f \in C_{ub}(\mathbb{R}, \mathbf{E}) : \text{sp}(f) \subset [-n, n]\}.$$

Moreover, for a subspace $\mathcal{M}$ of $C_{ub}(\mathbb{R}, \mathbf{E})$, we define

$$\Lambda(\mathcal{M}) := \{f \in \mathcal{M} : \text{sp}(f) \subset \Lambda\},$$

and for each $n \in \mathbb{N}$,

$$\Lambda_n(\mathcal{M}) := \{f \in \mathcal{M} : \text{sp}(f) \subset [-n, n]\}.$$

Note that $\Lambda(\mathbf{E})$ and $\Lambda(\mathcal{M})$ are examples of translation invariant subspaces of $C_{ub}(\mathbb{R}, \mathbf{E})$ which satisfy condition (H). We collect some properties we shall need in this paper:

**Proposition 2.1** ([18], p.143; [19], Proposition 0.5, p.23; [15], Theorem 2.1.)**.** *Let $f, g \in L^\infty(\mathbb{R}, \mathbf{E})$, $n \in \mathbb{N}$ such that $g_n \to f$ in the $L^\infty$-norm as $n \to \infty$. Then*

(i) $\mathrm{sp}(f)$ *is closed;*
(ii) $\mathrm{sp}(S(t)f) = \mathrm{sp}(f), \forall t \in \mathbb{R}$*;*
(iii) *If $\alpha \in \mathbb{C} \setminus \{0\}$, then* $\mathrm{sp}(\alpha f) = \mathrm{sp}(f)$*;*
(iv) $\mathrm{sp}(f) = \phi \Longleftrightarrow f \equiv 0$;
(v) $\mathrm{sp}(f)$ *is compact* $\Longleftrightarrow$ *$f$ admits extension to an entire function of exponential growth;*
(vi) *$\varphi * f = f$ for each $\varphi \in L^1(\mathbb{R})$ such that $\tilde{\varphi} \equiv 1$ on a neighborhood of* $\mathrm{sp}(f)$*;*
(vii) $\mathrm{sp}(dc * f) \subset \mathrm{sp}(f)$ *for each* $c \in BV(\mathbb{R}, B(\mathbf{E}))$;
(viii) $\mathrm{sp}(\psi * f) \subset \mathrm{sp}(f) \cap \mathrm{supp}\tilde{\psi}, \forall \psi \in L^1(\mathbb{R})$*;*
(ix) *If* $\mathrm{sp}(g_n) \subset \Lambda, \forall n \in \mathbb{N}$*, then* $\mathrm{sp}(f) \subset \overline{\Lambda}$*.*

The following results, which adopted from [21] (p. 36-37), are the keys of our main theorem.

**Proposition 2.2.** *If $h$ and $f$ are functions in $L^1(\mathbb{R})$ such that $\tilde{h}$ has compact support and $\tilde{f}(\xi) \neq 0$ for each $\xi \in supp\tilde{h}$, then the convolution equation*

$$f * u = h$$

*has a solution $u \in L^1(\mathbb{R})$ which depends continuously on $f$ and $h$, and $\tilde{u} = \tilde{h}/\tilde{f}$.*

**Proposition 2.3.** *Let $f \in L^\infty(\mathbb{R}, \mathbf{E})$ and $b \in L^1(\mathbb{R})$ such that*

$$(b * f)(t) = \int_{-\infty}^{\infty} b(s) f(t-s) ds = 0 \quad \forall t \in \mathbb{R}.$$

*Suppose that the Fourier transformation $\tilde{b}$ never vanishes on an open interval $J$. Then the Carleman transform $\hat{f}$ has analytic continuation throughout $J$, i.e.,* $\mathrm{sp}(f) \cap J = \phi$.

The following definition is adapted from [13].

**Definition 2.4.** A subset $S \subset \mathbb{R}$ is said to be relatively dense if there exists a number $l > 0$ (inclusion length) such that every interval $[a, a+l]$ contains at least one point of $S$. Let $f$ be a continuous function on $\mathbb{R}$ taking values in a Banach space $\mathbf{E}$. $f$ is said to be almost periodic if to every $\epsilon > 0$ there corresponds a relatively dense set $T(\epsilon, f)$ (of $\epsilon$-translations, or $\epsilon$-periods) such that

$$\sup_{t \in \mathbb{R}} \|f(t+\tau) - f(t)\| \leq \epsilon, \forall \tau \in T(\epsilon, f).$$

The space of all almost periodic $\mathbf{E}$-valued functions is denoted by $AP(\mathbf{E})$. It is known that all trigonometric polynomials

$$P(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t}, (a_k \in \mathbf{E}, \lambda_k \in \mathbb{R})$$

are almost periodic, and that

**Theorem 2.5** ([13]). *Let $f$ be an almost periodic function. Then for every $\epsilon > 0$ there exists a trigonometric polynomial $P(t)$ such that*

$$\sup_{t\in\mathbb{R}} \|f(t) - P(t)\| < \epsilon$$

Now, we recall the Trotter-Kato approximation theorem which will be used to prove our main result. The following definition comes from [3].

**Definition 2.6.** Let $(\mathbf{E}_n, \|\cdot\|_n)$, $n = 0, 1, 2, \dots$ be a sequence of Banach spaces and let $P_n : \mathbf{E}_0 \to \mathbf{E}_n$, $n \in \mathbb{N}$, be continuous linear maps such that $\|x\|_0 = \lim_{n\to\infty} \|P_n x\|_n$ for each $x$ in $\mathbf{E}_0$. For each $n \in \mathbb{N}$, let $A_n$ be a linear operator with domain and range in $\mathbf{E}_n$. The limit graph $\mathrm{G}((A_n))$ of the sequence $\{A_n\}$ is defined by

$$\begin{aligned}\mathrm{G}((A_n)) := \{&(x, y) \in \mathbf{E}_0 \times \mathbf{E}_0 : \text{there is a sequence } (x_n) \text{ with } x_n \in D(A_n)\\ &\text{such that } \lim_{n\to\infty} \|x_n - P_n x\|_n = 0 \text{ and } \lim_{n\to\infty} \|A_n x_n - P_n y\|_n = 0\}.\end{aligned}$$

**Theorem 2.7** (see [3]). *Let $(\mathbf{E}_n, \|\cdot\|_n)$, $\{P_n\}$ and $\{A_n\}$ be defined as in Definition 2.6. Let $M \geq 1$ and $\omega \in \mathbb{R}$ and suppose that each $A_n$ generates a $C_0$-semigroup $\{T_n(t)\}$ on $\mathbf{E}_n$ such that*

$$\|T_n(t) x_n\|_n \leq M e^{\omega t} \|x_n\|_n$$

*for all $t \geq 0$, $n \in \mathbb{N}$, and $x_n \in E_n$. Then the following are equivalent.*

(i) *There exists a $C_0$-semigroup $\{T(t)\}_{t\geq 0}$ on $E_0$ such that*

$$\lim_{n\to\infty} \sup_{0\leq s\leq t} \|T_n(s) P_n x - P_n T(s) x\|_n = 0$$

*for each $x \in \mathbf{E}_0$ and $t \geq 0$.*

(ii) *There exists a densely defined linear operator $A$ in $\mathbf{E}_0$ such that $\mathrm{G}(A) \subset \mathrm{G}((A_n))$ and $R(\lambda - A)$ is dense in $E_0$ for some $\lambda > \omega$.*

*Moreover, if (i) is valid and $A_0$ is the generator of $\{T(t)\}$, then $\mathrm{G}(A_0) = \mathrm{G}((A_n))$ and there is a $\lambda_0$ with $\mathrm{Re}\lambda_0 > \omega$ such that*

$$\lim_{n\to\infty} \left\|(\lambda_0 - A_n)^{-1} P_n x - P_n(\lambda_0 - A_0)^{-1} x\right\| = 0$$

*for all $x \in \mathbf{E}_0$.*

## 3. Main results and an application

We state and prove, in this section, our main results and the content will be divided into several subsections.

### 3.1. The convolution operator $\mathcal{K}_a$.

In this subsection we discuss the spectrum of the convolution operator $\mathcal{K}_a$. To this end, we first cite the following result.

**Proposition 3.1** ( [19], Proposition 11.1(iii), p.285.). *Let $F \in BV(\mathbb{R}, B(\mathbf{E}))$. Then the mapping $\mathcal{K}f := dF * f$ is a bounded linear operator on $C_{ub}(\mathbb{R}, \mathbf{E})$ with $\|\mathcal{K}\|_{B(C_{ub}(\mathbb{R},\mathbf{E}))} \leq VarF|_{-\infty}^{\infty}$, and $\mathcal{K}f = \int_{-\infty}^{\infty} d\mathcal{F}(\tau) S(-\tau) f$ for all $f \in C_{ub}(\mathbb{R}, \mathbf{E})$.*

Proposition 3.1 shows that any integrable function $F : \mathbb{R} \to B(\mathbf{E})$ induces via $\mathcal{K}f := F * f$ an operator $\mathcal{K} \in B(C_{ub}(\mathbb{R}, \mathbf{E}))$. In particular, we have the following corollary which follows immediately from Proposition 3.1.

**Corollary 3.2.** *For each $a \in L^1(\mathbb{R})$, let $\mathcal{K}_a$ be the operator defined on $C_{ub}(\mathbb{R}, \mathbf{E})$ by*

$$\mathcal{K}_a f = a * f. \tag{3.1}$$

*Then $\mathcal{K}_a \in B(C_{ub}(\mathbb{R}, \mathbf{E}))$ and can be expressed as*

$$\mathcal{K}_a f = \int_{-\infty}^{\infty} S(-\tau) f d\alpha(\tau), \qquad f \in C_{ub}(\mathbb{R}, \mathbf{E}),$$

*where $\alpha(t) = a(0) + \int_0^t a(\tau)d\tau$ for all $t \in \mathbb{R}$.*

If $a \in L^1(\mathbb{R})$, then the operator $\mathcal{K}_a$ defined by (3.1) will be called the convolution operator induced by $a$. We introduce some properties of $\mathcal{K}_a$ in the following proposition, the proof of which will be omitted.

**Proposition 3.3.** *Let $a \in L^1(\mathbb{R})$, $\mathcal{K}_a$ given by (3.1), and $Z \in B(C_{ub}(\mathbb{R}, \mathbf{E}), \mathbf{E})$.*

(i) *$a * f \in \mathcal{M}$ for all $f \in \mathcal{M}$, i.e., $\mathcal{M}$ is invariant under $\mathcal{K}_a$.*
(ii) *$S(t)\mathcal{K}_a f = \mathcal{K}_a S(t) f$ for all $t \in \mathbb{R}$ and $f \in C_{ub}(\mathbb{R}, \mathbf{E})$, i.e., $\mathcal{K}_a$ commutes with the translation group. In particular, $S(t)\mathcal{K}_a f = \mathcal{K}_a S(t) f$ for all $t \in \mathbb{R}$ and $f \in \mathcal{M}$.*

Consider the set

$$\mathcal{M}_a := \mathcal{K}_a \mathcal{M} := \{a * g : g \in \mathcal{M}\}.$$

which is included by Proposition 3.3(i), in $\mathcal{M}$. We shall show that $\mathcal{M}_a$ is dense in $\mathcal{M}$. To this end, recall that for each $m \in \mathbb{N}$,

$$D(\mathcal{D}^m) := \{f \in D(\mathcal{D}^{m-1}) : \mathcal{D}^{m-1} f \in D(\mathcal{D})\},$$

$$\|f\|_{D(\mathcal{D}^m)} := \sum_{k=0}^{m} \|\mathcal{D}^k f\|_\infty,$$

and

$$D^\infty(\mathcal{D}) := \bigcap_{m \geq 1} D(\mathcal{D}^m).$$

Of interest is the space $A(\mathcal{D})$ of entire vectors with respect to the translation group, i.e.,

$$A(\mathcal{D}) := \Big\{f \in D^\infty(\mathcal{D}) : \overline{\lim_{n\to\infty}} \|\mathcal{D}^n f\|_\infty^{1/n} < \infty\Big\}.$$

This is precisely the space of all functions $f \in C_{ub}(\mathbb{R}, \mathbf{E})$ such that $S(t)f$ extends to an entire function on $\mathbb{C}$ of exponential growth. Since $\{S(t)\}_{t\in\mathbb{R}}$ is a bounded $C_0$-group on $C_{ub}(\mathbb{R}, \mathbf{E})$, it is well-known that $A(\mathcal{D})$ is dense in $C_{ub}(\mathbb{R}, \mathbf{E})$ ( [7], p.81, Excercise 3.12; see also [4], p.134). Set

$$A(\mathcal{M}) := \mathcal{M} \cap A(\mathcal{D})$$

and

$$C(\mathcal{M}) := \{f \in \mathcal{M} : \mathrm{sp}(f) \text{ is compact}\}.$$

Then we see that $A(\mathcal{M})$ is dense in $\mathcal{M}$ and $A(\mathcal{M}) \subset C(\mathcal{M})$, by Proposition 2.5(v). It follows that $C(\mathcal{M})$ is dense in $\mathcal{M}$.

**Proposition 3.4.** *If $a \in L^1(\mathbb{R})$ satisfies that $\tilde{a}(\rho) \neq 0$ for all $\rho \in \mathbb{R}$, then the following assertions hold.*

(i) *$\mathcal{M}_a$ is dense in $\mathcal{M}$. In particular, the operator $\mathcal{K}_a$ has a dense range for all $a \in L^1(\mathbb{R})$.*
(ii) *If $\Lambda$ is a compact subset of $\mathbb{R}$, then $\mathcal{K}_a$ maps $\Lambda(\mathcal{M})$ onto itself.*
(iii) *$\mathcal{M}_a$ is translation-invariant.*

*Proof.* (i) Let $f \in C(\mathcal{M}) \subset \mathcal{M}$ be arbitrary and choose $\varphi_0 \in L^1(\mathbb{R})$ so that $\tilde{\varphi}_0 \in C_0^\infty(\mathbb{R}), \tilde{\varphi}_0 \equiv 1$ on an $\epsilon$-neighborhood of sp$(f)$, $\tilde{\varphi}_0 \equiv 0$ outside $2\epsilon$-neighborhood of sp$(f)$, and $0 \leq \tilde{\varphi}_0 \leq 1$ (see [9], p.236, Exercise 8(b) for an example of such $\varphi_0$). Then by Proposition 2.2, there is $c \in L^1(\mathbb{R})$ such that

$$a * c = \varphi_0$$

Put $g = c * f$. Then $g \in \mathcal{M}$ by Proposition 3.3(i) and sp$(g) \subset$ sp$(f)$ is compact by Proposition 2.5(viii); hence $g \in C(\mathcal{M})$. Thus by Proposition 2.5(vi), we have

$$f = \varphi_0 * f = (a * c) * f = a * g \in \mathcal{M}_a.$$

This shows that

$$C(\mathcal{M}) \subset \mathcal{M}_a$$

which implies the density of $\mathcal{M}_a$ in $\mathcal{M}$.

Replace $C(\mathcal{M})$ by $\Lambda(\mathcal{M})$ in the proof of (i) and one sees that (ii) immediately follows. Moreover, since $\mathcal{M}_a = \mathcal{K}_a\mathcal{M}$, (iii) follows immediately from Proposition 3.3(ii). □

To investigate the spectrum of the convolution operators, we need the following result which is the vector-valued version of a theorem from Paley and Weiner.

**Theorem 3.5** ([16], Proposition 0.3 and Corollary 0.4 of [19], see also [12])**.** *Let $k \in L^1(\mathbb{R})$ . Then for every $f \in C_{ub}(\mathbb{R}, \mathbf{E})$, the convolution equation on the line*

$$u(t) = f(t) + \int_{-\infty}^{\infty} k(s)u(t-s)ds, \qquad t \in \mathbb{R}$$

*has a unique solution $u \in C_{ub}(\mathbb{R}, \mathbf{E})$ if and only if $1 - \tilde{k}(\rho) \neq 0$ for all $\rho \in \mathbb{R}$.*

Since $\lambda \in \rho(\mathcal{K}_a)\backslash\{0\}$ if and only if for every $f \in C_{ub}(\mathbb{R}, \mathbf{E})$, the equation

$$\lambda u - a * u = f \quad \text{or} \quad u - \frac{a}{\lambda} * u = f$$

has a unique solution $u \in C_{ub}(\mathbb{R}, \mathbf{E})$, then it follows by Theorem 3.5 that $\lambda \in \rho(\mathcal{K}_a)\backslash\{0\}$ if and only if $\lambda - \tilde{a}(\rho) \neq 0$ for all $\rho \in \mathbb{R}$. This proves the following theorem.

**Theorem 3.6.** $\sigma(\mathcal{K}_a) = \overline{\{\tilde{a}(\rho) : \rho \in \mathbb{R}\}}$

Note that Lemma 3.6 implies that $0 \in \sigma(\mathcal{K}_a)$ by the Riemann-Lebesgue lemma.

3.2. **The inverse of the convolution operator $\mathcal{K}_a$.** Let $\mathcal{W}(\mathbb{R})$ be the Wiener class, i.e.,

$$\mathcal{W}(\mathbb{R}) = \{a \in L^1(\mathbb{R}) : \tilde{a}(\rho) \neq 0 \text{ for all } \rho \in \mathbb{R}\} \cup \{\delta_0\},$$

where $\delta_0$ is the Dirac measure at the origin. From now on, we will always assume that $a \in \mathcal{W}(\mathbb{R})$ unless stated otherwise. If $a \in \mathcal{W}(\mathbb{R})$ and $f \in L^\infty(\mathbb{R}, \mathbf{E})$, then by Proposition 2.1(iv) and Proposition 2.3, we see that

$$a * f \equiv 0 \text{ implies } f \equiv 0,$$

Thus the convolution operator $\mathcal{K}_a$ has an inverse $\mathcal{D}_a$ , which is obviously a closed operator with domain

$$D(\mathcal{D}_a) = R(\mathcal{K}_a) = \{a * f : f \in C_{ub}(\mathbb{R}, \mathbf{E})\}.$$

It is easy to see that if $f \in D(\mathcal{D}_a) \cap \mathcal{M}$ and $\mathcal{D}_a f \in \mathcal{M}$ if and only if $f \in \mathcal{M}$ and $f = a * g$ for some $g \in \mathcal{M}$. This shows that $\mathcal{D}_a^{\mathcal{M}}$, the part of $\mathcal{D}_a$ in $\mathcal{M}$, has domain $D\left(\mathcal{D}_a^{\mathcal{M}}\right) = \mathcal{M}_a$ which is dense in $\mathcal{M}$.

**Proposition 3.7.** *Let $a \in \mathcal{W}(\mathbb{R})$. Then*

(i) *$R\left(\mathcal{D}_a^{\mathcal{M}}\right) = \mathcal{M}$, i.e., $\mathcal{D}_a^{\mathcal{M}}$ is surjective*;

(ii) *$\mathcal{D}_a|_{\Lambda(\mathcal{M})} = \mathcal{D}_a^{\Lambda(\mathcal{M})}$ is a bounded bijection for all compact $\Lambda \subset \mathbb{R}$*;

(iii) *If $\Lambda_1 \subset \Lambda_2$ are compact sets in $\mathbb{R}$, then $\sigma\left(\mathcal{D}_a^{\Lambda_1(\mathcal{M})}\right) \subset \sigma\left(\mathcal{D}_a^{\Lambda_2(\mathcal{M})}\right)$; in particular, $\sigma\left(\mathcal{D}_a^{\Lambda(\mathcal{M})}\right) \subset \sigma\left(\mathcal{D}_a^{\mathcal{M}}\right)$ for all compact $\Lambda \subset \mathbb{R}$.*

*Proof.* (i) This follows since for each $f \in \mathcal{M}$, $a * f \in \mathcal{M}_a = D(\mathcal{D}_a^{\mathcal{M}})$ and $\mathcal{D}_a(a * f) = f$.

(ii) This follows immediately from Proposition 3.4(ii) and the closed range theorem ( [20], Theorem IV.5.8, p.216).

(iii) If $\lambda \in \rho\left(\mathcal{D}_a^{\Lambda_2(\mathcal{M})}\right)$ and $g \in \Lambda_1(\mathcal{M}) \subset \Lambda_2(\mathcal{M})$, then there is an $f \in \Lambda_2(\mathcal{M})$ such that $\left(\lambda - \mathcal{D}_a^{\Lambda_2(\mathcal{M})}\right) f = g$. By taking Fourier-Carleman transform we have

$$\hat{f} = \left(\lambda - \mathcal{D}_a^{\Lambda_2(\mathcal{M})}\right)^{-1} \hat{g},$$

and it follows that every regular point of $g$ is also a regular point of $f$. Thus, $\mathrm{sp}(f) \subset \mathrm{sp}(g)$ and $f \in \Lambda_1(\mathcal{M})$ which shows that $\rho\left(\mathcal{D}_a^{\Lambda_2(\mathcal{M})}\right) \subset \rho\left(\mathcal{D}_a^{\Lambda_1(\mathcal{M})}\right)$. □

The following result is a trivial consequence of Theorem 3.6 and the general spectral operator theory.

**Theorem 3.8.** $\sigma(\mathcal{D}_a) = \overline{\left\{\frac{1}{\tilde{a}(\rho)} : \rho \in \mathbb{R}\right\}}$.

3.3. **Spectral conditions for $\mathcal{K}_a$ and $\mathcal{D}_a$ to generate semigroups.** Let $x \neq \mathbf{0} \in \mathbf{E}$, $\rho \in \mathbb{R}$ and $\epsilon_{\rho,x}(t) = e^{i\rho t}x$ for all $t \in \mathbb{R}$. Then

$$(\mathcal{K}_a \epsilon_{\rho,x})(t) = \int_{-\infty}^{\infty} e^{i\rho(t-s)} a(s) x ds = e^{i\rho t}\tilde{a}(\rho)x = \tilde{a}(\rho)\epsilon_{\rho,x}(t) \tag{3.2}$$

In view of (3.2), it is clear that

$$e^{t\mathcal{K}_a}\epsilon_{\rho,x} = e^{t\tilde{a}(\rho)}\epsilon_{\rho,x}.$$

Thus, for a trigonometric polynomial $p(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t}$, where $a_k \in \mathbf{E}$, and $\lambda_k \in \mathbb{R}$, we have

$$e^{t\mathcal{K}_a}p(t) = e^{t\tilde{a}(\rho)} \sum_{k=1}^{n} a_k e^{i\lambda_k t} \tag{3.3}$$

Now, we are in the position to prove our main result.

**Theorem 3.9.** *If there is an $\omega \in \mathbb{R}$ such that $\Re\big(\frac{1}{\tilde{a}(\rho)}\big) < \omega$ for all $\rho \in \mathbb{R}$, then the operator $\mathcal{D}_a$ generates a $C_0$-semigroup on $AP(\mathbf{E})$.*

*Proof.* Put $\mathcal{M} = AP(\mathbf{E})$. By Proposition 3.7(ii), we see that the operator $\mathcal{D}_a|_{\Lambda_n(\mathcal{M})} = \mathcal{D}_a^{\Lambda_n(\mathcal{M})}$ is bounded on $\Lambda_n(\mathcal{M})$ and hence each $\mathcal{D}_a^{\Lambda_n(\mathcal{M})}$ generates a uniformly continuous semigroup $\{e^{t\mathcal{D}_a^{\Lambda_n(\mathcal{M})}}\}_{t\geq 0}$ on $\Lambda_n(\mathcal{M})$. Since the growth bound and the spectral bound of $\{e^{t\mathcal{D}_a^{\Lambda_n(\mathcal{M})}}\}_{t\geq 0}$ coincide, $\sigma(\mathcal{D}_a^{\Lambda_n(\mathcal{M})}) \subset \sigma(\mathcal{D}_a) = \overline{\{1/\tilde{a}(\rho) : \rho \in \mathbb{R}\}}$ and $\Re(1/\tilde{a}(\rho)) < \omega$ for all $\rho \in \mathbb{R}$, it follows that for each $n \in \mathbb{N}$ there is an $M_n > 0$

$$\left\| e^{t\mathcal{D}_a^{\Lambda_n(\mathcal{M})}} \right\| \leq M_n e^{\omega t}.$$

However, to apply Trotter-Kato approximation theorem, one shall show that these $M_n$ can be taken uniformly; i.e., there is an $M > 0$ such that

$$\left\| e^{t\mathcal{D}_a^{\Lambda_n(\mathcal{M})}} \right\| \leq M e^{\omega t} \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

To see this, it suffices to show that the collection of semigroups

$$\left\{ e^{\cdot\mathcal{D}_a^{\Lambda_n(\mathcal{M})}} \right\}_{n=1}^{\infty}$$

is uniformly bounded on the interval $[0,1]$. In fact, it follows by (3.3) that for all trigonometric polynomial $p$, we have

$$\left\| e^{t\mathcal{D}_a^{\Lambda_n(\mathcal{M})}} p \right\| = e^{\omega}\|p\| \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [0,1], \tag{3.5}$$

which in turn implies by theorem 2.5, that

$$\left\| e^{t\mathcal{D}_a^{\Lambda_n(\mathcal{M})}} \right\| \leq e^{\omega} \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [0,1], \tag{3.6}$$

This proves (3.4).

Now, choose a rapidly decreasing function $\psi \geq 0$ such that $\int_{\mathbb{R}} \psi(t)dt = 1$ and supp$\tilde{\psi}$, the support of $\tilde{\psi}$, is contained in $[-1,1]$. For each $n \in \mathbb{N}$, let $\psi_n(t) = n\psi(nt)$.

Then $\psi_n$ is a sequence of functions in $L^1(\mathbb{R})$ such that $\int_{\mathbb{R}} \psi_n(t)dt = 1$ and supp $\widetilde{\psi_n} \subset [-n, n]$. For each $n \in \mathbb{N}$, let $P_n : \mathcal{M} \to \Lambda_n(\mathcal{M})$ be defined by

$$P_n f = \psi_n * f, \qquad f \in \mathcal{M}.$$

Note that $P_n f \in \Lambda_n(\mathcal{M})$ by Proposition 2.1(viii) and that the $P_n$ are uniformly bounded by 1 by Proposition 3.1.

Now, let $(f, g) \in \mathrm{G}(\mathcal{D}_a)$ be arbitrary. Then $f \in D(\mathcal{D}_a)$ and $g = \mathcal{D}_a f$. Since for each $n \in \mathbb{N}$,

$$\mathcal{D}_a^{\Lambda_n(\mathcal{M})} P_n f = \mathcal{D}_a^{\Lambda_n(\mathcal{M})} \psi_n * f = \mathcal{D}_a^{\Lambda_n(\mathcal{M})} \psi_n * \mathcal{K}_a g = \mathcal{D}_a^{\Lambda_n(\mathbf{E})} \mathcal{K}_a \psi_n * g = P_n g,$$

then by Definition 2.6 and the closedness of $\mathcal{D}_a^{\mathcal{M}}$

$$\mathrm{G}\left(\mathcal{D}_a^{\mathcal{M}}\right) = \mathrm{G}\left(\left(\mathcal{D}_a^{\Lambda_n(\mathcal{M})}\right)\right).$$

On the other hand, choose $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > \omega$. Hence $\lambda \in \rho(\mathcal{D}_a^{\Lambda_n(\mathcal{M})})$ for all $n \in \mathbb{N}$. Let $g \in \mathcal{M}$ be arbitrary and let $g_n := P_n g \in \Lambda_n(\mathcal{M})$ for all $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, there is an $f_n \in \Lambda_n(\mathcal{M})$ such that

$$\lambda f_n - \mathcal{D}_a^{\Lambda_n(\mathcal{M})} f_n = g_n.$$

Since $g_n \to g$ in sup norm, as $n \to \infty$ (see [8], p.53, Proposition 2.42), then

$$\overline{\left(\lambda - \mathcal{D}_a^{\mathcal{M}}\right) \Lambda^{\infty}(\mathcal{M})} = \mathcal{M},$$

where $\Lambda^{\infty}(\mathcal{M}) := \bigcup_{n \in \mathbb{N}} \Lambda_n(\mathcal{M})$. Therefore, $R(\lambda - \mathcal{D}_a^{\mathcal{M}})$ is dense in $\mathcal{M}$ and it follows by Theorem 2.7 that $\mathcal{D}_a^{\mathcal{M}}$ generates a $C_0$-semigroup $\{\mathcal{T}_a(t)\}_{t \geq 0}$ on $\mathcal{M}$ such that

$$\lim_{n \to \infty} \sup_{0 \leq s \leq t} \left\| \mathcal{T}_a(s)^{\Lambda_n(\mathcal{M})} P_n f - P_n \mathcal{T}_a(s) f \right\|_n = 0$$

for each $f \in \mathcal{M}$ and $t \geq 0$. This completes the proof of the theorem. □

Finally, we give an application of our main result Theorem 3.9. For this we first recall a well-known formula as follows: Let $-A$ be the generator of a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ with negative growth bound on a Banach space. Then

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha - 1} T(t) dt, \tag{3.7}$$

where the integral converges in the uniform operator topology for every $\alpha > 0$ (See [17], p.70). Since the operator $-(I + \mathcal{D})$ generates a uniformly exponentially bounded semigroup $\{e^{-t(I+\mathcal{D})}\}_{t \geq 0}$ in $C_{ub}(\mathbb{R}, \mathbf{E})$, where

$$e^{-t(I+\mathcal{D})} f = e^{-t} f(\cdot - t), \quad f \in C_{ub}(\mathbb{R}, \mathbf{E}),$$

then we see by (3.7) that

$$(I + \mathcal{D})^{-n} f(t) = \frac{1}{\Gamma(n)} \int_0^{\infty} s^{n-1} e^{-s} f(t - s) ds = b_{\alpha} * f(t), \quad t \in \mathbb{R}$$

where $b_n(t) := H(t) e^{-t} t^{n-1} / \Gamma(n)$ and $H$ is the Heaviside function. In our terminologies,

$$\mathcal{K}_{b_n} = (I + \mathcal{D})^{-n} \text{ and } \mathcal{D}_{b_n} = (I + \mathcal{D})^n.$$

Note that for $\alpha > 0$

$$\widetilde{b_n}(\rho) = \frac{\Gamma(n)}{(1+i\rho)^n} \neq 0 \quad \text{for all } \rho \in \mathbb{R},$$

In view of binomial expansion, we have

$$\begin{aligned} \mathrm{Re}\left(\frac{1}{\widetilde{b_n}(\rho)}\right) &= \mathrm{Re}\left(\frac{(1+i\rho)^n}{\Gamma(n)}\right) = \mathrm{Re}\left(\frac{1}{\Gamma(n)}\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}(i\rho)^k\right) \\ &= \frac{1}{\Gamma(n)}\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\begin{pmatrix} n \\ 2k \end{pmatrix}\rho^{2k}, \end{aligned}$$

where $[\cdot]$ denotes the greatest integer function. Hence the set

$$\left\{\mathrm{Re}\left(\frac{1}{\widetilde{b_n}(\rho)}\right) : \rho \in \mathbb{R}\right\}$$

is bounded above whenever $n = 1,\ 4m+2$ or $4m+3$, for $m \in \mathbb{N}\cup\{0\}$ and is bounded below whenever $n = 4m$ or $4m+1$ for $m \in \mathbb{N}$. Therefore, the following result follows immediately from Theorem 3.9.

**Theorem 3.10.** *In $AP(\mathbf{E})$, the operators $(I+\mathcal{D})^n$ generate $C_0$-semigroups, whenever $n = 1,\ 4m+2$ or $4m+3$, for some $m \in \mathbb{N}\cup\{0\}$. Moreover, if $n = 4m$ or $4m+1$ for some $m \in \mathbb{N}$, then $-(I+\mathcal{D})^n$ generates a $C_0$-semigroup in $AP(\mathbf{E})$.*

A related but more general proposition about Theorem 3.10 can be found in the book [2] (see Proposition 8.1.3). However, this proposition is proved in the spaces $L^p(\mathbb{R})(1 \leq p < \infty)$ and $C_0(\mathbb{R})$, the space of continuous functions vanishing at infinity, while Theorem 3.10 concerns the space of almost periodic functions, a subspace of $L^\infty(\mathbb{R})$, and the result seems to be new, to the best knowledge of the authors.

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H. Liu
Department of Information Management, Yuan Ze University, Chung-Li 32003, Taiwan
*E-mail address*: liusean1965@gmail.com

S.-M. Guu
Graduate Institute of Business and Management, College of Management, Chang Gung University and Department of Neurology, Chang Gung Memorial Hospital LinKou, Kwei-Shan District, Taoyuan City, Taiwan
*E-mail address*: iesmguu@gmail.com

C.-T. Pang
Corresponding author. Department of Information Management, and Innovation Center for Big Data and Digital Convergence, Yuan Ze University, Chung-Li 32003, Taiwan
*E-mail address*: imctpang@saturn.yzu.edu.tw