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BARRIER METHOD FOR CONVEX OPTIMIZATION PROBLEM WITHOUT REGULARITY OF CONSTRAINT FUNCTIONS

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Abstract. We consider the convex optimization problem when the objective function may not be smooth and the constraint set is represented by constraint functions that are locally Lipschitz and directionally differentiable, but neither necessarily concave nor continuously differentiable. The obtained results improve and extend those results that have been presented in [Dutta, J., Lalitha, C.S.: Optimality conditions in convex optimization revisited. Optim. Lett. 7(2),221-229 (2013)], and [Dutta, J.: Barrier method in nonsmooth convex optimization without convex representation. Optim. Lett. 9(6), 1177-1185 (2015)], by removing the regularity and continuously differentiable assumptions on the constraint functions from the considering.

1. Introduction and preliminaries

In this paper, we are currently interested in the following convex optimization problem:

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in C, \end{aligned} \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and set C is a feasible convex subset of $\mathbb{R}^n$ given by

$$C = \{x \in \mathbb{R}^n : g_i(x) \geq 0, i = 1, ..., m\},$$

for some constraint functions $g_i : \mathbb{R}^n \to \mathbb{R}, \forall i = 1, ..., m$. The set $\{g_i : i = 1, .., m\}$ is called a representation set of C. If each g_i is a concave function, we say that C has a convex representation. In this case, we know that the Karush-Kuhn-Tucker

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(**KKT**) optimality condition[1] is both necessary and sufficient for a point to be a minimizer, under the Slater condition[2]. Further, let us notice that if C has a convex representation then the dual method for the problem (1.1) of type

$$\sup_{\lambda \in \mathbb{R}^m} \big\{ \inf_x f(x) - \sum_{i=1}^m \lambda_i g_i(x) \big\},$$

are well defined, because $x \mapsto f(x) - \sum_{i=1}^m \lambda_i g_i(x)$ is convex function. In particular, the Lagrangian $x \mapsto f(x) - f^* - \sum_{i=1}^m \lambda_i g_i(x)$, where $f^* = \inf\{f(x) : x \in C\}$, defined from an arbitary **KKT** point $(x^*, \lambda) \in C \times \mathbb{R}^m_+$, is convex and nonnegative on $\mathbb{R}^n$, with x^* being a global minimizer and $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m) \in \mathbb{R}^m_+$. However, if the g_i's are not concave this is not true in general see([7]).

In the case that each representation of C always has one or more g_i's which are not concave functions, we will call the problem of type (1.1) as a convex optimization without convex representation. In this situation, it is naturally interesting to ask that whether under the Slater condition, the **KKT** condition still continue to be both necessary and sufficient. Motivated by this point, Lasserre [7] showed that if f and g_i's are differentiable functions and under an additional suitable condition, so-called the non-degeneracy condition, that is, for all $i = 1, .., m, \nabla g_i(x) \neq 0, \forall x \in C$ with $g_i(x) = 0$, then the **KKT** condition is both necessary and sufficient.

Later on, Lasserre [8] considered the optimality conditions for convex optimization problem without convex representation and focussed on the algorithmic issues of such type problem, when f and g_i's are continuously differentiable functions, by using the following so-called a *barrier (or a log-barrier)* function: for each $\mu > 0$, a barrier function $\varphi_\mu : C \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\varphi_\mu(x) = \begin{cases} f(x) - \mu \ \sum_{i=1}^m \ln(g_i(x)) & \text{if } \ x \in \mathcal{S}, \\ +\infty & \text{otherwise}, \end{cases} \tag{1.2}$$

where $\mathcal{S} := \bigcap_{i=1}^m \{x : g_i(x) > 0\}$. Using this function, in that paper, he showed that even though the considered constraint set C does not have a convex representation, the barrier method can be used for finding a solution of the problem (1.1), if the data of the considered problem is smooth and satisfied the Slater condition and non-degeneracy condition.

Recently, in [5], Dutta continuously focussed on the algorithmic issues of convex optimization problem without convex representation, by removing the continuously differentiability of the objective function and presented the following theorem.

Theorem 1.1. *Consider the problem (1.1), where the constraint set C satisfies the Slater condition and non-degeneracy condition, but may not have a convex representation. If C is a compact set and each g_i is continuously differentiable then for each*

[1]A point $x \in C$ is a **KKT** point if there exist $\lambda_i \geq 0$ for all $i = 1, ..., m$ such that

$$\lambda_i g_i(x) = 0 \quad \text{and} \quad \nabla f(x) + \sum_{i=1}^m \lambda_i \nabla g_i(x) = 0,$$

where $\nabla f(x)$ and $\nabla g_i(x)$ are denoted for the *gradient vectors* of the function f and g_i at x, respectively.

[2]The Slater condition holds for C if there exists $x \in C$ such that $g_i(x) > 0$ for all $i = 1, .., m$.

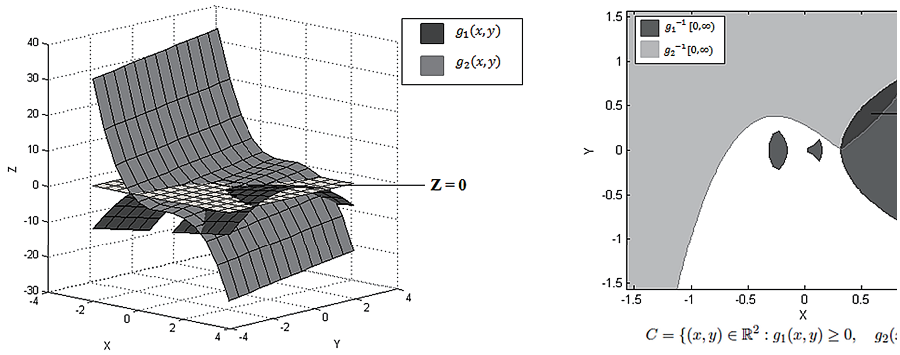

FIGURE 1. Illustration of example 1.2.

$\mu > 0$ there exists a global minimizer x_μ of the φ_μ in the interior of C. Moreover, every accumulation point x^* of $\{x_\mu\}$ with $\mu \to 0$ is a global minimizer of f.

Recall that a function $g : \mathbb{R}^n \to \mathbb{R}$ is called a *locally Lipschitz* at a point $x \in \mathbb{R}^n$ if there exist a real number $L_x > 0$ and $\epsilon_x > 0$ such that

$$|g(y) - g(z)| \leq L_x \|y - z\| \quad \text{for all} \quad y, z \in B(x, \epsilon_x),$$

where $B(x, \epsilon_x) = \{w \in \mathbb{R}^n : \|x - w\| < \varepsilon_x\}$. It is well known that each continuously differentiable function is a locally Lipschitz function. Inspried by this relation, we are convinced to improve the Theorem 1.1, by replacing the continuously differentiability of the representative functions by the locally Lipschitz assumption.

Example 1.2. Let C be a convex subset of $\mathbb{R}^2$, which is represented by

$$C = \{(x, y) \in \mathbb{R}^2 : g_1(x, y) \geq 0, \quad g_2(x, y) \geq 0\},$$

where

$$g_1(x, y) = \begin{cases} x^2 \sin(1/x) - y^2 & \text{if} \quad x \neq 0 \\ -y^2 & \text{if} \quad x = 0, \end{cases}$$

and

$$g_2(x, y) = y - \left| x - \frac{1}{\pi} \right| - \left(x - \frac{1}{\pi} \right)^3,$$

for all $(x, y) \in \mathbb{R}^2$. Ones can see that g_1 and g_2 are locally Lipschitz functions and we notice that a convex set C may be represented by different choices of the representative functions (not necessary convex or smooth). So this attractive feature is not specific to representations of C with smooth functions. See figure 1.

Next, we will recall some important concepts that will be used in this work.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. A *directional derivative* of f at x in the feasible direction h, denoted by $f'(x, h)$, is defined by

$$f'(x, h) = \lim_{t \downarrow 0} \frac{f(x + th) - f(x)}{t}.$$

We say that the directional derivative of f in the direction h at x exists if the above limit exists. It is well known that if f is a convex function, then the directional derivative $f'(x,h)$ exists in every direction $h \in \mathbb{R}^n$.

Now, let us focus on the class of locally Lipschitz functions, which is the main target of this paper. We start by recalling the concepts of generalized directional derivative and generalized subdifferential of a locally Lipschitz function in the sense of Clarke [3].

If $g : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function at $x \in \mathbb{R}^n$, then the *generalized directional derivative* or *Clarke derivative* of g at x in the direction $h \in \mathbb{R}^n$ is defined by

$$g^\circ(x,h) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{g(y+th) - g(y)}{t}.$$

Remind that under the locally Lipschitz assumption of g, the Clarke derivative always exists. For a locally Lipschitz function g and $x \in \mathbb{R}^n$, the *Clarke subdifferential* of g at x, denoted by $\partial^\circ g(x)$, is defined by

$$\partial^\circ g(x) = \{v \in \mathbb{R}^n : g^\circ(x,h) \geq \langle v,h\rangle, \forall h \in \mathbb{R}^n\}.$$

It is well known that, the Clarke subdifferential $\partial^\circ g(x)$ is a nonempty, convex and compact set. Furthermore, for each $h \in \mathbb{R}^n$, we know that $g^\circ(x,h) = \max\{\langle v,h\rangle : v \in \partial^\circ g(x)\}$. For more information on these concepts, the readers may consult [1–3, 9].

According to the definition of Clarke derivative, for each $x \in \mathbb{R}^n$, it is worth noting that $g^\circ(x,h) \geq g'(x,h)$, for all $h \in \mathbb{R}^n$, and under locally Lipschitz setting we also know that $g^\circ(x,h)$ is a positively homogeneous, subadditive and upper semicontinuous, with respect to h. Further, if g is a locally Lipschitz function which is also directionally differentiable at $x \in \mathbb{R}^n$ and $g'(x,h) = g^\circ(x,h)$ for all $h \in \mathbb{R}^n$, then g is said to be *regular in the sense of Clarke* at $x \in \mathbb{R}^n$. Note that if g is a convex function, then g is a locally Lipschitz function and regular in the sense of Clarke.

Under the framework of locally Lipschitz constraint functions, Dutta and Lalitha [4] considered the optimality conditions for the convex optimization problem (1.1), by assuming that each constraint function is regular in the sense of Clarke. They also introduced the following nonsmooth degeneracy type condition:

Condition NDC: Let C be a constraint set of problem (1.1). We say that the *non-degeneracy condition* holds if for all $i = 1, .., m$,

$$0 \notin \partial^\circ g_i(x), \quad \text{whenever} \quad x \in C \quad \text{and} \quad g_i(x) = 0.$$

In that work, Dutta and Lalitha [4] showed that if both the Slater condition and **NDC**-condition hold then the **KKT** condition is both necessary and sufficient. Further, the authors gave a noticeable question that whether the assumption of regularity of the constraint functions can be removed.

Remark 1.3. From the Example 1.2, we can check that the function g_1 is a locally Lipschitz function which is directionally differentiable but not regular in the sense of Clarke. Indeed, one can see that $(-1,0) \in \partial^\circ g_1((0,0))$ and this gives $g^\circ((0,0),(h_1,h_2)) \geq \langle(-1,0),(h_1,h_2)\rangle = -h_1$, for all direction $(h_1,h_2) \in \mathbb{R}^2$. This implies that there is a direction such that the Clarke derivative of g_1 at $(0,0)$ is a positive real number. However, since g_1 is differentiable at $(0,0)$ and its derivative is $(0,0)$, so we have $g_1'((0,0),(h_1,h_2)) = \langle (0,0),(h_1,h_2)\rangle = 0$, for all direction $(h_1,h_2) \in \mathbb{R}^2$. This shows that g_1 is not a regular in the sense of Clarke at $(0,0)$.

Motivated by above literature, in this paper, we will give an affirmative answer to the question which was asked in [4], by removing the regularity assumption of the constraint functions. Meanwhile, our obtained result shows that Dutta's framework [5] still works when the convex constraint set is described by locally Lipschitz functions which is directionally differentiable, but not necessarily continuously differentiable functions. To do this, in order to considering the case that function is not regular, we need the following important tool which can be found in [6].

Lemma 1.4 ([6])**.** *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function which is directionally differentiable over the open set $U \subset \mathbb{R}^n$. Then at each point $x \in U$ and in each direction $v \in \mathbb{R}^n$ we have*

$$g^\circ(x,v) = \limsup_{y\to x} g'(y,v).$$

2. Main result

From now on, we will always assume that the constraint set C in the problem (1.1) is represented by a set of locally Lipschitz functions which are directionally differentiable. We begin by providing an optimality condition for the problem (1.1).

Lemma 2.1. *Let C be defined as in Problem (1.1) and $x^* \in C$. Assume that there exist scalars $\lambda_i \geq 0$ such that for all $x \in C$ we have*

(a) $f'(x^*, x - x^*) \geq \sum_{i=1}^m \ \lambda_i g_i^\circ(x^*, x - x^*)$;
(b) $\lambda_i g_i(x^*) = 0$, *for all* $i \in \{1,2,...,m\}$.

Then x^* is a global minimizer of the problem (1.1).

Proof. Firstly, by the convexity of C, we observe that

$$x^* + \alpha(x - x^*) = \alpha x + (1-\alpha)x^* \in C,$$

for all $x \in C$ and $\alpha \in (0,1)$. Thus,

$$g_i(x^* + \alpha(x - x^*)) \ = \ g_i(\alpha x + (1-\alpha)x^*) \ \geq 0,$$

for each $i \in \{1,...,m\}$. Consequently, by the assumption (b), we can deduce that

$$\lambda_i \left[\frac{g_i(x^* + \alpha(x - x^*)) - g_i(x^*)}{\alpha}\right] \ \geq 0,$$

for each $i \in \{1,...,m\}$. Letting $\alpha \to 0$, in view of a relationship between directional derivative and Clarke derivative of g_i at x^* in direction $x - x^*$, we obtain

$$\lambda_i g_i^\circ(x^*, x - x^*) \geq 0,$$

for each $i \in \{1,...,m\}$. Subsequently, by the assumption (a), we have

$$f'(x^*, x - x^*) \geq \sum_{i=1}^m \ \lambda_i g_i^\circ(x^*, x - x^*) \geq 0.$$

Hence, since f is a convex function, we can conclude that x^* is a global minimizer of f over C. This completes the proof.

$\square$

The next result presents an implication of **NDC**-condition.

Lemma 2.2. *Let C be defined as in Problem (1.1). If **NDC**-condition is satisfied then for each $x \in C \setminus \mathcal{S}$ and $y \in intC$, we have*

$$g_i^\circ(x, y - x) > 0, \quad \forall i \in I(x),$$

where $I(x) = \{i \in \{1, .., m\} : g_i(x) = 0\}$, and $intC$ is denoted for the set of all interior points of C.

Proof. Suppose, on the contrary, that there exists $i \in I(x)$ such that $g_i^\circ(x, y-x) \leq 0$.
Note that, since $y \in intC$, we can find a $\delta > 0$ such that $y + u \in C$ for all $u \in B(0, \delta)$. Let us set $w := y + u$. Subsequently, by convexity of C, we have

$$x + \alpha(w - x) = \alpha w + (1 - \alpha)x \in C,$$

for all $\alpha \in (0, 1)$. This gives,

$$\frac{g_i(x + \alpha(w - x)) - g_i(x)}{\alpha} \geq 0,$$

for all $\alpha \in (0, 1)$. So, by a relation between the directional derivative and Clarke derivative, we have

$$g_i^\circ(x, w - x) \geq 0.$$

Then, by the subadditivity property of $g_i^\circ(x, \cdot)$, we get

$$0 \leq g_i^\circ(x, y + u - x) \leq g_i^\circ(x, y - x) + g_i^\circ(x, u) \leq g_i^\circ(x, u),$$

for all $u \in B(0, \delta)$. Using this one together with the positive homogeneity of $u \mapsto g_i(x, u)$, we would have $0 \in \partial^\circ g_i(x)$. This contradicts to the **NDC**-condition, and the proof is completed. $\square$

The following result relates to a property of the barrier function φ_μ, which was defined by (1.2). Note that, in fact, the following result was presented in [8], when each g_i is a continuously differentiable function. While, here, we are pointing that the continuity condition of each g_i is sufficient.

Lemma 2.3. *Let C be defined as in Problem (1.1). Assume that C is a compact set and the Slater condition holds. Then, for every $\mu > 0$ the barrier function φ_μ has a minimizer, which is an element of $intC$.*

Proof. Let $\mu > 0$ be given and φ_μ be the corresponding barrier function, which was defined in (1.2). Firstly, let us notice that by the Slater condition, we can guarantee that the function φ_μ is a proper function, that is $\mathrm{Dom}(\varphi_\mu) \neq \emptyset$. Next, we will show that φ_μ is a continuous extended real valued function on C. That is we have to prove that for each sequence $\{x_k\}$ in $\mathcal{S}$ such that $x_k \to x$ for some $x \in C \setminus \mathcal{S}$, it holds $\varphi_\mu(x_k) \to \infty$.

Note that, since $x \in C \setminus \mathcal{S}$ and each g_i is a continuous function, there is an index $j \in \{1, 2, ..., m\}$ such that $g_j(x_k) \to 0$ as $k \to \infty$. Focusing on this such index j, we now consider

$$\begin{aligned} \varphi_\mu(x_k) =& f(x_k) - \mu \sum_{i=1}^{m} \ln(g_i(x_k)) \\ \geq& f^* - \mu(m-1)\ln K - \mu \ln(g_j(x_k)), \end{aligned} \tag{2.1}$$

where f^* is the minimum of f on C, and all the g_i's are bounded above on C by K. Subsequently, the inequality (2.1) implies that $\varphi_\mu(x_k) \to \infty$ as $k \to \infty$. This asserts that φ_μ is a continuous extended real valued function on C. Then, since C is a compact set, we know that φ_μ must has a minimizer on $\mathcal{S}$. Finally, since $\mathcal{S} \subseteq intC$, the minimizer must be an element of $intC$. □

According to the Lemma 2.3, under the Slater condition, we can construct a sequence $\{x_\mu\} \subset intC$ by

$$x_\mu = \arg\min_{x \in C} \varphi_\mu(x), \tag{2.2}$$

for each positive real number μ. In our next theorem, which is our main result, under the additional condition as **NDC**-condition, we show that every accumulation point of $\{x_\mu\}$ as $\mu \to 0$ is a global minimizer of the problem (1.1).

Theorem 2.4. *Let C in the Problem (1.1) be a compact set and $\{x_\mu\}$ defined as in (2.2). If the Slater condition and **NDC**-condition hold, then every accumulation point of $\{x_\mu\}$ as $\mu \to 0$ is a global minimizer of the problem (1.1).*

Proof. Notice that, by carefully reading the proof of Lemma 2.3, one can see that $\{x_\mu\}$ is a sequence in $\mathcal{S}$. Moreover, let us observe that φ_μ is directionally differentiable at each $x \in \mathcal{S}$. Thus, from the basic necessary optimality conditions, we must have

$$\varphi'_\mu(x_\mu, v) \geq 0,$$

for all $v \in \mathbb{R}^n$. Subsequently, by applying the chain rule for the directionally differentiable function, we have

$$f'(x_\mu, v) \geq \sum_{i=1}^{m} \frac{\mu}{g_i(x_\mu)} g'_i(x_\mu, v), \tag{2.3}$$

for all $v \in \mathbb{R}^n$.

Now, let x^* be an accumulation point of $\{x_\mu\}$ as $\mu \to 0$. It follows that there is a null sequence $\{\mu_k\} \subset (0,1)$ such that $x_{\mu_k} \to x^*$ as $k \to \infty$. To complete the proof, we shall show that x^* is a global minimizer of f. We will now consider the following two possible cases.

Case(I) $g_i(x^*) > 0$ for each $i \in \{1, ...m\}$.

Case(II) $g_i(x^*) = 0$ for some $i \in \{1, ...m\}$.

Let us discuss Case(I). Since $x_{\mu_k} \to x^*$ as $k \to \infty$, by the continuity of each g_i, for $i \in \{1, 2, ..., m\}$, we have $g_i(x_{\mu_k}) \to g_i(x^*) > 0$. Then, for each $i \in \{1, 2, ..., m\}$, we have

$$\lim_{k\to\infty} \frac{\mu_k}{g_i(x_{\mu_k})} = 0,$$

since $\mu_k \to 0$ as $k \to \infty$. Moreover, by Lemma 1.4, we know that $\{g_i'(x_{\mu_k}, x-x^*)\}_{k=1}^{\infty}$ is a bounded sequence, for each $i \in \{1, 2, ..., m\}$, and these imply that

$$\lim_{k\to\infty} \sum_{i=1}^{m} \frac{\mu_k}{g_i(x_{\mu_k})} g_i'(x_{\mu_k}, x - x^*) = 0. \tag{2.4}$$

Again, by using Lemma 1.4, we also have

$$\limsup_{k\to\infty} f'(x_{\mu_k}, v) = f^\circ(x^*, v) = f'(x^*, v), \tag{2.5}$$

since f is a convex function. Thus, by considering $v = x - x^*$ in (2.3), we see that (2.4) and (2.5) give

$$f'(x^*, x - x^*) \geq 0,$$

for all $x \in C$. This means that x^* is a global minimizer of the problem (1.1).

Next we consider Case(II). Let us pick an element $x_0 \in intC$. By Lemma 2.2, we know that

$$g_i^\circ(x^*, x_0 - x^*) > 0, \tag{2.6}$$

for all $i \in I(x^*)$. Subsequently, since $g_i^\circ(x^*, x_0 - x^*)$ is the superior limit of $\{g_i'(x_{\mu_k}, x_0 - x^*)\}_{k=1}^{\infty}$, we may assume without loss of generality (passing to a subsequence if necessary) that $g_i'(x_{\mu_k}, x_0 - x^*) \to g_i^\circ(x^*, x_0 - x^*)$ and $g_i'(x_{\mu_k}, x_0 - x^*) > 0$, for all $k \in \mathbb{N}$ and $i \in I(x^*)$.

Now, by considering $v = x_0 - x^*$, we rewritten (2.3) as

$$\begin{aligned} f'(x_{\mu_k}, x_0 - x^*) \geq & \sum_{i \notin I(x^*)} \frac{\mu_k}{g_i(x_{\mu_k})} g_i'(x_{\mu_k}, x_0 - x^*) \\ & + \sum_{i \in I(x^*)} \frac{\mu_k}{g_i(x_{\mu_k})} g_i'(x_{\mu_k}, x_0 - x^*), \end{aligned} \tag{2.7}$$

for each $k \in \mathbb{N}$. For the sake of simplicity, for each $k \in \mathbb{N}$, let us put

$$B_k := \sum_{i \notin I(x^*)} \frac{\mu_k}{g_i(x_{\mu_k})} g_i'(x_{\mu_k}, x_0 - x^*).$$

Notice that, by following the lines as proving the Case(I), we know that $B_k \to 0$ as $k \to \infty$. Subsequently, in view of (2.7), we can choose a positive real number B and its corresponding natural number $k_0 \in \mathbb{N}$ such that

$$f'(x_{\mu_k}, x_0 - x^*) + B \geq \sum_{i \in I(x^*)} \frac{\mu_k}{g_i(x_{\mu_k})} g_i'(x_{\mu_k}, x_0 - x^*), \tag{2.8}$$

for all $k \geq k_0$. Thus, by using Lemma 1.4, we have

$$\begin{aligned} f'(x^*, x_0 - x^*) + B &\geq \limsup_{k\to\infty} \sum_{i\in I(x^*)} \frac{\mu_k}{g_i(x_{\mu_k})} g_i'(x_{\mu_k}, x_0 - x^*) \\ &\geq \limsup_{k\to\infty} \frac{\mu_k}{g_i(x_{\mu_k})} g_i'(x_{\mu_k}, x_0 - x^*), \end{aligned}$$

for all $i \in I(x^*)$. This implies that the sequence $\left\{\frac{\mu_k}{g_i(x_{\mu_k})}\right\}_{k=1}^{\infty}$ is a bounded sequence, for each $i \in I(x^*)$. Invoking this fact, it allows us to define a function $\lambda : \{1, 2, ..., m\} \to [0, \infty)$ by

$$\lambda_i = \begin{cases} \lim\limits_{k\to\infty} \frac{\mu_k}{g_i(x_{\mu_k})}, & \text{if } i \in I(x^*); \\ 0, & \text{otherwise.} \end{cases}$$

Then it immediately follows that $\lambda_i g_i(x^*) = 0$, for all $i \in \{1, 2, ..., m\}$.

Next, let $x \in C$ be arbitrarity given. In view of (2.3), with $v = x - x^*$, and by using Lemma 1.4 we obtain

$$f'(x^*, x - x^*) \geq \limsup_{k\to\infty} \sum_{i=1}^{m} \frac{\mu_k}{g_i(x_{\mu_k})} g_i'(x_{\mu_k}, x - x^*) \tag{2.9}$$

$$= \sum_{i=1}^{m} \lambda_i g_i^{\circ}(x^*, x - x^*) \tag{2.10}$$

Hence, by using Lemma 2.1, we can conclude that x^* is a global minimizer of the problem (1.1). This completes the proof.

$\square$

Remark 2.5. (a) Theorem 2.4 improves a presented result in [4], by removing the regularity assumption from the considered constraint functions.

(b) Since Clarke subdifferential of a smooth function will consists only the gradient vector, so in this situation the **NDC**-condition is coincided with the non-degeneracy condition in the sense of Lasserre [7]. Further, since every continuously differentiable function is a locally Lipschitz and regular function, we can deduce that Theorem 2.4 contains Theorem 1.1 as a special case.

3. Conclusion

In this work, we consider the convex optimization problem when the objective function may not be smooth and the constraint functions are just locally Lipschitz and directionally differentiable, and need not be continuously differentiable or regular. It is worth to pointing that, we are giving an affirmative answer to a problem that was proposed in [4]. Moreover, after carefully considering, one may observe that the Slater condition is used only for guaranteeing that the barrier function φ_μ is a proper function and its minimizer is an element of interior of the constraint set, see Lemma 2.3. This may rise an interesting following question: can we define a barrier function by using a condition that weaker than the Slater condition such that it is still a proper function and its minimizer belongs to the set of all interior points of the constraint set? In order to develop this research area, this question should be considered in the future works.

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