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A SHRINKING PROJECTION ALGORITHM FOR VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS

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Abstract. A shrinking projection algorithm is presented for solving the variational inequality and fixed point problem of the pseudocontractive operator. Strong convergence analysis of the suggested algorithm is given.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Let $A : C \to H$ be a nonlinear operator. By definition, the variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \ \forall v \in C. \tag{1.1}$$

The set of solutions of the variational inequality (1.1) is denoted by $VI(A, C)$.

Variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral and equilibrium problems, which arise in several branches of pure and applied sciences in a unified and general framework. Several numerical methods have been developed for solving variational inequalities and related optimization problems, see [1, 4–8, 10, 11, 14–16, 19, 21] and the references therein.

Recall that a mapping $T : C \to C$ is said to be pseudocontractive if

$$\langle Tu - Tu^{\dagger}, u - u^{\dagger} \rangle \leq \|u - u^{\dagger}\|^2 \tag{1.2}$$

for all $u, u^{\dagger} \in C$.

The interest of pseudocontractions lies in their connection with monotone operators; namely, T is a pseudocontraction if and only if the complement $I-T$ is a monotone operator. There are a large number references associated with algorithmic approaches to the fixed points of pseudocontractive operators, see, e.g., [2,3,17,18,22].

The main purpose of this paper is to study the variational inequality problem (1.1) and the fixed point problem of the pseudocontractive operator. We suggest an iterative algorithm by using the shrinking projection method in Hilbert spaces. We demonstrate the strong convergence of the presented the shrinking projection

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algorithm for finding the common element of the variational inequality problem (1.1) and the fixed point problem of the pseudocontractive operator.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonlinear mapping. Denote the set of fixed points of T by $Fix(T)$, that is, $Fix(T) := \{x \in C | x = Tx\}$.

A mapping $T : C \to C$ is said to be L-Lipschitzian if

$$\|Tu - Tu^\dagger\| \leq L\|u - u^\dagger\|$$

for all $u, u^\dagger \in C$, where $L > 0$ is a constant. If $L = 1$, T is said to be nonexpansive.

A mapping $A : C \to H$ is said to be inverse strongly monotone if there exists $\zeta > 0$ such that

$$\langle u - v, Au - Av\rangle \geq \zeta\|Au - Av\|^2$$

for all $u, v \in C$.

Note that (1.2) is equivalent to the following:

$$\|Tu - Tu^\dagger\|^2 \leq \|u - u^\dagger\|^2 + \|(I - T)u - (I - T)u^\dagger\|^2 \tag{2.1}$$

for all $u, u^\dagger \in C$.

The metric (or nearest point) projection from H onto C is the mapping $proj_C : H \to C$ which assigns to each point $x \in C$ the unique point $proj_C x \in C$ satisfying the property

$$\|x - proj_C x\| = \inf_{y \in C} \|x - y\|.$$

It is clear that the metric projection $proj$ is a typical firmly nonexpansive mapping. The characteristic inequality of the projection is

$$\langle u - proj_C u, u^\dagger - proj_C u\rangle \leq 0$$

for all $u \in H$ and $u^\dagger \in C$.

A mapping T is said to be demiclosed if, for any sequence $\{x_n\}$ which weakly converges to $\tilde{x}$, if the sequence $\{T(x_n)\}$ strongly converges to $x^\dagger$, then $T(\tilde{x}) = x^\dagger$.

It is well-known that in a real Hilbert space H, the following equality holds:

$$\|\xi u + (1 - \xi)u^\dagger\|^2 = \xi\|u\|^2 + (1 - \xi)\|u^\dagger\|^2 - \xi(1 - \xi)\|u - u^\dagger\|^2 \tag{2.2}$$

for all $u, u^\dagger \in H$ and $\xi \in [0, 1]$.

We need the following lemmas for our main results:

Lemma 2.1 ([22]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudocontractive mapping. Then*

(i) *$Fix(T)$ is a closed convex subset of C.*

(ii) *$(I - T)$ is demiclosed at zero.*

Lemma 2.2 ([18]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be an L-Lipschitz pseudocontractive operator. Then, the operator $(1 - \xi)I + \xi T((1 - \eta)I + \eta T)$ is quasi-nonexpansive when $0 < \xi < \eta < \frac{1}{\sqrt{1+L^2}+1}$. That is,*

$$\|(1 - \xi)x + \xi T((1 - \eta)x + \eta Tx) - u^\dagger\| \leq \|x - u^\dagger\|,$$

for all $x \in C$ *and* $u^\dagger \in Fix(T)$.

For the convenience, in the sequel we use the following expressions:

- $x_n \rightharpoonup x^\dagger$ denotes the weak convergence of x_n to $x^\dagger$;
- $x_n \to x^\dagger$ denotes the strong convergence of x_n to $x^\dagger$.

Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of a Hilbert space H. We define $s - Li_n C_n$ and $w - Ls_n C_n$ as follows.

(i) $x \in s - Li_n C_n$ if and only if there exists $\{x_n\} \subset C_n$ such that $x_n \to x$.
(ii) $x \in w - Ls_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\}$ in C_{n_i} such that $y_i \rightharpoonup y$.

If C_0 satisfies the following:

$$C_0 = s - Li_n C_n = w - Ls_n C_n,$$

then we say that $\{C_n\}$ converges to C_0 in the sense of Mosco [10] and we write $C_0 = M - \lim_{n\to\infty} C_n$. It is easy to show that, if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco.

Tsukada [13] proved the following theorem for the metric projection:

Lemma 2.3 ([13]). *Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of a Hilbert space H. If $C_0 = M - \lim_{n\to\infty} C_n$ exists and is nonempty, then, for each $x \in H$, $\{proj_{C_n}(x)\}$ converges strongly to $proj_{C_0}(x)$, where $proj_{C_n}$ and $proj_{C_0}$ are the metric projections of H onto C_n and C_0, respectively.*

Let (X, d) be a complete metric space. A mapping $f : X \to X$ is called a *Meir-Keeler contraction* ([9]) if, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \epsilon + \delta \implies d(f(x), f(y)) < \epsilon, \ \forall x, y \in X.$$

It is well known that the Meir-Keeler contraction is a generalization of the contraction.

Lemma 2.4 ([9]). *A Meir-Keeler contraction defined on a complete metric space has a unique fixed point.*

Lemma 2.5 ([12]). *Let f be a Meir-Keeler contraction on a convex subset C of a Banach space E. Then, for any $\epsilon > 0$, there exists $r \in (0, 1)$ such that*

$$\|x - y\| \geq \epsilon \Longrightarrow \|f(x) - f(y)\| \leq r\|x - y\|$$

for all $x, y \in C$.

Lemma 2.6 ([12]). *Let C be a convex subset of a Banach space E. Let T be a nonexpansive mapping on C and f be a Meir-Keeler contraction on C. Then the following hold:*

(i) *Tf is a Meir-Keeler contraction on C.*
(ii) *For each $\alpha \in (0, 1)$, $(1 - \alpha)T + \alpha f$ is a Meir-Keeler contraction on C.*

3. Main results

First, we suggest a shrinking projection algorithm for finding the common element of variational inequality problem and fixed point problem. Subsequently, we show the strong convergence of the presented algorithm.

Algorithm 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a Meir-Keeler contractive operator and $A : C \to H$ be a δ-inverse strongly monotone operator. Let $T : C \to C$ be an L-Lipschitz pseudocontractive operator with $L > 1$. For $x_0 \in C_0 = C$ arbitrarily, define a sequence $\{x_n\}$ iteratively by*

$$\begin{cases} z_n = proj_C(x_n - \lambda_n A x_n), \\ y_n = (1-\alpha_n)z_n + \alpha_n T((1-\beta_n)z_n + \beta_n T z_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = proj_{C_{n+1}} f(x_n), \ n \ge 0, \end{cases} \tag{3.1}$$

where $\{\lambda_n\} \subset [a,b] \subset (0,2\delta)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$.

Set $\Omega := VI(A,C) \cap Fix(T)$. Suppose $\Omega \neq \emptyset$. Since f is a Meir-Keeler contraction of C, it follows that $proj_\Omega f$ is a Meir-Keeler contraction of C by Lemma 2.6. According to Lemma 2.4, there exists a unique fixed point $x^\dagger \in C$ such that $x^\dagger = proj_\Omega f(x^\dagger)$.

Conclusion 3.2. $\Omega \subset C_n$ *for all* $n \ge 0$

Proof. In fact, $\Omega \subset C_0$ is obvious. Suppose that $\Omega \subset C_k$ for some $k \in \mathbb{N}$. Set $v_n = (1-\beta_n)z_n + \beta_n T z_n$ for all $n \ge 0$. Then $y_n = (1-\alpha_n)z_n + \alpha_n T v_n$ for all $n \ge 0$. Let $x^* \in \Omega \subset C_k$. Then, we have

$$\begin{aligned} \|z_n - x^*\| &= \|proj_C(x_n - \lambda_n A x_n) - proj_C(x^* - \lambda_n A x^*)\| \\ &\le \|(x_n - \lambda_n A x_n) - (x^* - \lambda_n A x^*)\| \\ &\le \|x_n - x^*\|. \end{aligned} \tag{3.2}$$

By Lemma 2.2 and (3.2), we have

$$\begin{aligned} \|y_n - x^*\| &= \|(1-\alpha_n)z_n + \alpha_n T((1-\beta_n)z_n + \beta_n T z_n) - x^*\| \\ &\le \|z_n - x^*\| \\ &\le \|x_n - x^*\| \end{aligned} \tag{3.3}$$

and hence $x^* \in C_{k+1}$. This indicates that $\Omega \subset C_n$ for all $n \ge 0$. □

Conclusion 3.3. *C_n is closed and convex for all $n \ge 0$.*

Proof. In fact, it is obvious from the assumption that $C_0 = C$ is closed convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$. For any $z \in C_k$, $\|y_k - z\| \le \|x_k - z\|$ is equivalent to

$$\|y_k - x_k\|^2 + 2\langle y_k - x_k, x_k - z\rangle \le 0.$$

So C_{k+1} is closed and convex. By induction, we deduce that C_n is closed and convex for all $n \ge 0$. □

Remark 3.4. By Conclusions 3.2 and 3.3, we deduce that $\{x_n\}$ is well-defined. Since $\bigcap_{n=1}^{\infty} C_n$ is closed convex, we also have that $proj_{\bigcap_{n=1}^{\infty} C_n}$ is well-defined and so $proj_{\bigcap_{n=1}^{\infty} C_n} f$ is a Meir-Keeler contraction on C. By Lemma 2.4, there exists a unique fixed point $u \in \bigcap_{n=1}^{\infty} C_n$ of $proj_{\bigcap_{n=1}^{\infty} C_n} f$. Since C_n is a nonincreasing sequence of nonempty closed convex subsets of H with respect to inclusion, it follow that

$$\emptyset \neq \Omega \subset \bigcap_{n=1}^{\infty} C_n = M - \lim_{n\to\infty} C_n.$$

Setting $u_n := proj_{C_n} f(u)$ and applying Lemma 2.3, we can conclude that

$$\lim_{n\to\infty} u_n = proj_{\bigcap_{n=1}^{\infty} C_n} f(u) = u.$$

Conclusion 3.5. $\lim_{n\to\infty} \|x_n - u\| = 0$

Proof. Assume that $M = \limsup_{n\to\infty} \|x_n - u\| > 0$. Then, for any ϵ with $0 < \epsilon < M$, we can choose $\delta_1 > 0$ such that

$$\limsup_{n\to\infty} \|x_n - u\| > \epsilon + \delta_1. \tag{3.4}$$

Since f is a Meir-Keeler contraction, for the positive ϵ, there exists another $\delta_2 > 0$ such that

$$\|x - y\| < \epsilon + \delta_2 \Longrightarrow \|f(x) - f(y)\| < \epsilon \tag{3.5}$$

for all $x, y \in C$.

In fact, we can choose a common $\delta > 0$ such that (3.4) and (3.5) hold. If $\delta_1 > \delta_2$, then

$$\limsup_{n\to\infty} \|x_n - u\| > \epsilon + \delta_1 > \epsilon + \delta_2.$$

If $\delta_1 \leq \delta_2$, then, from (3.5), it follows that

$$\|x - y\| < \epsilon + \delta_1 \implies \|f(x) - f(y)\| < \epsilon$$

for all $x, y \in C$. Thus we have

$$\limsup_{n\to\infty} \|x_n - u\| > \epsilon + \delta \tag{3.6}$$

and

$$\|x - y\| < \epsilon + \delta \Longrightarrow \|f(x) - f(y)\| < \epsilon \tag{3.7}$$

for all $x, y \in C$. Since $u_n \to u$, there exists $n_0 \in \mathbb{N}$ such that

$$\|u_n - u\| < \delta \tag{3.8}$$

for all $n \geq n_0$.

Now, we now consider two possible cases:

Case 1. There exists $n_1 \geq n_0$ such that

$$\|x_{n_1} - u\| \leq \epsilon + \delta.$$

By (3.7) and (3.8), we get

$$\begin{aligned}\|x_{n_1+1}-u\| &\le \|x_{n_1+1}-u_{n_1+1}\|+\|u_{n_1+1}-u\|\\ &= \|proj_{C_{n_1+1}}f(x_{n_1})-proj_{C_{n_1+1}}f(u)\|+\|u_{n_1+1}-u\|\\ &\le \|f(x_{n_1})-f(u)\|+\|u_{n_1+1}-u\|\\ &\le \epsilon+\delta.\end{aligned}$$

By induction, we can obtain that

$$\|x_{n_1+m}-u\|\le \epsilon+\delta$$

for all $m\ge 1$, which implies that

$$\limsup_{n\to\infty}\|x_n-u\|\le\epsilon+\delta,$$

which contradicts (3.6). Therefore, we conclude that $\|x_n-u\|\to 0$ as $n\to\infty$.

Case 2. $\|x_n-u\|>\epsilon+\delta$ for all $n\ge n_0$.

Now, we prove that Case 2 is impossible. Suppose that Case 2 is true. By Lemma 2.5, there exists $r\in(0,1)$ such that

$$\|f(x_n)-f(u)\|\le r\|x_n-u\|$$

for all $n\ge n_0$. Thus we have

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &= \|proj_{C_{n+1}}f(x_n)-proj_{C_{n+1}}f(u)\|\\ &\le \|f(x_n)-f(u)\|\\ &\le r\|x_n-u\|\end{aligned}$$

for all $n\ge n_0$. It follows that

$$\begin{aligned}\lim_{n\to\infty}\|x_{n+1}-u\| &= \limsup_{n\to\infty}\|x_{n+1}-u_{n+1}\|\\ &\le r\limsup_{n\to\infty}\|x_n-u\|\\ &< \limsup_{n\to\infty}\|x_n-u\|,\end{aligned}$$

which gives a contradiction. Hence we obtain

$$\lim_{n\to\infty}\|x_n-u\|=0$$

□

Theorem 3.6. *If* $0<c<\alpha_n\le\beta_n<d<\frac{1}{\sqrt{1+L^2}+1}$, *then the sequence* $\{x_n\}$ *defined by* (3.1) *converges strongly to* $x^\dagger$.

Proof. By Conclusion 3.5, we get that $\{x_n\}$ is bounded. Observe that

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \|x_n-u\|+\|u-u_{n+1}\|+\|u_{n+1}-x_{n+1}\|\\ &= \|x_n-u\|+\|u-u_{n+1}\|+\|proj_{C_{n+1}}f(x_n)-proj_{C_{n+1}}f(u)\|\\ &\le \|x_n-u\|+\|u-u_{n+1}\|+\|f(x_n)-f(u)\|.\end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.9}$$

Since $x_{n+1} \in C_{n+1}$, we have

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|.$$

This together with (3.9) implies that

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = \lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.10}$$

Note that

$$\begin{aligned}
\|y_n - x^*\|^2 &\leq \|z_n - x^*\|^2 \\
&\leq \|(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*)\|^2 \\
&= \|x_n - x^*\|^2 + \lambda_n^2\|Ax_n - Ax^*\|^2 - 2\langle Ax_n - Ax^*, x_n - x^*\rangle \\
&= \|x_n - x^*\|^2 + \lambda_n(\lambda_n - 2\delta)\|Ax_n - Ax^*\|^2 \\
&\leq \|x_n - x^*\|^2 + (b - 2\delta)b\|Ax_n - Ax^*\|^2.
\end{aligned} \tag{3.11}$$

Then we have

$$\begin{aligned}
(2\delta - b)b\|Ax_n - Ax^*\|^2 &\leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2 \\
&\leq \|x_n - y_n\|(\|x_n - x^*\| + \|y_n - x^*\|).
\end{aligned} \tag{3.12}$$

By (3.10) and (3.12), we obtain

$$\lim_{n\to\infty} \|Ax_n - Ax^*\| = 0. \tag{3.13}$$

Since $proj_C$ is firmly-nonexpansive, we have

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|proj_C(x_n - \lambda_n Ax_n) - proj_C(x^* - \lambda_n Ax^*)\|^2 \\
&\leq \langle (x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*), z_n - x^*\rangle \\
&= \frac{1}{2}(\|(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*)\|^2 + \|z_n - x^*\|^2 \\
&\quad - \|(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*) + x^* - z_n\|^2) \\
&\leq \frac{1}{2}(\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|(x_n - z_n) - \lambda_n(Ax_n - Ax^*)\|^2) \\
&= \frac{1}{2}(\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|x_n - z_n\|^2 \\
&\quad + 2\lambda_n\langle x_n - z_n, Ax_n - Ax^*\rangle - \lambda_n^2\|Ax_n - Ax^*\|^2).
\end{aligned} \tag{3.14}$$

It follows that

$$\begin{aligned}
\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 &+ 2\lambda_n\langle x_n - z_n, Ax_n - Ax^*\rangle \\
&- \|x_n - z_n\|^2 - \lambda_n^2\|Ax_n - Ax^*\|^2.
\end{aligned} \tag{3.15}$$

From (3.3) and (3.15), we get

$$\begin{aligned}
\|y_n - x^*\|^2 &\leq \|z_n - x^*\|^2 \\
&\leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\langle x_n - z_n, Ax_n - Ax^*\rangle \\
&\quad - \lambda_n^2\|Ax_n - Ax^*\|^2 \\
&\leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Ax^*\|
\end{aligned}$$

and so

$$\begin{aligned}\|x_n - z_n\|^2 &\le \|x_n - x^*\|^2 - \|y_n - x^*\|^2 + 2\lambda_n \|x_n - z_n\| \|Ax_n - Ax^*\| \\ &\le \|x_n - y_n\|(\|x_n - x^*\| + \|y_n - x^*\|) + 2\lambda_n \|x_n - z_n\| \|Ax_n - Ax^*\|.\end{aligned}$$

This together with (3.10) and (3.13) implies that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.16}$$

Next, we prove that $u \in Fix(T) \cap VI(A, C)$. Note that

$$\begin{aligned}\|z_n - Tz_n\| &\le \|z_n - y_n\| + \|y_n - Tz_n\| \\ &\le \|z_n - y_n\| + (1-\alpha_n)\|z_n - Tz_n\| + \alpha_n \|Tv_n - Tz_n\| \\ &\le \|z_n - y_n\| + (1-\alpha_n)\|z_n - Tz_n\| + \alpha_n L \|v_n - z_n\| \\ &\le \|z_n - y_n\| + (1-\alpha_n)\|z_n - Tz_n\| + \alpha_n \beta_n L \|z_n - Tz_n\|.\end{aligned}$$

It follows that

$$\|z_n - Tz_n\| \le \frac{1}{\alpha_n(1-\beta_n L)} \|z_n - y_n\| \le \frac{1}{c(1-dL)} \|z_n - y_n\| \to 0. \tag{3.17}$$

Since $x_n \to u$, we have $z_n \to u$ by (3.16). So, from (3.17) and Lemma 2.1, we deduce that $u \in Fix(T)$. By the same argument as that in the proof of [4, Theorem 3.1], we can show that $u \in VI(A, C)$. This implies that $u \in VI(A, C)$. Therefore, we have $u \in \Omega$. Since $x_{n+1} = proj_{C_{n+1}} f(x_n)$, we have

$$\langle f(x_n) - x_{n+1}, x_{n+1} - y \rangle \ge 0$$

for all $y \in C_{n+1}$. Since $\Omega \subset C_{n+1}$, we get

$$\langle f(x_n) - x_{n+1}, x_{n+1} - y \rangle \ge 0$$

for all $y \in \Omega$. Noting that $x_n \to u \in \Omega$, we deduce

$$\langle f(u) - u, u - y \rangle \ge 0$$

for all $y \in \Omega$. Thus $u = proj_\Omega f(u) = x^\dagger$. This completes the proof. □

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