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# COMMENTS ON A FIXED POINT THEOREM OF MIRON NICOLESCU

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Abstract. In what turn out to be his last published paper [*Un théoréme de triplet fixe.* Collection of articles dedicated to Mauro Picone on the occasion of his ninetieth birthday, Rend. Mat. (6) 8 (1975), 17–20.], Miron Nicolescu established a very interesting fixed point theorem for a triplet of mappings $f = (f', f'', f''')$, where $f' : G'' \times G''' \to G'$; $f'' : G'' \times G' \to G''$; $f''' : G' \times G'' \to G'''$ and $(G', d')$, $(G'', d'')$, $(G''', d''')$ are metric spaces. The main aim of this note is to present a slightly simplified proof of that result, to discuss some other related aspects and to establish some extensions and generalizations of Nicolescu's fixed point theorem.

## 1. Introduction

In [9], Miron Nicolescu (1903-1975), a proeminent Romanian mathematician, former Director of the Institute of Mathematics in Bucharest (1963-1975), President of Romanian Academy (1966-1975) and Vice-President of the International Mathematical Union (1974-1975), established an interesting fixed point theorem for a triplet of mappings. Our aim in this note is to present a slightly simpler proof of Nicolescu's result, to discuss various aspects related to it and to establish some extensions and generalizations.

We first introduce all notions and reformulate the results in [9], by using a slightly different notation.

Let $G = (G', G'', G''')$ be a triplet of metric spaces, $(G', d')$, $(G'', d'')$, $(G''', d''')$, and $f = (f', f'', f''')$ be a triplet of mappings defined as follows

$$f' : G'' \times G''' \to G'; \quad f'' : G'' \times G' \to G''; \quad f''' : G' \times G'' \to G'''.$$

According to [9], it is said that the triplet $f$ is a *contraction* of the triplet $G$ if there exists $\alpha' \in (0, 1)$ such that

$$\max\{d''(x'', x''_1), d'''(x''', x'''_1)\} \leq d \Longrightarrow d'(f'(x'', x'''), f'(x''_1, x'''_1)) \leq \alpha' \cdot d; \tag{1.1}$$

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and the analogue relations for $f''$ and $f'''$, i.e., there exists $\alpha'' \in (0,1)$ such that

$$\max\{d'''(x''', x_1'''), d'(x', x_1')\} \leq d \Longrightarrow d''(f''(x''', x'), f''(x_1''', x_1')) \leq \alpha'' \cdot d; \tag{1.2}$$

and there exists $\alpha''' \in (0,1)$ such that

$$\max\{d'(x', x_1'), d''(x'', x_1'')\} \leq d \Longrightarrow d'''(f'''(x', x''), f'''(x_1', x_1'')) \leq \alpha''' \cdot d. \tag{1.3}$$

The main result in [9] is the following interesting fixed point theorem.

**Theorem 1.1.** *If $(G', d')$, $(G'', d'')$, $(G''', d''')$ are complete metric spaces and $f = (f', f'', f''')$ is a contraction of $G = G' \times G'' \times G'''$, then there exists a unique triplet $(a', a'', a''') \in G$ such that*

$$f'(a'', a''') = a'; \quad f''(a''', a') = a''; \quad f'''(a', a'') = a'''. \tag{1.4}$$

*Proof.* We adapt the original proof given in [9]. First we note that, in order to simplify calculations, one can consider the contraction coefficient

$$\alpha = \max\{\alpha', \alpha'', \alpha'''\}$$

for all contraction conditions (1.1), (1.2), (1.3) satisfied by the mappings $f', f'', f'''$.

Now let $x_1'' \in G''$, $x_1''' \in G'''$ and compute $x_1' = f'(x_1'', x_1''')$ and continue the process:

$$x_2'' = f''(x_1''', x_1'); \quad x_3'' = f''(x_2''', x_2'); \quad x_4'' = f''(x_3''', x_3'); \tag{1.5}$$

$$x_2''' = f'''(x_1', x_1''); \quad x_3''' = f'''(x_2', x_2''); \quad x_4''' = f'''(x_3', x_3''); \tag{1.6}$$

$$x_2' = f'(x_2'', x_2'''); \quad x_3' = f'(x_3'', x_3'''); \quad x_4' = f'(x_4'', x_4'''). \tag{1.7}$$

In this way we obtain the sequences

$$\{x_n'\} \subset G'; \quad \{x_n''\} \subset G''; \quad \{x_n'''\} \subset G'''$$

defined by the following recurrence relations

$$x_{n+1}'' = f''(x_n''', x_n'); \quad x_{n+1}''' = f'''(x_n', x_n''); \quad x_{n+1}' = f'(x_{n+1}'', x_{n+1}'''), n = 1, 2, 3, \dots, \tag{1.8}$$

respectively.

We prove that $\{x_n'\}, \{x_n''\}, \{x_n'''\}$ are Cauchy sequences. Let $d > 0$ be arbitrary such that

$$\max\{d'(x_1', x_2'), d''(x_1'', x_2''), d'''(x_1''', x_2''')\} \leq d. \tag{1.9}$$

Then

$$\max\{d'''(x_1''', x_2'''), d'(x_1', x_2')\} \leq d$$

which, by (1.1), implies

$$d''(x_2'', x_3'') = d''(f''(x_1''', x_1'), f''(x_2''', x_2')) \leq \alpha \cdot d.$$

Similarly, we have by (1.9)

$$\max\{d'(x_1', x_2'), d''(x_1'', x_2'')\} \leq d$$

which implies, by means of (1.2),

$$d'''(x_2''', x_3''') = d'''(f'''(x_1', x_1''), f'''(x_2', x_2'')) \leq \alpha \cdot d.$$

while

$$\max\{d''(x_2'', x_3''), d'''(x_2''', x_3''')\} \leq d$$

implies, by means of (1.3),

$$d'(x_2', x_3') = d'(f'(x_2'', x_2'''), f'(x_3'', x_3''')) \leq \alpha \cdot d.$$

Now, further,

$$\max\{d'(x_2', x_3'), d''(x_2'', x_3''), d'''(x_2''', x_3''')\} \leq \alpha \cdot d$$

implies by (1.1)

$$d''(x_3'', x_4'') = d''(f''(x_2''', x_2'), f''(x_3''', x_3')) \leq \alpha^2 \cdot d$$

and the analogous relations,

$$d'''(x_3''', x_4''') = d'''(f'''(x_2', x_2''), f'''(x_3', x_3'')) \leq \alpha^2 \cdot d$$

and

$$d'(x_3', x_4') = d'(f'(x_3'', x_3'''), f'(x_4', x_4'')) \leq \alpha^2 \cdot d,$$

respectively.

By induction and in view of the recurrences (1.8), we obtain

$$d''(x_n'', x_{n+1}'') = d''(f''(x_{n-1}''', x_{n-1}'), f''(x_n''', x_n')) \leq \alpha^{n-1} \cdot d, \tag{1.10}$$

$$d'''(x_n''', x_{n+1}''') = d'''(f'''(x_{n-1}', x_{n-1}''), f'''(x_n', x_n'')) \leq \alpha^{n-1} \cdot d, \tag{1.11}$$

and

$$d'(x_n', x_{n+1}') = d'(f'(x_n'', x_n'''), f'(x_{n+1}'', x_{n+1}''')) \leq \alpha^{n-1} \cdot d, \tag{1.12}$$

for all $n = 1, 2, 3, \ldots$

Now, by a routine computation, it follows from (1.10), (1.11), (1.12) that $\{x_n'\}, \{x_n''\}, \{x_n'''\}$ are Cauchy sequences in the complete metric spaces $(G', d')$, $(G'', d'')$, $(G''', d''')$, respectively. Denote

$$a' = \lim_{n\to\infty} x_n'; \quad a'' = \lim_{n\to\infty} x_n''; \quad a''' = \lim_{n\to\infty} x_n'''. \tag{1.13}$$

In view of the fact that $f', f'', f'''$ are uniformly continuous, by letting $n \to \infty$ in (1.8), we obtain (1.4), i.e.,

$$f'(a'', a''') = a'; \quad f''(a''', a') = a''; \quad f'''(a', a'') = a'''.$$

To prove the uniqueness, assume that there exists $(b', b'', b''') \in G$ with the same property, i.e.,

$$f'(b'', b''') = b'; \quad f''(b''', b') = b''; \quad f'''(b', b'') = b''',$$

and let $h > 0$ be such that

$$\max\{d'(a', b'), d''(a'', b''), d'''(a''', b''')\} \leq h.$$

Then, the following implications hold

$$\max\{d''(a'', b''), d'''(a''', b''')\} \leq h \Longrightarrow d'(a', b') \leq \alpha \cdot h,$$

$$\max\{d'''(a''', b'''), d'(a', b')\} \leq h \Longrightarrow d''(a'', b'') \leq \alpha \cdot h,$$

$$\max\{d'(a', b'), d''(a'', b'')\} \leq h \Longrightarrow d'''(a''', b''') \leq \alpha \cdot h,$$

and hence

$$\max\{d'(a', b'), d''(a'', b''), d'''(a''', b''')\} \leq \alpha \cdot h.$$

By induction one obtains

$$\max\{d'(a',b'),d''(a'',b''),d'''(a''',b''')\} \le \alpha^n \cdot h, n=1,2,3,\dots. \tag{1.14}$$

Now, by letting $n \to \infty$ in (1.14), one obtains

$$\max\{d'(a',b'),d''(a'',b''),d'''(a''',b''')\} \le 0,$$

which implies $d'(a',b')=0, d''(a'',b'')=0, d'''(a''',b''')=0$, that is,

$$a'=b', a''=b'', a'''=b''',$$

which proves the uniqueness of the fixed point of $f$. □

**Remark 1.2.** The contractiveness property introduced by conditions (1.1), (1.2) and (1.3) must be interpreted in the following sense: $f'$ is a *contraction* if there exists $\alpha' \in (0,1)$ such that for **any** $d$ satisfying $\max\{d''(x'',x_1''),d'''(x''',x_1''')\} \le d$, one has

$$d'(f'(x'',x'''),f'(x_1'',x_1''')) \le \alpha' \cdot d, \tag{1.15}$$

and the analogous ones for $f''$ and $f'''$. This also includes the limit case

$$d = \max\{d''(x'',x_1''),d'''(x''',x_1''')\}.$$

This means that the interpretation of that condition as: $f'$ is a *contraction* if there exists $\alpha' \in (0,1)$ such that for **some** $d$ satisfying $\max\{d''(x'',x_1''),d'''(x''',x_1''')\} \le d$, one has

$$d'(f'(x'',x'''),f'(x_1'',x_1''')) \le \alpha' \cdot d, \tag{1.16}$$

would be wrong, as illustrated by Example 1.1 below.

**Example 1.1.** Let $G'=G''=G'''=[0,1]$ with the usual distance and let $f' : G'' \times G''' \to G'$ be given by $f'(x'',x''') = \dfrac{x''+x'''}{2}$, for all $x'' \in G''$ and $x''' \in G'''$. Then, for $x''=x'''=1$ and $x_1''=x_1'''=0$ one has

$$1 = \max\{d''(x'',x_1''),d'''(x''',x_1''')\} \le d = 2$$

and

$$d'(f'(x'',x'''),f'(x_1'',x_1''')) \le \alpha' \cdot d$$

for $\alpha' = \dfrac{1}{2}$ but condition (1.16) is not satisfied, since

$$1 = d'(f'(x'',x'''),f'(x_1'',x_1''')) > \alpha' \max\{d''(x'',x_1''),d'''(x''',x_1''')\} = \frac{1}{2}.$$

It is easy to see that, if $f''$ and $f'''$ are defined similarly, i.e., $f''(x''',x') = \frac{x'''+x'}{2}$, for all $x''' \in G'''$ and $x' \in G'$, $f'''(x',x'') = \frac{x'+x''}{2}$, for all $x' \in G'$ and $x'' \in G''$, then any triplet $(a,a,a)$ with $a \in [0,1]$ is a fixed point of the triplet $f=(f',f'',f''')$ .

**Remark 1.3.** In view of the previous Remark 1.2, if $f=(f',f'',f''')$ is a contraction of $G$, then this implies that

$$d'(f'(x'',x'''),f'(x_1'',x_1''')) \le \alpha' \cdot \{d''(x'',x_1''),d'''(x''',x_1''')\}, \forall x'',x_1'' \in G'', x''',x_1''' \in G'''; \tag{1.17}$$

$\forall x'', x''_1 \in G'', x''', x'''_1 \in G'''$;

(1.18) $$d''(f''(x''', x'), f''(x'''_1, x'_1)) \le \alpha'' \cdot \max\{d'''(x''', x'''_1), d'(x', x'_1)\},$$
$$\forall x''', x'''_1 \in G''', x', x'_1 \in G';$$

(1.19) $$d'''(f'''(x', x''), f'''(x'_1, x''_1)) \le \alpha''' \cdot \max\{d'(x', x'_1), d''(x'', x''_1)\},$$
$$\forall x', x'_1 \in G', x'', x'''_1 \in G'',$$

Note that in the presence of the contraction conditions (1.17), (1.18), (1.19) for $f = (f', f'', f''')$, the proof of Theorem 1.1 follows by the Banach contraction principle.

**A simpler proof of Theorem 1.1.**

In this case we consider the mapping $F : G \to G$, defined by

(1.20) $$F(x', x'', x''') = (f'(x'', x'''), f''(x''', x'), f'''(x', x'')), \forall (x', x'', x''') \in G,$$

and consider on the product space $G$ the max metric $d$ given by

(1.21) $$d((x', x'', x'''), (y', y'', y''')) = \max\{d'(x', y'), d''(x'', y''), d'''(x''', y''')\},$$

for all $(x', x'', x'''), (y', y'', y''') \in G$.

Now, by denoting
$$\alpha = \max\{\alpha', \alpha'', \alpha'''\},$$
it is easy to prove that, in view of (1.17), (1.18) and (1.19), $F$ is an $\alpha$-contraction on the complete metric space $(G, d)$. Therefore, by Banach contraction mapping principle [1], we conclude that there exists a unique fixed point $(a', a'', a''')$ of $F$ in $G$, that is,
$$F(a', a'', a''') = (a', a'', a'''),$$
which, by virtue of (1.20), implies (1.4).

**Remark 1.4.** Note that the second proof of Theorem 1.1 has been also indicated by Ivanov [6] in the review he has written in Zentralblatt für Mathematik to the article of Miron Nicolescu [9].

At a certain step, in the proofs of Theorem 1.1 we used the continuity of the mappings $f', f'', f'''$. But, as it is well known, see for example [8, 10] and the monograph [3], it is possible to get the conclusion of the contraction mapping principle for discontinuous mappings, too. This has been done for the first time by Kannan [7], see also [4] and [5].

The next theorem is an extension of Theorem 1.1 to Kannan type contractive mappings.

**Theorem 1.5.** *Let $(G', d')$, $(G'', d'')$, $(G''', d''')$ be complete metric spaces and suppose $f = (f', f'', f''')$ is a Kannan type contraction of $G$, that is, there exists $\alpha', \alpha'', \alpha''' \in (0, 1/2)$ such that*

(1.22) $$d'(f'(x'', x'''), f'(y'', y''')) \le \alpha' \left[d'(x', f'(x'', x''')) + d'(y', f'(y'', y'''))\right],$$

*for all $x', y' \in G', x'', y'' \in G'', x''', y''' \in G'''$;*

(1.23) $$d''(f''(x''', x'), f''(y''', y')) \le \alpha'' \left[d''(x'', f''(x''', x')) + d''(y'', f''(y''', y'))\right],$$

*for all* $x', y' \in G', x'', y'' \in G'', x''', y''' \in G'''$*;*

$$(1.24)\quad d'''(f'''(x', x''), f'''(y', y'')) \le \alpha''' \left[ d'''(x''', f'''(x', x'')) + d''(y''', f'''(y', y'')) \right],$$

*for all* $x', y' \in G', x'', y'' \in G'', x''', y''' \in G'''$*.*

*Then there exists a unique triplet* $(a', a'', a''') \in G' \times G'' \times G'''$ *such that*

$$(1.25)\quad f'(a'', a''') = a'; \quad f''(a''', a') = a''; \quad f'''(a', a'') = a'''.$$

*Proof.*

Consider the mapping $F : G \to G$, defined by

$$(1.26)\quad F(x', x'', x''') = (f'(x'', x'''), f''(x''', x'), f'''(x', x'')), \forall (x', x'', x''') \in G,$$

and consider on the product space $G$ the metric $d$ given by

$$(1.27)\quad d((x', x'', x'''), (y', y'', y''')) = d'(x', y') + d''(x'', y'') + d'''(x''', y'''),$$

for all $(x', x'', x'''), (y', y'', y''') \in G$. Since, by hypothesis, $(G', d')$, $(G'', d'')$, $(G''', d''')$ are complete metric spaces, it follows that $(G, d)$ is a complete metric space, too.

Now, by denoting

$$\alpha = \max\{\alpha', \alpha'', \alpha'''\},$$

it is easy to see that the contractions conditions (1.22), (1.23) and (1.24), also hold with $\alpha', \alpha'', \alpha'''$ replaced by $\alpha$:

$$(1.28)\quad d'(f'(x'', x'''), f'(y'', y''')) \le \alpha \left[ d'(x', f'(x'', x''')) + d'(y', f'(y', y''')) \right],$$

for all $x', y' \in G', x'', y'' \in G'', x''', y''' \in G'''$;

$$(1.29)\quad d''(f''(x''', x'), f''(y''', y')) \le \alpha \left[ d''(x'', f''(x''', x')) + d''(y'', f''(y''', y')) \right],$$

for all $x', y' \in G', x'', y'' \in G'', x''', y''' \in G'''$;

$$(1.30)\quad d'''(f'''(x', x''), f'''(y', y'')) \le \alpha \left[ d'''(x''', f'''(x', x'')) + d''(y''', f'''(y', y'')) \right],$$

for all $x', y' \in G', x'', y'' \in G'', x''', y''' \in G'''$. Now, by summing up (1.28), (1.29) and (1.30) we obtain a Kannan type contraction condition for the mapping $F : G \to G$:

(1.31)

$$d(F(x), F(y)) \le \alpha \left[ d(x, F(x)) + d(y, F(y)) \right], \forall x = (x', x'', x'''), y = (y', y'', y''') \in G.$$

Now, conclusion follows by Kannan fixed point theorem [7].

However, for the sake of completeness and especially in order to illustrate the way we proceed when $F$ is not more continuous, we shall present a detailed proof, different from that given in [3].

Take $x_0 \in G$ arbitrary and consider the Picard orbit of $F$ starting from $x_0$, i.e., the sequence $\{x_n\}$, given by $x_n = F^n(x_0)$, $n \ge 1$.

By taking $x := x_{n-1}$, $y := x_n$ in (1.31), we obtain

$$d(F(x_{n-1}), F(x_n)) \le \alpha \left[ d(x_{n-1}, F(x_{n-1})) + d(x_n, F(x_n)) \right]$$

which yields

$$d(x_n, x_{n+1}) \le \alpha \left[ d(x_{n-1}, x_n) + d(x_n, x_{n+1}) \right],$$

i.e.,

$$(1.32)\quad d(x_n, x_{n+1}) \le \frac{\alpha}{1-\alpha} d(x_{n-1}, x_n), n \ge 1.$$

Denote $\delta = \frac{\alpha}{1-\alpha}$. Since $0 < \alpha < 1/2$, we get $0 < \delta < 1$ and therefore

(1.33) $$d(x_n, x_{n+1}) \leq \delta \cdot d(x_{n-1}, x_n), n \geq 1.$$

Now, by (1.33) we obtain

(1.34) $$d(x_n, x_{n+1}) \leq \delta^n \cdot d(x_0, x_1), n \geq 1$$

and

(1.35) $$d(x_n, x_{n+p}) \leq (\delta + \delta^2 + \cdots + \delta^p) \cdot d(x_{n-1}, x_n), n \geq 1,$$

respectively. By (1.34) and (1.35) we get

$$d(x_n, x_{n+p}) \leq \frac{\delta^n}{1-\delta} \cdot d(x_0, x_1), n, p \geq 1,$$

which shows that $\{x_n\}$ is a Cauchy sequence in the complete metric space $(G, d)$. Denote

$$a = \lim_{n\to\infty} F(x_n).$$

Next,

$$d(a, F(a)) \leq d(a, x_{n+1}) + d(x_{n+1}, F(a)) = d(a, x_{n+1}) + d(F(x_n), F(a))$$
$$\leq d(a, x_{n+1}) + \alpha[d(x_n, x_{n+1}) + d(a, F(a))].$$

The previous inequalities show that

$$d(a, F(a)) \leq \frac{1}{1-\alpha} d(a, x_{n+1}) + \frac{\alpha}{1-\alpha} d(x_n, x_{n+1}, n \geq 1.$$

By letting $n \to \infty$ in the previous inequality we obtain $d(a, F(a)) = 0$, i.e., $a$ is a fixed point of $F$. If we denote $a = (a', a'', a''')$, this means

$$f'(a'', a''') = a'; \quad f''(a''', a') = a''; \quad f'''(a', a'') = a'''.$$

To prove that the fixed point $a = (a', a'', a''')$ is unique, assume there exists another fixed point $b = (b', b'', b''')$ of $F$ in $G$. Then, by (1.31) we get

$$d(a, b) = d(F(a), F(b)) \leq \alpha[d(a, F(a)) + d(b, F(b))] = 0,$$

which proves the uniqueness. □

**Remark 1.6.** From the proof of Theorem 1.5 we could obtain, like in [3], both *a priori* and *a posteriori* error estimates for Picard iteration in a unique formula:

(1.36) $$d(x_{n+i-1}, a) \leq \frac{\delta^i}{1-\delta} d(x_n, x_{n-1}), \quad n = 0, 1, 2, \ldots; i = 1, 2, \ldots$$

Indeed, if we take $i = 1$ in (1.36), then we get the a posteriori error estimate

$$d(x_n, a) \leq \frac{\delta}{1-\delta} d(x_n, x_{n-1}), \quad n = 1, 2, \ldots,$$

while, for $n = 1$ in (1.36), we get the a priori error estimate

$$d(x_i, a) \leq \frac{\delta^i}{1-\delta} d(x_1, x_0), \quad i = 1, 2, \ldots,$$

where $\delta = \frac{\alpha}{1-\alpha}$ and

$$\alpha = \max\{\alpha', \alpha'', \alpha'''\}.$$

**Remark 1.7.** A similar result to the fixed point theorem given by Theorem 1.5 can be established by using a Chatterjea type contraction condition, see [4, 8, 10].

**Theorem 1.8.** *Let $(G', d')$, $(G'', d'')$, $(G''', d''')$ be complete metric spaces and suppose $f = (f', f'', f''')$ is a Chatterjea type contraction of $G$, that is, there exists $\alpha', \alpha'', \alpha''' \in (0, 1/2)$ such that*

$$d'(f'(x'', x'''), f'(y'', y''')) \le \alpha' \left[d'(x', f'(y'', y''')) + d'(y', f'(x'', x'''))\right], \tag{1.37}$$

*for all $x', y' \in G', x'', y'' \in G'', x''', y''' \in G'''$;*

$$d''(f''(x''', x'), f''(y''', y')) \le \alpha'' \left[d''(x'', f''(y''', y')) + d''(y'', f''(x''', x'))\right], \tag{1.38}$$

*for all $x', y' \in G', x'', y'' \in G'', x''', y''' \in G'''$;*

$$d'''(f'''(x', x''), f'''(y', y'')) \le \alpha''' \left[d'''(x''', f'''(y', y'')) + d''(y''', f'''(x', x''))\right], \tag{1.39}$$

*for all $x', y' \in G', x'', y'' \in G'', x''', y''' \in G'''$.*

*Then there exists a unique triplet $(a', a'', a''') \in G' \times G'' \times G'''$ such that*

$$f'(a'', a''') = a'; \quad f''(a''', a') = a''; \quad f'''(a', a'') = a'''. \tag{1.40}$$

*Proof.*

Similarly to the proof of Theorem 1.5, consider the mapping $F : G \to G$, defined by

$$F(x', x'', x''') = (f'(x'', x'''), f''(x''', x'), f'''(x', x'')), \forall (x', x'', x''') \in G, \tag{1.41}$$

and on the product space $G$ consider the metric $d$ given by

$$d((x', x'', x'''), (y', y'', y''')) = d'(x', y') + d''(x'', y'') + d'''(x''', y'''), \tag{1.42}$$

for all $(x', x'', x'''), (y', y'', y''') \in G$. Since, by hypothesis, $(G', d')$, $(G'', d'')$, $(G''', d''')$ are complete metric spaces, it follows that $(G, d)$ is a complete metric space, too.

Now, by denoting

$$\alpha = \max\{\alpha', \alpha'', \alpha'''\},$$

it is easy to see that the contractions conditions (1.37), (1.38) and (1.39), also hold with $\alpha', \alpha'', \alpha'''$ replaced by $\alpha$:

$$d'(f'(x'', x'''), f'(y'', y''')) \le \alpha \left[d'(x', f'(y'', y''')) + d'(y', f'(x'', x'''))\right], \tag{1.43}$$

for all $x', y' \in G', x'', y'' \in G'', x''', y''' \in G'''$;

$$d''(f''(x''', x'), f''(y''', y')) \le \alpha \left[d''(x'', f''(y''', y')) + d''(y'', f''(x''', x'))\right], \tag{1.44}$$

for all $x', y' \in G', x'', y'' \in G'', x''', y''' \in G'''$;

$$d'''(f'''(x', x''), f'''(y', y'')) \le \alpha \left[d'''(x''', f'''(y', y'')) + d''(y''', f'''(x', x''))\right], \tag{1.45}$$

for all $x', y' \in G', x'', y'' \in G'', x''', y''' \in G'''$. Now, by summing up (1.43), (1.44) and (1.45) we obtain a Chatterjea type contraction condition [4] for the mapping $F : G \to G$:

(1.46)

$$d(F(x), F(y)) \le \alpha \left[d(x, F(y)) + d(y, F(x)\right], \forall x = (x', x'', x'''), y = (y', y'', y''') \in G.$$

From (1.46), we obtain the following two inequalities

$$d(F(x), F(y)) \le \delta d(x, y) + 2\delta \cdot d(y, F(x)), \forall x, y \in G, \tag{1.47}$$

$$d(F(x), F(y)) \leq \delta d(x, y) + 2\delta \cdot d(x, F(x)), \forall x, y \in G, \tag{1.48}$$

where $\delta = \frac{\alpha}{1-\alpha}$. Now, in a standard way, see [2], by (1.47) we obtain the existence of the fixed point of $F$, while, by means of (1.48) we obtain its uniqueness. □

**Remark 1.9.** We can prove Theorem 1.8 in a similar manner to the first proof of Theorem 1.1, by working separately in each complete metric space $(G', d')$, $(G'', d'')$ and $(G''', d''')$. By using the same technique, one can prove the following interesting result.

**Theorem 1.10.** *Let $(G', d')$, $(G'', d'')$, $(G''', d''')$ be complete metric spaces and suppose $f', f'', f'''$ are hybrid type contractions, that is, there exists $\alpha', \alpha''' \in (0, 1)$ and $\alpha'' \in (0, 1/2)$ such that*

$$d'(f'(x'', x'''), f'(y'', y''')) \leq \alpha' \cdot \max\{d''(x'', y''), d'''(x''', y''')\},$$

*for all $x'', y'' \in G''$, $x''', y''' \in G'''$;*

$$d''(f''(x''', x'), f''(y''', y')) \leq \alpha'' \left[d''(x'', f''(x''', x')) + d''(y'', f''(y''', y'))\right],$$

*for all $x', y' \in G'$, $x'', y'' \in G''$, $x''', y''' \in G'''$;*

$$d'''(f'''(x', x''), f'''(x'_1, x''_1)) \leq \alpha''' \cdot \max\{d'(x', x'_1), d''(x'', x''_1)\}, \forall x', x'_1 \in G', x'', x'''_1 \in G''.$$

*Then there exists a unique triplet $(a', a'', a''') \in G' \times G'' \times G'''$ such that*

$$f'(a'', a''') = a'; \quad f''(a''', a') = a''; \quad f'''(a', a'') = a'''.$$

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