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HÖLDER CONTINUITY OF SOLUTION MAPS TO PARAMETRIC PRIMAL AND DUAL WEAK GENERALIZED KY FAN INEQUALITIES

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Abstract. In this paper we consider primal and dual parametric vector Ky Fan inequality in metric linear spaces. Using the strong convexity and Hölder continuity for vector valued mappings, without strong monotonicity assumption, we establish the sufficient conditions for the solution mappings to these problems to be Hölder continuous around the reference point, when the solution of these problems is not unique. Many examples are provided to illustrate that the imposed assumptions are essential and our results are different from the existed ones in the literature.

1. Introduction

The Ky Fan inequality has been intensively studied in [16], where the author proposed the existence of a solution for a class of bifunctions. The mathematical formulation of this problem includes various important problems related to optimization, namely, optimization problems, variational inequality, Nash equilibrium problems, fixed-point and coincidence-point problems, etc. It is known that the general form of this probem was investigated earlier by Nikaido and Isoda [26] in 1955. At that time, this problem was carried out as an auxiliary problem to study the existence conditions for Nash equilibrium points in non-cooperative games. In the existence theory of the problem, the key contributions were made by Ky Fan, whose new existence results contained the original techniques which became a basis for most further existence theorems [16]. Hence, this model was also called a Ky Fan inequality.

The Ky Fan inequality has been extended and generalized to vector-valued mappings. This formulation is also known as the generalized Ky Fan inequality. In the literature, existence results for Ky Fan inequalities and the various types of generalized Ky Fan inequalities have been investigated intensively; see [7, 8, 24] and the references therein, where the generalized Ky Fan inequalities are called generalized systems or vector equilibrium problems.

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It is well known that stability and sensitivity analysis of solution mappings for (generalized) Ky Fan inequalities is an important topic for (vector) optimization theory and its applications. There have been a large number of contributions to kinds of semicontinuity, continuity and Hölder continuity of solutions with respect to parameters. Anh and Khanh [3] introduced some definitions related to semicontinuity of multivalued mappings and discussed various kinds of semicontinuity. They studied the sufficient conditions for the solution sets of parametric multivalued symmetric vector quasiequilibrium problems to have these properties. Huang et al. [20] further considered a class of parametric implicit vector equilibrium problems in Hausdorff topological vector spaces. They established sufficient conditions for the upper semicontinuity and lower semicontinuity of the solution set mapping. Furthermore, the continuity (or Hausdorff continuity) of solution maps was studied in [4, 21]. Kimura and Yao [21] investigated the existence of solutions and continuity of for the parametric vector quasi-equilibrium problem. Anh and Khanh [4] considered a parametric vector quasiequilibrium problem in topological vector spaces and given some sufficient conditions for solution maps to be lower and Hausdorff lower semicontinuous, upper semicontinuous and continuous. Using scalarization approaches, Chen and Li [13] obtained new results on the lower semicontinuity and upper semicontinuity properties of the Pareto solutions to a parametric vector variational inequality with a polyhedral constraint set. For related results on this field one can also refer to [9–11, 14, 17, 28].

However, this continuity was obtained under assumptions related to strong monotonicity or strong pseudomonotonicity. Then the results have been extended to quasiequilibrium problems. The case of non-unique solutions was dealt with recently in Anh et al. [5], avoiding assumptions related to strong monotonicity, but Hölder continuity was obtained only for approximate solutions which are as close as we want to exact solutions. In [6], Anh et al. established the sufficient conditions for Hölder continuity of solution mappings of Ky Fan inequalities without using the strong monotonicity assumptions. The technique is based on strong convexity and strong convex-likeness. In [**?**], Li and Li established the Hölder continuity of a set-valued solution mapping to a parametric weak generalized Fan Ky inequality. It is worth pointing that their method is based on a scalarization representation of the solution mapping for a parametric Ky Fan inequality and the Hölder strong monotonicity assumptions. To the best of our knowledge, there is no existing result on Hölder continuity of primal and dual Ky Fan inequalities for the generalized Ky Fan inequalities without the virtue of Hölder strong monotonicity assumptions.

In this paper, we will first present the new concepts of the strong convexity and Hölder continuity for vector-valued mappings. Further, without the virtue of strong monotonicity assumptions, we next establish also certain sufficient conditions for Hölder continuity of both solution maps of parametric primal and dual weak generalized Ky Fan inequalities, when the solution set may not be singleton. Our technique is based on the strong convexity for vector-valued maps. Examples are provided for cases where our results are applicable while existing ones under the Hölder strong monotonicity assumptions are out of use.

2. Preliminaries

In the sequel, $\|\cdot\|$ and $d(\cdot,\cdot)$ denote the norm and metric in any normed space and metric space, respectively (by the context, no confusion occurs). $B(0,\delta)$ is the closed ball with center $0 \in X$ and radius $\delta \geq 0$. We use $\text{int}A, \text{conv}(A)$, and $\text{diam}A := \sup_{x,z\in A} d(x,z)$ for the interior, the convex hull, and the diameter, respectively, of a subset A. For a set-valued map $G : X \to 2^Y$, $\text{gr}\, G := \{(x,y) \in X \times Y : y \in G(x)\}$ is the graph of G. Recall that X is called a metric linear space iff it is both a metric space and a linear space and the metric d of X is translation invariant (i.e., $d(x+z, y+z) = d(x,y), \forall x,y,z \in X$) and, for any convergent sequences $\{\lambda_m\}$ in $\mathbb{R}$ and $\{x^m\}$ in X, we have $\lim_{m\to\infty}(\lambda_m x^m) = (\lim_{m\to\infty}\lambda_m)(\lim_{m\to\infty} x^m)$.

Throughout this paper, if not otherwise specified, X will denote a metric linear space, Λ and M two metric spaces, and Y a normed space. Let Y^* be the topological dual space of Y. For any $\xi \in Y^*$, the norm of ξ is defined by

$$\|\xi\| := \sup\{\langle \xi, x\rangle : \|x\| = 1\},$$

where $\langle \xi, x\rangle$ denotes the value of ξ at x. Let $C \subset Y$ be a pointed, closed and convex cone with $\text{int}\, C \neq \emptyset$ and $C^* := \{\xi \in Y^* : \langle \xi, x\rangle \geq 0, \forall x \in C\}$ be the dual cone of C. Since $\text{int}\, C \neq \emptyset$, the dual cone C^* of C has a weak* compact base. For any given point $e \in \text{int}\, C$, $B^*_e := \{\xi \in C^* : \langle \xi, e\rangle = 1\}$ is a weak* compact base of C^*. We first recall some notions needed in the sequel.

Definition 2.1. Let (E,d) be a metric space. For $A, B \subseteq E$, the Hausdorff distance between A and B is defined by

$$H(A,B) = \max\{H^*(A,B), H^*(B,A)\},$$

where $H^*(A,B) = \sup_{a\in A} d(a,B)$ and $d(x,A) = \inf_{y\in A} d(x,y)$.

Note that $H(.,.)$ may not be a metric in the space of the subsets of E, since it can take the value ∞, and H is a metric in the space $\text{CB}(E)$, the collection of all nonempty, closed and bounded subsets of E.

Now we recall and introduce some concepts related to Hölder continuity for vector valued mappings.

Definition 2.2 (Hölder continuity)**.** Let X, Y, M and C be given as above, and l, α be given positive numbers.

(i) (classical) A mapping $g : X \to Y$, is called $(l\cdot\alpha)$-Hölder continuous around $\bar{x} \in X$ iff there is a neighborhood U of $\bar{x}$ such that, for all $x_1, x_2 \in U$,

$$d(g(x_1), g(x_2)) \leq l d^\alpha(x_1, x_2).$$

(ii) (classical) A set-valued mapping $G : X \to 2^Y$ is $(l\cdot\alpha)-$Hölder continuous around $\bar{x}$ iff there exists a neighborhood U of $\bar{x}$ such that for any $x_1, x_2 \in U$,

$$G(x_1) \subset G(x_2) + lB(0, d^\alpha(x_1,x_2)).$$

(iii) (see [?]) A vector-valued mapping $g : X \to Y$ is said to be $(l\cdot\alpha)-$Hölder continuous with respect to $e \in \text{int}C$ around $\bar{x}$ iff, there exists a neighborhood U of $\bar{x}$ such that for any $x_1, x_2 \in U$,

$$g(x_1) \in g(x_2) + l d^\alpha(x_1,x_2)[-e, e],$$

where $[-e, e] := (e - C) \cap (-e + C)$.

(iv) For $\theta \geq 0$, a given mapping f, $f : X \times X \times M \to \mathbb{R}$, is called $(l \cdot \alpha)$-Hölder continuous with respect to $e \in \text{int}C$ around $\bar{\mu} \in M$, θ-uniformly in $A \subseteq X$ iff, there is a neighborhood N of $\bar{\mu}$ such that, for all $\mu_1, \mu_2 \in N$ and $x, y \in A : x \neq y$,

$$f(x, y, \mu_1) \in f(x, y, \mu_2) + ld^{\alpha}(\mu_1, \mu_2)d^{\theta}(x, y)[-e, e].$$

If the degree of Hölder continuity is equal to 1, then Hölder continuity is called Lipschitz continuity. If a certain property is true at every point of $B \subseteq X$, we will said that it is satisfied in B.

Definition 2.3 (Monotonicity for vector valued mapping)**.** Let X, Y, C be as in Definition 2.2, $g : X \times X \to Y$ be a vector valued mapping, and h, β be positive numbers

(i) ([2]) g is said to be $(h \cdot \beta)$-Hölder strongly monotone on $A \subset X$ iff for each $x, y \in A : x \neq y$,

$$g(x, y) + g(y, x) + hd^{\beta}(x, y)B(0, 1) \subset -C.$$

(ii) ([?]) g is said to be $(h \cdot \beta)-$Hölder strongly monotone with respect to $e \in \text{int}\, C$ on $A \subset X$ iff for each $x, y \in A : x \neq y$,

$$g(x, y) + g(y, x) + hd^{\beta}(x, y)e \in -C.$$

(iii) (classical) g is said to be monotone on $A \subseteq X$ iff for each $x, y \in A$,

$$g(x, y) + g(y, x) \in -C.$$

Lemma 2.4 ([18, Lemma 3.21])**.** *If Y is a real topological linear space and C is a convex cone with int $C \neq \emptyset$, then*

$$\text{int}\, C = \Big\{y \in Y : \langle \xi, y \rangle > 0, \forall \xi \in C^* \setminus \{0\}\Big\}.$$

Now, we give the topological version of the separation theorem which is also known as Eidelheit's separation theorem.

Lemma 2.5 ([19])**.** *Let S and T be nonempty convex subsets of a real topological linear space X with int $S \neq \emptyset$. Then int $S \cap T \neq \emptyset$ if and only if there are a continuous linear functional $l \in X^* \setminus \{0_{X^*}\}$ and a real number α with*

$$l(s) \leq \alpha \leq l(t), \text{ for all } s \in S \text{ and } t \in T$$

and

$$l(s) < \alpha, \text{ for all } s \in \text{ int } S.$$

Now, we introduce new definitions of the strong convexity and convexity-likeness for the vector valued mappings which are need in this paper.

Definition 2.6 (Strong convexity)**.** Let (X, d) be a metric linear space, Y a normed space. Let $A \subset X$ be a convex subset of X, $C \subset Y$ a convex cone with $\text{int}\, C \neq \emptyset$. Let $g : X \to Y$ be a vector valued mapping. g is said to be *C-strongly convex with modulus* h and β on A iff there exists two positive real number h and β and a point $e \in$ int C such that for all $x, y \in A$ and $t \in [0, 1]$,

$$tg(x) + (1 - t)g(y) - ht(1 - t)d^{\beta}(x, y)e \in g(tx + (1 - t)y) + C. \quad (2.1)$$

In this case g is said to be $(h \cdot \beta)$-C-*strongly convex with respect to* e.

Remark 2.7. (i) A mapping $g : X \to Y$ is called C-convex if for all $x, y \in A$ and $t \in [0, 1]$,

$$tg(x) + (1 - t)g(y) \in g(tx + (1 - t)y) + C.$$

Clearly, every C-strongly convex function is C-convex. However, the converse is not true. For example, if $X = Y = \mathbb{R}, C = \mathbb{R}_+ \cup \{0\}$ and g is defined by $g(x) = x^4$. It is clear that g is C-convex but it is not C-strongly convex as there is no $h > 0$ that satisfies (2.1).

(ii) If $Y = \mathbb{R}, C = \mathbb{R}_+ \cup \{0\}$ in the above definition, then g is called $(h \cdot \beta)$-*strongly convex* on A (see Definition 2.2 [5] and Definition 2.1 [25]).

(iii) The examples of C-strongly convex vector valued functions can be found in Example 3.5 and Example 3.7.

Definition 2.8 (Strong convex-likeness). Let (X, d) be a metric linear space, Y a normed space. Let $B \subset X$ be a nonempty subset of X, $C \subset Y$ a convex cone with $\operatorname{int} C \neq \emptyset$. Let $g : X \to Y$ be a vector valued mapping.

(i) g is said to be C-*convex-like* in B iff for all $x_1, x_2 \in B$, and $t \in [0, 1]$, there is $z \in B$

$$g(z) \in tg(x_1) + (1 - t)g(x_2) - C.$$

(ii) g is said to be C-*strongly convex-like with respect to* e in B iff there are two positive real numbers h and β and a point $e \in \operatorname{int} C$ such that for all $x_1, x_2 \in B$ and $t \in [0, 1]$ there is $z \in B$,

$$g(z) + ht(1 - t)d^{\beta}(x_1, x_2)e \in tg(x_1) + (1 - t)g(x_2) - C.$$

As usual g is said to be $(h \cdot \beta)$-C-*strongly concave* (C-concave-like, resp.) on A if $-g$ is $(h \cdot \beta)$-C-*strongly convex* (C-convex-like, resp.) on A.

3. Hölder continuity of solution maps of parametric primal Ky Fan inequalities

Let X, Y, Λ, M, C and e be as in Section 2, and let $K : \Lambda \to 2^X$ be a set-valued mapping with nonempty convex values and $f : X \times X \times M \to Y$ be a vector valued mapping. For each $\lambda \in \Lambda$ and $\mu \in M$, we consider the following parametric weak generalized Ky Fan inequality (for short, (GKF)) :

$$\text{(GKF)} \begin{cases} \text{Find } x_0 \in K(\lambda) \text{ such that} \\ f(x_0, y, \mu) \notin -\operatorname{int} C, \forall y \in K(\lambda). \end{cases} \tag{3.1}$$

For each $(\lambda, \mu) \in \Lambda \times M$, we denote the solution set of by

$$S(\lambda, \mu) := \{x \in K(\lambda) : f(x, y, \mu) \notin -\operatorname{int} C, \forall y \in K(\lambda)\}.$$

For each $\xi \in C^* \backslash \{0\}, \lambda \in \Lambda$ and $\mu \in M$, the ξ−solution set of the (GKF) is denoted by

$$S_{\xi}(\lambda, \mu) := \{x \in K(\lambda) : \langle \xi, f(x, y, \mu) \rangle \geq 0, \forall y \in K(\lambda)\}.$$

In this section, we consider Hölder continuity of solution mappings of (GKF). Since existence conditions of solutions have been studied much in the literature, we do not include existence investigations and always assume that $S(\lambda, \mu)$ is nonempty for the considered point (λ, μ). Firstly, by using the idea given in [15], we establish the relationships between the solution sets and the $\xi-$solution sets under new mild conditions imposed on the vector valued mappings.

Lemma 3.1. *If for each $x \in K(\Lambda)$ and $(\lambda, \mu) \in \Lambda \times M$, $f(x, \cdot, \mu)$ is C-convex-like in $K(\lambda)$, then*

$$S(\lambda, \mu) = \bigcup_{\xi \in C^* \setminus \{0\}} S_\xi(\lambda, \mu) = \bigcup_{\xi \in B_e^*} S_\xi(\lambda, \mu).$$

Proof. (a) We first prove that

$$S(\lambda, \mu) = \bigcup_{\xi \in C^* \setminus \{0\}} S_\xi(\lambda, \mu).$$

Let $x \in \bigcup_{\xi \in C^* \setminus \{0\}} S_\xi(\lambda, \mu)$ be arbitrary given. Then, there exists $\bar{\xi} \in C^* \setminus \{0\}$ such that $x \in S_{\bar{\xi}}(\lambda, \mu)$. By the definition of $S_{\bar{\xi}}(\lambda, \mu)$, for each $y \in K(\lambda)$,

$$\langle \bar{\xi}, f(x, y, \mu) \rangle \geq 0.$$

Applying Lemma 2.4, we can see that, for all $y \in K(\lambda)$, $f(x, y, \mu) \notin -\text{int}C$. This shows that $x \in S(\lambda, \mu)$. Hence, we have that

$$\bigcup_{\xi \in C^* \setminus \{0\}} S_\xi(\lambda, \mu) \subseteq S(\lambda, \mu).$$

Conversely, let $x \in S(\lambda, \mu)$ be arbitrary given. Then, we have that $f(x, y, \mu) \notin -\text{int}C$ for all $y \in K(\lambda)$. Obviously, we have

$$f(x, K(\lambda), \mu) \cap (-\text{int}C) = \emptyset. \tag{3.2}$$

Next, we prove that

$$(f(x, K(\lambda), \mu) + C) \cap (-\text{int}C) = \emptyset.$$

Suppose on the contrary that, there exists $w \in (f(x, K(\lambda), \mu) + C) \cap (-\text{int}C)$. Hence, there exist $z \in f(x, K(\lambda), \mu)$ and $c \in C$ such that $w = z + c \in -\text{int}C$. So, one has $z = -c - w \in -\text{int}C$, which is a contradiction with (3.2). Now, we will show that $f(x, K(\lambda), \mu) + C$ is a convex subset of Y. To this end, let $z_1, z_2 \in f(x, K(\lambda), \mu) + C$ and $t \in [0, 1]$ be arbitrary given. Then, there are $y_1, y_2 \in K(\lambda)$ and $c_1, c_2 \in C$ such that

$$z_i = f(x, y_i, \mu) + c_i, \text{ for all } i = 1, 2.$$

Then, the convex-likeness of the mapping $f(x, \cdot, \mu)$ in $K(\lambda)$ implies that there exist $s \in K(\lambda)$ and $c \in C$ such that

$$tf(x, y_1, \mu) + (1 - t)f(x, y_2, \mu) = f(x, s, \mu) + c.$$

Since C is a convex cone, we have

$$\begin{aligned} tz_1 + (1 - t)z_2 &= tf(x, y_1, \mu) + tc_1 + (1 - t)f(x, y_2, \mu) + (1 - t)c_2 \\ &= f(x, s, \mu) + [c + tc_1 + (1 - t)c_2] \\ &\in f(x, K(\lambda), \mu) + C \end{aligned}$$

Hence, we have that $f(x, K(\lambda), \mu) + C$ is convex. Applying Lemma 2.5, there exist a continuous linear functional $\xi \in Y^* \setminus \{0\}$ and a number $\gamma \in \mathbb{R}$ such that

$$\langle \xi, \bar{c} \rangle < \gamma < \langle \xi, z + c \rangle, \forall \bar{c} \in -\text{int}C, \forall z \in f(x, K(\lambda), \mu), \forall c \in C.$$

Since C is a cone, we imply that, for all $\bar{c} \in -\text{int}C$ and $\alpha > 0$, $\alpha\bar{c} \in -\text{int}C$. So, $\langle \xi, \bar{c} \rangle \le 0$ for all $\bar{c} \in -\text{int}C$. Moreover, ξ is a continuous linear function, one yields $\langle \xi, c \rangle \ge 0$ for all $c \in C$, i.e., $\xi \in C^* \setminus \{0\}$. Since $\bar{c} \in -\text{int}C$ can chosen arbitrarily close to 0, we imply that $\gamma \ge 0$. Similarly, since $c \in C$ also can taken arbitrarily close to 0 and ξ is a continuous linear function, we derive that $\langle \xi, z \rangle \ge 0$ for all $z \in f(x, K(\lambda), \mu)$, i.e.,

$$\langle \xi, f(x, y, \mu) \rangle \ge 0, \forall y \in K(\lambda),$$

and hence $x \in S_\xi(\lambda, \mu)$. So, $S(\lambda, \mu) = \bigcup_{\xi \in C^* \setminus \{0\}} S_\xi(\lambda, \mu)$.

(b) Next we show that

$$\bigcup_{\xi \in C^* \setminus \{0\}} S_\xi(\lambda, \mu) = \bigcup_{\xi \in B_e^*} S_\xi(\lambda, \mu).$$

Since $B_e^* \subset (C^* \setminus \{0\})$, one has $\bigcup_{\xi \in C^* \setminus \{0\}} S_\xi(\lambda, \mu) \supset \bigcup_{\xi \in B_e^*} S_\xi(\lambda, \mu)$. Conversely, for each $\xi \in C^* \setminus \{0\}$, from Lemma 2.4 we have $\langle \xi, e \rangle > 0$ as $e \in \text{int}C$. Putting $\bar{\xi} = \frac{1}{\langle \xi, e \rangle}\xi$, then $\bar{\xi} \in B_e^*$ and $S_\xi(\lambda, \mu) = S_{\bar{\xi}}(\lambda, \mu)$, for all $(\lambda, \mu) \in \Lambda \times M$, and hence $S_\xi \subset \cup_{\xi \in B_e^*} S_\xi(\lambda, \mu)$, for all $\xi \in C^* \setminus \{0\}$. So, $\cup_{\xi \in C^* \setminus \{0\}} S_\xi(\lambda, \mu) \subset \cup_{\xi \in B_e^*} S_\xi(\lambda, \mu)$. □

Remark 3.2. Compared with the results (Lemma 3.1) obtained in [15], C-convexity of $f(x, \cdot, \mu)$ in $K(\lambda)$ is weakened by C-convex-likeness of $f(x, \cdot, \mu)$ in $K(\lambda)$.

Theorem 3.3. *Suppose that for each $\xi \in B_e^*$, $S_\xi(\lambda, \mu)$ is nonempty in a neighborhood of the considered point (λ_0, μ_0). Furthermore, assume the following :*

(i) *K is $(l \cdot \alpha)$-Hölder continuous in a neighborhood $N(\lambda_0)$ of λ_0;*
(ii) *there is a neighborhood $N(\mu_0)$ of μ_0 such that for all $x \in K(N(\lambda_0))$ and $\mu \in N(\mu_0), f(x, \cdot, \mu)$ is $(h \cdot \beta)$-C-strongly convex wiyh respect to e as well as $(m \cdot 1)$-Hölder continuous with respect to e in $conv(K(N(\lambda_0)))$;*
(iii) *for each $\mu \in N(\mu_0), f(\cdot, \cdot, \mu)$ is monotone in $K(N(\lambda_0)) \times K(N(\lambda_0))$;*
(iv) *f is $(n \cdot \gamma)$-Hölder continuous with respect to e around μ_0, θ-uniformly in $K(N(\lambda_0))$ with $\theta < \beta$.*

Then, there exist neighborhoods of $N'(\lambda_0)$ and $N'(\mu_0)$ such that S is single-valued and satisfies the following Hölder condition:

$$H(S(\lambda_1, \mu_1), S(\lambda_2, \mu_2)) \le \Big(\frac{4ml}{h}\Big)^{\frac{1}{\beta}} d^{\frac{\alpha}{\beta}}(\lambda_1, \lambda_2) + \Big(\frac{n}{h}\Big)^{\frac{1}{\beta}} d^{\frac{\gamma}{\beta - \theta}}(\mu_1, \mu_2). \tag{3.3}$$

for each $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in N'(\lambda_0) \times N'(\mu_0)$.

Proof. (a) We first prove that, for each $\bar{\xi} \in B_e^*$ there are neighborhoods $N(\bar{\xi})$, $N_{\bar{\xi}}(\lambda_0)$ and $N_{\bar{\xi}}(\mu_0)$ such that for each $\xi \in N(\bar{\xi})$ and $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in N_{\bar{\xi}}(\lambda_0) \times N_{\bar{\xi}}(\mu_0)$,

$$\rho(S_\xi(\lambda_1, \mu_1), S_\xi(\lambda_2, \mu_2)) \le \Big(\frac{4ml}{h}\Big)^{\frac{1}{\beta}} d^{\frac{\alpha}{\beta}}(\lambda_1, \lambda_2) + \Big(\frac{n}{h}\Big)^{\frac{1}{\beta}} d^{\frac{\gamma}{\beta - \theta}}(\mu_1, \mu_2), \tag{3.4}$$

where $\rho(A, B) := \sup\{d(a, b) : a \in A, b \in B\}$ for each $A, B \subset Y$.

The proof of (3.4) is separated into three steps:

Step I: For any given two points $x_{11} \in S_\xi(\lambda_1, \mu_1)$ and $x_{21} \in S_\xi(\lambda_2, \mu_1)$, we claim that

$$d_1 := d(x_{11}, x_{21}) \le \Big(\frac{4ml}{h}\Big)^{\frac{1}{\beta}} d^{\frac{\alpha}{\beta}}(\lambda_1, \lambda_2).$$

From the definition of S_ξ, we have that, for all $y \in K(\lambda_1)$ and $z \in K(\lambda_2)$,

$$\min\Big\{\langle \xi, f(x_{11}, y, \mu_1)\rangle, \langle \xi, f(x_{21}, z, \mu_1)\rangle\Big\} \geq 0. \tag{3.5}$$

It follows from the $(l\cdot\alpha)$-Hölder continuity of K imposed in (i), there are $x_1 \in K(\lambda_1)$ and $x_2 \in K(\lambda_2)$ satisfying

$$\max\Big\{d(x_{11}, x_2), d(x_{21}, x_1)\Big\} \leq l d^\alpha(\lambda_1, \lambda_2). \tag{3.6}$$

Taking $\bar{x} = \frac{1}{2}(x_{11} + x_{21})$, it is clear that $\bar{x} \in \operatorname{conv}(K(N(\lambda_0)))$. Since $f(x_{11}, \cdot, \mu_1)$ is $(h \cdot \beta)$-C-strong convexity with respect to e, we have that

$$f(x_{11}, \bar{x}, \mu_1) - \frac{1}{2} f(x_{11}, x_{11}, \mu_1) - \frac{1}{2} f(x_{11}, x_{21}, \mu_1) + \frac{h}{4} d_1^\beta e \in -C.$$

Then, it follows from the definition of B_e^* that

$$\begin{aligned} 0 \;\geq\; & \Big\langle \xi, f(x_{11}, \bar{x}, \mu_1) - \frac{1}{2} f(x_{11}, x_{11}, \mu_1) - \frac{1}{2} f(x_{11}, x_{21}, \mu_1) + \frac{h}{4} d_1^\beta e \Big\rangle \\ &= \langle \xi, f(x_{11}, \bar{x}, \mu_1)\rangle - \Big\langle \xi, \frac{1}{2} f(x_{11}, x_{11}, \mu_1)\Big\rangle - \\ &\quad - \Big\langle \xi, \frac{1}{2} f(x_{11}, x_{21}, \mu_1)\Big\rangle + \Big\langle \xi, \frac{h}{4} d_1^\beta e\Big\rangle, \end{aligned}$$

which arrives that

$$\frac{h}{4} d_1^\beta \leq \Big\langle \xi, \frac{1}{2} f(x_{11}, x_{11}, \mu_1)\Big\rangle + \Big\langle \xi, \frac{1}{2} f(x_{11}, x_{21}, \mu_1)\Big\rangle - \langle \xi, f(x_{11}, \bar{x}, \mu_1)\rangle. \tag{3.7}$$

Since $f(\cdot, \cdot, \mu_1)$ is monotone and $x_{11} \in S_\xi(\lambda_1, \mu_1)$, we have

$$\Big\langle \xi, \frac{1}{2} f(x_{11}, x_{11}, \mu_1)\Big\rangle = 0 \tag{3.8}$$

and

$$\Big\langle \xi, f(x_{11}, x_{21}, \mu_1)\Big\rangle \leq -\langle \xi, f(x_{21}, x_{11}, \mu_1)\rangle. \tag{3.9}$$

Using (3.7),(3.8) and (3.9), we imply that

$$\frac{h}{4} d_1^\beta \leq -\Big\langle \xi, \frac{1}{2} f(x_{21}, x_{11}, \mu_1)\Big\rangle - \Big\langle \xi, f\,\Big(x_{11}, \frac{x_{11} + x_{21}}{2}, \mu_1\Big)\Big\rangle. \tag{3.10}$$

Taking $z = x_2$ and $y = \frac{1}{2}(x_{11} + x_1)$ in (3.5), we can get that

$$\min\Big\{\Big\langle \xi, f\Big(x_{11}, \frac{1}{2}(x_{11} + x_1), \mu_1\Big)\Big\rangle, \langle \xi, f(x_{21}, x_2, \mu_1)\rangle\Big\} \geq 0. \tag{3.11}$$

Hence, (3.10) and (3.11) together yield that

$$\begin{aligned}
\frac{h}{4}d_1^{\beta} &\leq -\Big\langle \xi, \frac{1}{2}f(x_{21}, x_{11}, \mu_1)\Big\rangle - \Big\langle \xi, f\Big(x_{11}, \frac{x_{11}+x_{21}}{2}, \mu_1\Big)\Big\rangle \\
&\quad + \Big\langle \xi, f\Big(x_{11}, \frac{x_{11}+x_1}{2}, \mu_1\Big)\Big\rangle + \Big\langle \xi, \frac{1}{2}f(x_{21}, x_2, \mu_1)\Big\rangle \\
&\leq \Big\langle \xi, f\Big(x_{11}, \frac{x_{11}+x_1}{2}, \mu_1\Big)\Big\rangle - \Big\langle \xi, f\Big(x_{11}, \frac{x_{11}+x_{21}}{2}, \mu_1\Big)\Big\rangle \\
&\quad + \Big\langle \xi, \frac{1}{2}f(x_{21}, x_2, \mu_1)\Big\rangle - \Big\langle \xi, \frac{1}{2}f(x_{21}, x_{11}, \mu_1)\Big\rangle \\
&\leq \Big|\Big\langle \xi, f\Big(x_{11}, \frac{x_{11}+x_1}{2}, \mu_1\Big)\Big\rangle - \Big\langle \xi, f\Big(x_{11}, \frac{x_{11}+x_{21}}{2}, \mu_1\Big)\Big\rangle\Big| \\
&\quad + \Big|\Big\langle \xi, \frac{1}{2}f(x_{21}, x_2, \mu_1)\Big\rangle - \Big\langle \xi, \frac{1}{2}f(x_{21}, x_{11}, \mu_1)\Big\rangle\Big|.
\end{aligned}$$

Hence, using (3.6) and the $(m\cdot 1)$-Hölder continuity with respect to e of $f(x_{11}, \cdot, \mu_1)$ and $f(x_{21}, \cdot, \mu_1)$ given in (ii), we have

$$\begin{aligned}
\frac{h}{4}d_1^{\beta} &\leq \frac{1}{2}md(x_{11}, x_2) + \frac{1}{2}md(x_{21}, x_1) \\
&\leq \frac{1}{2}mld^{\alpha}(\lambda_1, \lambda_2) + \frac{1}{2}mld^{\alpha}(\lambda_1, \lambda_2) \\
&\leq mld^{\alpha}(\lambda_1, \lambda_2),
\end{aligned}$$

which gives that

$$d_1 \leq \Big(\frac{4ml}{h}\Big)^{\frac{1}{\beta}} d^{\frac{\alpha}{\beta}}(\lambda_1, \lambda_2).$$

Step II: We claim that for any given two points $x_{21} \in S_{\xi}(\lambda_2, \mu_1)$ and $x_{22} \in S_{\xi}(\lambda_2, \mu_2)$,

$$d_2 := d(x_{21}, x_{22}) \leq \Big(\frac{n}{h}\Big)^{\frac{1}{\beta}} d^{\frac{\gamma}{\beta-\theta}}(\mu_1, \mu_2).$$

By the definition of ξ-solutions, we have that, for all $y, z \in K(\lambda_2)$,

$$\min\Big\{\langle \xi, f(x_{21}, y, \mu_1)\rangle, \langle \xi, f(x_{22}, z, \mu_2)\rangle\Big\} \geq 0. \tag{3.12}$$

Putting $y = \frac{1}{2}(x_{21} + x_{22})$ in (3.12), it is clear that $y \in K(\lambda_2)$, and hence

$$\langle \xi, f(x_{21}, y, \mu_1)\rangle \geq 0.$$

By virtue of the $(h\cdot\beta)$-C-strong convexity with respect to e of $f(x_{21}, \cdot, \mu_1)$, one has

$$f\left(x_{21}, \frac{x_{21}+x_{22}}{2}, \mu_1\right) - \frac{1}{2}f(x_{21}, x_{22}, \mu_1) - \frac{1}{2}f(x_{21}, x_{21}, \mu_1) + \frac{h}{4}d_2^{\beta}e \in -C.$$

Therefore,

$$\begin{aligned}
0 &\geq \left\langle \xi, f\Big(x_{21}, \frac{x_{21}+x_{22}}{2}, \mu_1\Big) - \frac{1}{2}f(x_{21}, x_{22}, \mu_1) - \frac{1}{2}f(x_{21}, x_{21}, \mu_1) + \frac{h}{4}d_2^{\beta}e\right\rangle \\
&\geq \left\langle \xi, -\frac{1}{2}f(x_{21}, x_{22}, \mu_1) - \frac{1}{2}f(x_{21}, x_{21}, \mu_1) + \frac{h}{4}d_2^{\beta}e\right\rangle.
\end{aligned}$$

Since $\xi \in B_e^*$, one yields

$$\langle \xi, f(x_{21}, x_{22}, \mu_1) \rangle + \langle \xi, f(x_{21}, x_{21}, \mu_1) \rangle - \frac{h}{2} d_2^{\beta} \geq 0. \tag{3.13}$$

By the monotonicity of $f(\cdot, \cdot, \mu_1)$ and $x_{21} \in S_{\xi}(\lambda_2, \mu_1)$, we obtain

$$\xi(f(x_{21}, x_{21}, \mu_1)) = 0 \tag{3.14}$$

and

$$\langle \xi, f(x_{21}, x_{22}, \mu_1) \rangle \leq -\langle \xi, f(x_{22}, x_{21}, \mu_1) \rangle. \tag{3.15}$$

From (3.13), (3.14) and (3.15), we imply that

$$\frac{h}{2} d_2^{\beta} \leq -\langle \xi, f(x_{22}, x_{21}, \mu_1) \rangle. \tag{3.16}$$

Now, replacing z in (3.12) by $\frac{1}{2}(x_{22} + x_{21})$, we have

$$\Big\langle \xi, f\Big(x_{22}, \frac{x_{22} + x_{21}}{2}, \mu_2\Big) \Big\rangle \geq 0. \tag{3.17}$$

By the $(h \cdot \beta)$-C-strong convexity with respect to e of $f(x_{22}, \cdot, \mu_2)$, we obtain the following inequality which is similar to (3.13), concretely

$$\langle \xi, f(x_{22}, x_{21}, \mu_2) \rangle + \langle \xi, f(x_{22}, x_{22}, \mu_2) \rangle - \frac{h}{2} d_2^{\beta} \geq 0.$$

As $\xi(f(x_{22}, x_{22}, \mu_2)) = 0$,

$$\frac{h}{2} d_2^{\beta} \leq \langle \xi, f(x_{22}, x_{21}, \mu_2) \rangle. \tag{3.18}$$

Summing (3.16) and (3.18) and combining with assumption (iv), we have

$$\begin{aligned} h d_2^{\beta} &\leq \langle \xi, f(x_{22}, x_{21}, \mu_2) \rangle - \langle \xi, f(x_{22}, x_{21}, \mu_1) \rangle \\ &\leq |\langle \xi, f(x_{22}, x_{21}, \mu_2) - f(x_{22}, x_{21}, \mu_1) \rangle| \\ &\leq n d^{\gamma}(\mu_1, \mu_2) d_2^{\theta}, \end{aligned}$$

and thus,

$$d_2 \leq \Big(\frac{n}{h}\Big)^{\frac{1}{\beta - \theta}} d^{\frac{\gamma}{\beta - \theta}}(\mu_1, \mu_2).$$

Step III: Finally, applying Step I and Step II, for each $x_{11} \in S_{\xi}(\lambda_1, \mu_1)$ and $x_{22} \in S_{\xi}(\lambda_2, \mu_2)$, we have

$$d(x_{11}, x_{22}) \leq d_1 + d_2 \leq \Big(\frac{4ml}{h}\Big)^{\frac{1}{\beta}} d^{\frac{\alpha}{\beta}}(\lambda_1, \lambda_2) + \Big(\frac{n}{h}\Big)^{\frac{1}{\beta - \theta}} d^{\frac{\gamma}{\beta - \theta}}(\mu_1, \mu_2).$$

Hence, we get

$$\rho(S^{\xi}(\lambda_1, \mu_1), S^{\xi}(\lambda_2, \mu_2)) \leq \Big(\frac{4ml}{h}\Big)^{\frac{1}{\beta}} d^{\frac{\alpha}{\beta}}(\lambda_1, \lambda_2) + \Big(\frac{n}{h}\Big)^{\frac{1}{\beta - \theta}} d^{\frac{\gamma}{\beta - \theta}}(\mu_1, \mu_2). \tag{3.19}$$

(b) Now we prove (3.3). We see that B_e^* is a weak* compact set and hence it can be covered by finitely many open sets, i.e.,

$$B_e^* \subset \bigcup_{i=1}^{n} N(\xi_i), \tag{3.20}$$

where $\xi_i \in B_e^*$ and $N(\xi_i)$ is a neighborhood of ξ_i, defined in (a). For $N_{\xi_i}(\lambda_0)$ and $N_{\xi_i}(\mu_0)$ determined as in (a), let $N'(\lambda_0) = \bigcap_{i=1}^n N_{\xi_i}(\lambda_0)$ and $N'(\mu_0) = \bigcap_{i=1}^n N_{\xi_i}(\mu_0)$. For each $(\lambda, \mu) \in N'(\lambda_0) \times N'(\mu_0)$ and $\xi \in B_e^*$, it follows from (3.20) that there exists $i_0 \in \{1, \dots, n\}$ such that $\xi \in N(\xi_{i_0})$ and obviously, $(\lambda, \mu) \in N_{\xi_{i_0}}(\lambda_0) \times N_{\xi_{i_0}}(\mu_0)$. Combining the convexity of $f(x, ., \mu)$ assumed in (ii) with Lemma 3.2, we have

$$S(\lambda, \mu) = \bigcup_{\xi \in B_e^*} S_\xi(\lambda, \mu).$$

For each $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in N'(\lambda_0) \times N'(\mu_0)$, we now show that

$$H(S(\lambda_1, \mu_1), S(\lambda_2, \mu_2)) \le \left(\frac{4ml}{h}\right)^{\frac{1}{\beta}} d^{\frac{\alpha}{\beta}}(\lambda_1, \lambda_2) + \left(\frac{n}{h}\right)^{\frac{1}{\beta}} d^{\frac{\gamma}{\beta-\theta}}(\mu_1, \mu_2). \tag{3.21}$$

Indeed, for each $x_{11} \in S(\lambda_1, \mu_1) = \cup_{\xi \in B_e^*} S_\xi(\lambda_1, \mu_1)$, there is $\hat{\xi} \in B_e^*$ such that $x_{11} \in S_{\hat{\xi}}(\lambda_1, \mu_1)$. As $S_{\hat{\xi}}(\lambda_2, \mu_2) \subseteq S(\lambda_2, \mu_2)$ and applying (a), one has

$$\begin{aligned}
d(x_1, S(\lambda_2, \mu_2)) &\le d(x_1, S_{\hat{\xi}}(\lambda_2, \mu_2)) \\
&\le H^*(S_{\hat{\xi}}(\lambda_1, \mu_1), S_{\hat{\xi}}(\lambda_2, \mu_2)) \\
&\le \rho(S_{\hat{\xi}}(\lambda_1, \mu_1), S_{\hat{\xi}}(\lambda_2, \mu_2)). \\
&\le \left(\frac{4ml}{h}\right)^{\frac{1}{\beta}} d^{\frac{\alpha}{\beta}}(\lambda_1, \lambda_2) + \left(\frac{n}{h}\right)^{\frac{1}{\beta}} d^{\frac{\gamma}{\beta-\theta}}(\mu_1, \mu_2).
\end{aligned}$$

Therefore,

$$H^*(S(\lambda_1, \mu_1), S(\lambda_2, \mu_2)) \le \left(\frac{4ml}{h}\right)^{\frac{1}{\beta}} d^{\frac{\alpha}{\beta}}(\lambda_1, \lambda_2) + \left(\frac{n}{h}\right)^{\frac{1}{\beta-\theta}} d^{\frac{\gamma}{\beta-\theta}}(\mu_1, \mu_2). \tag{3.22}$$

Similarly, we also have

$$H^*(S(\lambda_2, \mu_2), S(\lambda_1, \mu_1)) \le \left(\frac{4ml}{h}\right)^{\frac{1}{\beta}} d^{\frac{\alpha}{\beta}}(\lambda_1, \lambda_2) + \left(\frac{n}{h}\right)^{\frac{1}{\beta-\theta}} d^{\frac{\gamma}{\beta-\theta}}(\mu_1, \mu_2). \tag{3.23}$$

Combining (3.22) with (3.23), we obtain (3.3). Furthermore, setting $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$ in the inequality (3.3), we can see that the diameter of the solution set $S(\lambda_1, \mu_1)$ is 0 (for arbitrary (λ_1, μ_2)), i.e., the solution map S of (GKF) is single-valued in $N'(\lambda_0) \times N'(\mu_0)$. The proof is complete. □

Remark 3.4. (i) Compared with the results obtained in [5], the real valued function f is generalized to the vector valued function. Further, the strong convexity for the real valued function is extended to C-strong convexity for the vector valued function.

(ii) Comparing Theorem 3.3 and the results obtained in [1, 12, 23, 27], we can see that the main difference is the assumption (ii), namely, we assume that $f(x, \cdot, \mu)$ is $(h \cdot \beta)$-C-strongly convex with respect to e in $\operatorname{conv}(K(N(\lambda_0)))$ in Theorem 3.3 but strong monotone in [23], or strong pseudomonotone in [1, 12] or f is satisfied a condition related to strong monotonicity in [27]. Hence, Theorem 3.3 can be applicable in the general case of non-unique solutions while the aforecited results do not work as in the following example.

Example 3.5. Let $X = Y = \mathbb{R}^2, \Lambda \equiv M = [0,1]$, $C = \mathbb{R}^2_+$, $e = (1,1)$, $K(\lambda) = [-1-\lambda, 1+\lambda], \bar{\lambda} = 0$, and

$$f(x, y, \lambda) = ((\lambda+1)(y_1^2 - x_1^2), (\lambda+1)(y_2^2 - x_2^2)),$$

where $x = (x_1, x_2), y = (y_1, y_2) \in X$. It is not hard to see that all assumptions of Theorem 3.3 are satisfied and $S(\lambda) \equiv \{(0,0)\}$ is Hölder continuous. However, we can see that

$$f((1,1), (-1,-1), \lambda) = f((-1,-1), (1,1), \lambda) = (0,0), \text{ for all } \lambda \in M.$$

Hence the assumptions on strong monotonicity in [23], strong pseudo-monotonicity in [1,12] and some conditions related to strong monotonicity in [27] are not satisfied. Thus, the main results given in [1, 12, 27] do not work while Theorem 3.3 can be applicable.

To illustrate the essential of imposed assumptions which do not relate to Hölder continuity in Theorem 3.3, we bring out some following examples.

Example 3.6 (the strong convexity is essential)**.** Let $X = Y = \mathbb{R}^2$, $\Lambda \equiv M = [0,1]$, $C = \mathbb{R}^2_+$, $e = (1,1)$, $K(\lambda) = [\lambda, 1], \bar{\lambda} = 0$, and $f(x, y, \lambda) = (\lambda(x_1^2 - y_1^2), \lambda(x_2^2 - y_2^2))$, where $x = (x_1, x_2), y = (y_1, y_2) \in X$. Then, all assumptions of Theorem 3.3 are fulfilled, except for the strong convexity of f. By direct calculations, we have

$$S(\lambda) = \begin{cases} [0,1] \times [0,1], & \text{if } \lambda = 0, \\ \{(1,1)\}, & \text{if } \lambda \neq 0, \end{cases}$$

which is not lower semicontinuous at $\bar{\lambda} = 0$.

Example 3.7 (the monotonicity is crucial)**.** Let $X = Y = \mathbb{R}^2$, $\Lambda \equiv M = [0,1]$, $C = \mathbb{R}^2_+$, $e = (1,1)$, $K(\lambda) \equiv [0,1], \bar{\lambda} = 0$, and $f(x, y, \lambda) = (y_1^2 - \lambda x_1, y_2^2 - \lambda x_2)$, where $x = (x_1, x_2), y = (y_1, y_2) \in X$. $f(x, y, \lambda) = y^2 - \lambda x$. Then, it is not hard to see that the assumptions of Theorem 3.3 are satisfied, except the monotonicity of f ($f(x, y, 0) + f(y, x, 0) = (y_1^2 + x_1^2, y_2^2 + x_2^2) > 0$ assumed in the theorem, for many $x, y \in [0,1] \times [0,1]$). Direct computations give us the solution set

$$S(\lambda) = \begin{cases} [0,1] \times [0,1], & \text{if } \lambda = 0, \\ \{(0,0)\}, & \text{if } \lambda \neq 0, \end{cases}$$

which is even not lower semicontinuous at $\bar{\lambda} = 0$.

In the special case where $K(\lambda) \equiv K$ (K is a nonempty set), the strong convexity in assumption (ii) of Theorem 3.3 can be reduced to the strong convex-likeness along with several other relaxations, and we obtain the following result.

Theorem 3.8. *For (KF) with $K(\lambda) \equiv K$. Suppose that for each $\xi \in B_e^*$, $S_\xi(\lambda, \mu)$ is nonempty in a neighborhood of the considered point (λ_0, μ_0). Furthermore, assume the following :*

(i) *there is a neighborhood $N(\mu_0)$ of μ_0 such that for all $x \in K$ and $\mu \in N(\mu_0)$, $f(x, \cdot, \mu)$ is $(h \cdot \beta)$-C-strongly convex-like with respect to e as well as $(m \cdot 1)$-Hölder continuous with respect to e in convK;*

(ii) *for each $\mu \in N(\mu_0)$, $f(\cdot, \cdot, \mu)$ is monotone in $K \times K$;*

(iii) *f is $(n \cdot \gamma)$-Hölder continuous with respect to e around μ_0, θ-uniformly in K with $\theta < \beta$.*

Then, there exist neighborhoods of $N'(\mu_0)$ such that S satisfies the following Hölder condition: For each $\mu_1, \mu_2 \in N'(\mu_0)$,

$$H(S(\mu_1), S(\mu_2)) \leq \left(\frac{n}{h}\right)^{\frac{1}{\beta}} d^{\frac{\gamma}{\beta-\theta}}(\mu_1, \mu_2). \tag{3.24}$$

Proof. (a) We first prove that, for each $\bar{\xi} \in B_e^*$ there are neighborhoods $N(\bar{\xi})$ and $N_{\bar{\xi}}(\mu_0)$ such that for each $\xi \in N(\bar{\xi})$ and $\mu_1, \mu_2 \in N_{\bar{\xi}}(\mu_0)$,

$$\rho(S_\xi(\mu_1), S_\xi(\mu_2)) \leq +\left(\frac{n}{h}\right)^{\frac{1}{\beta}} d^{\frac{\gamma}{\beta-\theta}}(\mu_1, \mu_2), \tag{3.25}$$

where $\rho(A, B) := \sup\{d(a, b) : a \in A, b \in B\}$ for each $A, B \subset Y$.
Firstly, we claim that for any given two points $x_1 \in S_\xi(\mu_1)$ and $x_2 \in S_\xi(\mu_2)$,

$$d(x_1, x_2) \leq \left(\frac{n}{h}\right)^{\frac{1}{\beta}} d^{\frac{\gamma}{\beta-\theta}}(\mu_1, \mu_2).$$

By the definition of ξ-solutions, we have that, for all $y, z \in K$,

$$\min\Big\{\langle \xi, f(x_1, y, \mu_1)\rangle, \langle \xi, f(x_2, z, \mu_2)\rangle\Big\} \geq 0. \tag{3.26}$$

Putting $y = x_2 \in K$ in (3.26), we have that

$$\langle \xi, f(x_1, y, \mu_1)\rangle \geq 0.$$

By virtue of the $(h \cdot \beta)$-C-strong convex-likeness with respect to e of $f(x_{21}, \cdot, \mu_1)$, we have that there is $v \in K$ such that

$$f(x_1, v, \mu_1) - \frac{1}{2} f(x_1, x_2, \mu_1) - \frac{1}{2} f(x_1, x_1, \mu_1) + \frac{h}{4} d^\beta(x_1, x_2)e \in -C.$$

Therefore,

$$\begin{aligned} 0 &\geq \left\langle \xi, f(x_1, v, \mu_1) - \frac{1}{2} f(x_1, x_2, \mu_1) - \frac{1}{2} f(x_1, x_1, \mu_1) + \frac{h}{4} d^\beta(x_1, x_2)e \right\rangle \\ &\geq \left\langle \xi, -\frac{1}{2} f(x_1, x_2, \mu_1) - \frac{1}{2} f(x_1, x_2, \mu_1) + \frac{h}{4} d^\beta(x_1, x_2)e \right\rangle. \end{aligned}$$

Since $\xi \in B_e^*$, one yields

$$\langle \xi, f(x_1, x_2, \mu_1)\rangle + \langle \xi, f(x_1, x_1, \mu_1)\rangle - \frac{h}{2} d^\beta(x_1, x_2) \geq 0. \tag{3.27}$$

By the monotonicity of $f(\cdot, \cdot, \mu_1)$ and $x_1 \in S_\xi(\mu_1)$, we obtain

$$\xi(f(x_1, x_1, \mu_1)) = 0 \tag{3.28}$$

and

$$\langle \xi, f(x_1, x_2, \mu_1)\rangle \le -\langle \xi, f(x_2, x_1, \mu_1)\rangle. \tag{3.29}$$

From (3.27), (3.28) and (3.29), we have that

$$\frac{h}{2} d^{\beta}(x_1, x_2) \le -\langle \xi, f(x_2, x_1, \mu_1)\rangle. \tag{3.30}$$

Now, replacing w in (3.26) by x_1, we have

$$\Big\langle \xi, f\Big(x_2, z, \mu_2\Big)\Big\rangle = \Big\langle \xi, f\Big(x_2, w, \mu_2\Big)\Big\rangle \ge 0. \tag{3.31}$$

By the $(h \cdot \beta)$-C-strong convex-likeness with respect to e of $f(x_2, \cdot, \mu_2)$, we obtain the following inequality which is similar to (3.27), concretely

$$\langle \xi, f(x_2, x_1, \mu_2)\rangle + \langle \xi, f(x_2, x_2, \mu_2)\rangle - \frac{h}{2} d_2^{\beta} \ge 0.$$

As $\xi(f(x_2, x_2, \mu_2)) = 0$,

$$\frac{h}{2} d_2^{\beta} \le \langle \xi, f(x_2, x_1, \mu_2)\rangle. \tag{3.32}$$

Summing (3.30) and (3.32) and combining with assumption (iv), we have

$$\begin{aligned} h d_2^{\beta} &\le \langle \xi, f(x_2, x_1, \mu_2)\rangle - \langle \xi, f(x_2, x_1, \mu_1)\rangle \\ &\le |\langle \xi, f(x_2, x_1, \mu_2) - f(x_2, x_1, \mu_1)\rangle| \\ &\le n d^{\gamma}(\mu_1, \mu_2) d_2^{\theta}, \end{aligned}$$

and thus,

$$d(x_1, x_2) \le \Big(\frac{n}{h}\Big)^{\frac{1}{\beta-\theta}} d^{\frac{\gamma}{\beta-\theta}}(\mu_1, \mu_2).$$

Hence, we get

$$\rho(S_{\xi}(\mu_1), S_{\xi}(\mu_2)) \le +\Big(\frac{n}{h}\Big)^{\frac{1}{\beta-\theta}} d^{\frac{\gamma}{\beta-\theta}}(\mu_1, \mu_2). \tag{3.33}$$

(b) Proving (3.24) is similar to the proof of (3.24) in Theorem 3.3 and hence we can get our result.

□

4. Hölder continuity of solution maps of parametric dual Ky Fan inequalities

Let $X, Y, \Lambda, M, C, e, K, f$ be as in Section 3. For each $(\lambda, \mu) \in \Lambda \times M$, we consider the following parametric dual weak generalized Ky Fan inequality:
(DKF) Find $x_0 \in K(\lambda)$, such that

$$f(y, x_0, \mu) \notin \text{int} C, \forall y \in K(\lambda). \tag{4.1}$$

As usual, for each $(\lambda, \mu) \in \Lambda \times M$, the solution set of (DKF) is denoted by

$$S^d(\lambda, \mu) := \{x \in K(\lambda) : f(y, x, \mu) \notin \text{int}\, C, \forall y \in K(\lambda)\}.$$

For each $\xi \in C^*\backslash\{0\}, \lambda \in N(\lambda_0)$ and $\mu \in N(\mu_0)$, the $\xi-$solution set of (DKF) is denoted by

$$S^d_\xi(\xi, \lambda, \mu) := \{x \in K(\lambda) : \langle \xi, f(y, x, \mu)\rangle \leq 0, \forall y \in K(\lambda)\}.$$

Note that we can not compare the solution sets of primal problems (KF) with that of the dual problems (DKF) in general (see, Example 2.1 in [6]). However, under some suitable assumptions we can get certain relationships between both of them (see [22]). In this section, we focuss on the Hölder continuity of solution mappings of (DKF) and always assume that all kinds of solution sets of the problems are nonempty in the neighborhood of the reference point. The following lemma plays an important role in the proof of our main result.

Lemma 4.1. *If for each $x \in K(\Lambda)$ and $(\lambda, \mu) \in \Lambda \times M$, $f(\cdot, x, \mu)$ is C-concave-like in $K(\lambda)$, then*

$$S^d(\lambda, \mu) = \bigcup_{\xi \in C^*\backslash\{0\}} S^d_\xi(\lambda, \mu) = \bigcup_{\xi \in B^*_e} S^d_\xi(\lambda, \mu).$$

Proof. We first show that

$$S^d(\lambda, \mu) = \bigcup_{\xi \in C^*\backslash\{0\}} S^d_\xi(\lambda, \mu).$$

Let $x \in \cup_{\xi\in C^*\backslash\{0\}} S^d_\xi(\lambda, \mu)$, there is $\bar{\xi} \in C^* \setminus \{0\}$, $x \in S^d_{\bar{\xi}}(\lambda, \mu)$. So,

$$\langle \bar{\xi}, f(y, x, \mu)\rangle \leq 0, \forall y \in K(\lambda).$$

Combining above affirmation with Lemma 2.4, we imply that $f(y, x, \mu) \notin \text{int}C$, for all $y \in K(\lambda)$. Hence $x \in S^d(\lambda, \mu)$. Conversely, let $x \in S^d(\lambda, \mu)$. By the definition of (DKF), we have

$$f(K(\lambda), x, \mu) \cap (\text{int}C) = \emptyset,$$

and hence,

$$(f(K(\lambda), x, \mu) - C) \cap (\text{int}C) = \emptyset.$$

Let $z_1, z_2 \in f(K(\lambda), x, \mu) - C$, there are $y_1, y_2 \in K(\lambda)$ and $c_1, c_2 \in C$ such that $z_i = f(y_i, x, \mu) - c_i, i = 1, 2$. Since $f(\cdot, x, \mu)$ is concave-like in $K(\lambda)$, for all $t \in [0, 1]$, there are $s \in K(\lambda)$ and $c \in C$ such that

$$tf(y_1, x, \mu) + (1 - t)f(y_2, x, \mu) = f(s, x, \mu) - c,$$

i.e., $tz_1 + (1 - t)z_2 = f(s, x, \mu) - tc_1 - (1 - t)c_2 - c \in f(K(\lambda), x, \mu) - C$, and hence $f(K(\lambda), x, \mu) - C$ is a convex subset of Y. By the virtue of Lemma 2.5, we obtain a continuous linear functional $\xi \in Y^* \setminus \{0\}$ and a number $\gamma \in \mathbb{R}$ such that

$$\langle \xi, z - c\rangle < \gamma < \langle \xi, \bar{c}\rangle, \forall z \in f(K(\lambda), x, \mu), \forall c \in C, \forall \bar{c} \in \text{int}C.$$

Since C is a cone, we have $\langle \xi, \bar{c}\rangle \geq 0$ for all $\bar{c} \in \text{int}C$, and hence we also have $\langle \xi, c\rangle \geq 0$ for all $c \in C$, i.e., $\xi \in C^* \setminus \{0\}$. Since $\bar{c} \in \text{int}C$ and $c \in C$ can chosen arbitrarily close to 0, we yield $\gamma \geq 0$ and $\langle \xi, z\rangle \leq 0$ for all $z \in f(K(\lambda), x, \mu)$, and hence $x \in S^d_\xi(\lambda, \mu)$. So, $S^d(\lambda, \mu) = \cup_{\xi\in C^*\backslash\{0\}} S^d_\xi(\lambda, \mu)$. Using the given arguments as in Lemma 3.1, we have

$$\bigcup_{\xi \in C^*\backslash\{0\}} S^d_\xi(\lambda, \mu) \subset \bigcup_{\xi \in B^*_e} S^d_\xi(\lambda, \mu).$$

□

Applying Lemma 4.1, we establish sufficient conditions for the solution mappings of (DKF) to be Hölder continuous.

Theorem 4.2. *Suppose that for each $\xi \in B_e^*$, the $\xi-$solution set $S_\xi^d(\lambda,\mu)$ exists in a neighborhood of the considered point (λ_0,μ_0). Furthermore, assume that the following assumptions are satisfied:*

(i) *K is $(l\cdot\alpha)$-Hölder continuous in a neighborhood $N(\lambda_0)$ of λ_0;*
(ii) *there is a neighborhood $N(\mu_0)$ of μ_0 such that for all $x\in K(N(\lambda_0))$ and $\mu\in N(\mu_0)$, $f(\cdot,x,\mu)$ is $(h\cdot\beta)$-C-strongly concave with respect to e as well as $(m\cdot 1)$-Hölder continuous with respect to e in $conv(K(N))$;*
(iii) *for each $\mu\in U(\mu_0), -f(\cdot,\cdot,\mu)$ is monotone on $K(N(\lambda_0))\times K(N(\lambda_0))$;*
(iv) *f is $(n\cdot\gamma)$-Hölder continuous with respect to e around μ_0, θ-uniformly in $K(N(\lambda_0))$ with $\theta<\beta$.*

Then, there exist neighborhoods of $N'(\lambda_0)$ and $N'(\mu_0)$ such that S^d is single-valued and satisfies the following Hölder condition: for each $(\lambda_1,\mu_1),(\lambda_2,\mu_2)\in N'(\lambda_0)\times N'(\mu_0)$,

$$H(S^d(\lambda_1,\mu_1),S^d(\lambda_2,\mu_2))\le\Big(\frac{4ml}{h}\Big)^{\frac{1}{\beta}}d^{\frac{\alpha}{\beta}}(\lambda_1,\lambda_2)+\Big(\frac{n}{h}\Big)^{\frac{1}{\beta}}d^{\frac{\gamma}{\beta-\theta}}(\mu_1,\mu_2). \tag{4.2}$$

Proof. We first show that, for each $\bar\xi\in B_e^*$ there are neighborhoods $N(\bar\xi)$, $N_{\bar\xi}(\lambda_0)$ and $N_{\bar\xi}(\mu_0)$ such that for each $\xi\in N(\bar\xi)$ and $(\lambda_1,\mu_1),(\lambda_2,\mu_2)\in N_{\bar\xi}(\lambda_0)\times N_{\bar\xi}(\mu_0)$

$$\rho(S_\xi^d(\lambda_1,\mu_1),S_\xi^d(\lambda_2,\mu_2))\le\Big(\frac{4ml}{h}\Big)^{\frac{1}{\beta}}d^{\frac{\alpha}{\beta}}(\lambda_1,\lambda_2)+\Big(\frac{n}{h}\Big)^{\frac{1}{\beta}}d^{\frac{\gamma}{\beta-\theta}}(\mu_1,\mu_2). \tag{4.3}$$

To this end, the proof of (4.3) is divided into three steps:
Step I: For any given two points $x_{11}\in S_\xi^d(\lambda_1,\mu_1)$ and $x_{21}\in S_\xi^d(\lambda_2,\mu_1)$, we claim that

$$d_1:=d(x_{11},x_{21})\le\Big(\frac{4ml}{h}\Big)^{\frac{1}{\beta}}d^{\frac{\alpha}{\beta}}(\lambda_1,\lambda_2).$$

By the construction of S_ξ^d, for all $y\in K(\lambda_1)$ and $z\in K(\lambda_2)$, we have

$$\max\Big\{\langle\xi,f(y,x_{11},\mu_1)\rangle,\langle\xi,f(z,x_{21},\mu_1)\rangle\Big\}\le 0. \tag{4.4}$$

Thanks to (i), there are $x_1\in K(\lambda_1)$ and $x_2\in K(\lambda_2)$ such that

$$\max\Big\{d(x_{11},x_2),d(x_{21},x_1)\Big\}\le l d^{\alpha}(\lambda_1,\lambda_2). \tag{4.5}$$

Putting $\bar x=\frac{1}{2}(x_{11}+x_{21})$, it follows from the $(h\cdot\beta)$-C-strongly concavity with respect to e of $f(\cdot,x_{11},\mu_1)$ that

$$-f(\bar x,x_{11},\mu_1)+\frac12 f(x_{11},x_{11},\mu_1)+\frac12 f(x_{21},x_{11},\mu_1)+\frac{h}{4}d_1^{\beta}e\in -C.$$

So,

$$\begin{aligned} 0 \;\geq\; & \Big\langle \xi, -f(\bar{x}, x_{11}, \mu_1) + \frac{1}{2} f(x_{11}, x_{11}, \mu_1) + \frac{1}{2} f(x_{21}, x_{11}, \mu_1) + \frac{h}{4} d_1^{\beta} e \Big\rangle \\ & = \langle \xi, -f(\bar{x}, x_{11}, \mu_1) \rangle + \Big\langle \xi, \frac{1}{2} f(x_{11}, x_{11}, \mu_1) \Big\rangle + \\ & \quad + \Big\langle \xi, \frac{1}{2} f(x_{21}, x_{11}, \mu_1) \Big\rangle + \Big\langle \xi, \frac{h}{4} d_1^{\beta} e \Big\rangle, \end{aligned}$$

which arrives that

$$\frac{h}{4} d_1^{\beta} \leq -\langle \xi, -f(\bar{x}, x_{11}, \mu_1) \rangle - \Big\langle \xi, \frac{1}{2} f(x_{11}, x_{11}, \mu_1) \Big\rangle - \Big\langle \xi, \frac{1}{2} f(x_{21}, x_{11}, \mu_1) \Big\rangle. \tag{4.6}$$

On the other hand, combining the monotonicity of $f(\cdot, \cdot, \mu_1)$ and $x_{11} \in S_{\xi}^{d}(\lambda_1, \mu_1)$, one has

$$\Big\langle \xi, f(x_{11}, x_{11}, \mu_1) \Big\rangle = 0 \tag{4.7}$$

and

$$\langle \xi, f(x_{11}, x_{21}, \mu_1) \rangle \geq -\langle \xi, f(x_{21}, x_{11}, \mu_1) \rangle. \tag{4.8}$$

Combining (4.6),(4.7) and (4.8), one gets

$$\frac{h}{4} d_1^{\beta} \leq -\langle \xi, -f(\bar{x}, x_{11}, \mu_1) \rangle + \Big\langle \xi, \frac{1}{2} f(x_{11}, x_{21}, \mu_1) \Big\rangle. \tag{4.9}$$

Now, setting $z = x_2$ and $y = \frac{1}{2}(x_{11} + x_1)$ in (4.4), we have

$$\min \Big\{ -\Big\langle \xi, f\Big(\frac{1}{2}(x_{11} + x_1), x_{11}, \mu_1\Big) \Big\rangle, -\langle \xi, f(x_2, x_{21}, \mu_1) \rangle \Big\} \geq 0. \tag{4.10}$$

Hence, from (4.9) and (4.10), we obtain that

$$\begin{aligned} \frac{h}{4} d_1^{\beta} \;\leq\; & \Big\langle \xi, f(\frac{1}{2}(x_{11} + x_{21}), x_{11}, \mu_1) \Big\rangle + \Big\langle \xi, \frac{1}{2} f(x_{11}, x_{21}, \mu_1) \Big\rangle \\ & - \Big\langle \xi, f\Big(\frac{1}{2}(x_{11} + x_1), x_{11}, \mu_1\Big) \Big\rangle - \Big\langle \xi, \frac{1}{2} f(x_2, x_{21}, \mu_1) \Big\rangle \\ = \; & \Big\langle \xi, f\Big(\frac{1}{2}(x_{11} + x_{21}), x_{11}, \mu_1\Big) \Big\rangle - \Big\langle \xi, f\Big(\frac{1}{2}(x_{11} + x_1), x_{11}, \mu_1\Big) \Big\rangle \\ & + \Big\langle \xi, \frac{1}{2} f(x_{11}, x_{21}, \mu_1) \Big\rangle - \Big\langle \xi, \frac{1}{2} f(x_2, x_{21}, \mu_1) \Big\rangle \\ \leq \; & \Big| \Big\langle \xi, f\Big(\frac{1}{2}(x_{11} + x_{21}), x_{11}, \mu_1\Big) \Big\rangle - \Big\langle \xi, f\Big(\frac{1}{2}(x_{11} + x_1), x_{11}, \mu_1\Big) \Big\rangle \Big| \\ & + \Big| \Big\langle \xi, \frac{1}{2} f(x_{11}, x_{21}, \mu_1 \Big) \Big\rangle - \Big\langle \xi, \frac{1}{2} f(x_2, x_{21}, \mu_1) \Big\rangle \Big|. \end{aligned}$$

By (ii), we have

$$\begin{aligned} \frac{h}{4} d_1^{\beta} \;\leq\; & \frac{1}{2} m d(x_{21}, x_1) + \frac{1}{2} m d(x_{11}, x_2) \\ \leq\; & \frac{1}{2} m d^{\alpha}(\lambda_1, \lambda_2) + \frac{1}{2} m d^{\alpha}(\lambda_1, \lambda_2) \\ =\; & m l d^{\alpha}(\lambda_1, \lambda_2), \end{aligned}$$

Hence, we conclude that

$$d_1 \le \Big(\frac{4ml}{h}\Big)^{\frac{1}{\beta}} d^{\frac{\alpha}{\beta}}(\lambda_1, \lambda_2).$$

Step II : Next we prove that for any given points $x_{21} \in S^d_\xi(\lambda_2, \mu_1)$ and $x_{22} \in S^d_\xi(\lambda_2, \mu_2)$,

$$d_2 := d(x_{21}, x_{22}) \le \Big(\frac{m}{h}\Big)^{\frac{1}{\beta}} d^{\frac{\gamma}{\beta-\theta}}(\mu_1, \mu_2).$$

By the definition of S^d_ξ, all $y, z \in K(\lambda_2)$, we get

$$\max\Big\{\langle \xi, f(y, x_{21}, \mu_1)\rangle, \langle \xi, f(z, x_{22}, \mu_2)\rangle\Big\} \le 0. \tag{4.11}$$

Substituting $y = \dfrac{1}{2}(x_{21} + x_{22})$ in (4.11), we obtain

$$\Big\langle \xi, f\Big(\frac{1}{2}(x_{21} + x_{22}), x_{21}, \mu_1\Big)\Big\rangle \le 0,$$

which implies that

$$-\Big\langle \xi, f\Big(\frac{1}{2}(x_{21} + x_{22}), x_{21}, \mu_1\Big)\Big\rangle \ge 0.$$

By virtue of the $(h \cdot \beta)$ -C-strong concavity with respect to e of $f(\cdot, x_{21}, \mu_1)$, we get

$$-f\Big(\frac{1}{2}(x_{21} + x_{22}), x_{21}, \mu_1\Big) + \frac{1}{2}f(x_{21}, x_{21}, \mu_1) + \frac{1}{2}f(x_{22}, x_{21}, \mu_1) + \frac{h}{4}d_2^\beta e \in -C.$$

By virtue of the linearity of ξ and above discussion, we imply that

$$\Big\langle \xi, -\frac{1}{2}f(x_{21}, x_{21}, \mu_1)\Big\rangle + \Big\langle \xi, -\frac{1}{2}f(x_{22}, x_{21}, \mu_1)\Big\rangle - \frac{h}{4}d_2^\beta \ge 0. \tag{4.12}$$

From the monotonicity of $-f(\cdot, \cdot, \mu_1)$ and $x_{21} \in S^d_\xi(\lambda_2, \mu_1)$, we have

$$\xi(f(x_{21}, x_{21}, \mu_1)) = 0 \tag{4.13}$$

and

$$\langle \xi, -f(x_{22}, x_{21}, \mu_1)\rangle \le \langle \xi, f(x_{21}, x_{22}, \mu_1)\rangle. \tag{4.14}$$

Hence, combining (4.12), (4.13) and (4.14), one has

$$\frac{h}{2}d_2^\beta \le \langle \xi, f(x_{21}, x_{22}, \mu_1)\rangle. \tag{4.15}$$

Similarly, by setting $z = \dfrac{1}{2}(x_{22} + x_{21})$ in (4.11), and using the above discussion, we also imply that

$$\frac{h}{2}d_2^\beta \le \langle \xi, -f(x_{21}, x_{22}, \mu_2)\rangle = -\langle \xi, f(x_{21}, x_{22}, \mu_2)\rangle. \tag{4.16}$$

Summing (4.15) and (4.16) and combining with (iv), we establish

$$\begin{aligned} hd_2^\beta &\le \langle \xi, f(x_{21}, x_{22}, \mu_1)\rangle - \langle \xi, f(x_{21}, x_{22}, \mu_2)\rangle \\ &\le |\langle \xi, f(x_{21}, x_{22}, \mu_1)\rangle - \langle \xi, f(x_{21}, x_{22}, \mu_2)\rangle| \\ &\le nd^\gamma(\mu_1, \mu_2)d_2^\theta, \end{aligned}$$

i.e.,

$$d_2 \leq \left(\frac{n}{h}\right)^{\frac{1}{\beta}} d^{\frac{\gamma}{\beta-\theta}}(\mu_1, \mu_2),$$

and thus, (4.3) is derived from the conclusions of Steps I and II. Finally, using the given the same argument in the proof of Theorem 3.3, we can arrive the desired conclusion. □

5. Conclusions

In this paper, by using the strong convexity and Hölder continuity for both the vector valued mappings of the primal and dual parametric vector Ky Fan inequality in metric linear spaces, without strong monotonicity assumption, we presented the sufficient conditions for the mentioned solution mappings to be Hölder continuous around the reference point, when the solution of these problems is not unique. Finally, we provided many examples to illustrate that the imposed assumptions are essential.

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