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HALPERN-TYPE METHOD FOR NON-SELF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. The purpose of this paper is to introduce a Halpern-type algorithm and prove that the algorithm converges strongly to a common fixed point of a finite family of non-self nonexpansive mappings in Banach spaces. The algorithm does not require metric projection or sunny nonexpansive retraction mapping. In addition, a numerical example which supports our main result is presented. Our results improve and unify most of the results that have been proved for this important class of nonlinear operators.

1. Introduction

Let K be a nonempty, closed and convex subset of a real Banach space E. A mapping $T : K \to E$ is called *contraction* if there exists $L \in [0, 1)$ such that

$$||Tx - Ty|| \leq L||x - y|| \text{ for all } x, y \in K. \tag{1.1}$$

If in this case, (1.1) is satisfied with $L = 1$, then the mapping T is called a *non-expansive*. Approximation of fixed points of nonlinear mappings is an active area of nonlinear functional analysis due to the fact that many nonlinear problems can be reformulated as fixed point equations of nonlinear mappings. This research area dates back to Picard's and Banach's time. In fact, the well-known Banach contraction principle states that the Picard iterates $\{T^n x_0\}$ converge to the unique fixed point of T, whenever T is a contraction of a complete metric space. However, if T is not a contraction (nonexpansive, say), then the Picard iterates $\{T^n x_0\}$ fail, in general, to converge; hence, other iterative methods are needed.

The most general iterative algorithm for nonexpansive mappings studied by many authors is the following scheme known as *Mann iteration scheme* in the light of Mann [8].

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, n \geq 1, \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in the unit interval $(0, 1)$, and satisfies certain mild conditions. Mann's algorithm (1.2) has been studied extensively (see, for example, [14, 18, 20, 24, 26–28]), and in particular, it is known that if T is nonexpansive and has a fixed point, then the sequence $\{x_n\}$ generated by Mann's algorithm (1.2)

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converges weakly to a fixed point of T provided that the sequence $\{\alpha_n\}$ satisfies certain conditions. This algorithm, however, does not converge in the strong topology in general (see, eg, [3]).

Other iteration processes have been also used to approximate fixed points of nonexpansive mappings in Hilbert and/or Banach spaces (see, eg, [7, 25]). One of them is now known as Halpern's iteration process [7].

Halpern [7] initiated the study of an (explicit) iterative method in Hilbert spaces:

$$x_{n+1} := \alpha_{n+1}u + (1-\alpha_{n+1})Tx_n, n \geq 0, \tag{1.3}$$

where u, x_0 are arbitrary initial points in K, T is nonexpansive mapping of K into itself and $\{\alpha_n\}$ is a control sequence in $(0,1)$. He proved the strong convergence of the sequence $\{x_n\}$ to a fixed point of T provided that $\{\alpha_n\} \subset (0,1)$ satisfies certain mild conditions.

Next, consider r nonexpansive self-mappings $T_1, T_2, ..., T_r$. For a sequence $\{\alpha_n\} \subseteq (0,1)$ and an arbitrary $u \in K$, let the sequence $\{x_n\}$ in K be iteratively defined by $x_0 \in K$

$$x_{n+1} := \alpha_{n+1}u + (1-\alpha_{n+1})T_{n+1}x_n, n \geq 0, \tag{1.4}$$

where $T_n = T_{n \pmod r}$.

In 1996, Bauschke [2] studied the iterative process (1.4) for finding a common fixed point of a finite family of nonexpansive mappings. Under suitable conditions he proved that the sequence $\{x_n\}$ converges strongly to a common fixed point $P_F u$ of $T_1, T_2, ..., T_r$ in K, where $P_F : H \to F = \cap_{i=1}^r F(T_i)$ is the metric projection, in Hilbert space.

In 2002, Takahashi *et al.* [19], extended Bauschke's result to uniformly convex Banach spaces. More precisely, they proved the following result.

Theorem 1.1. *Let K be a nonempty, closed and convex subset of a uniformly convex Banach space E which has a uniformly Gâteaux differentiable norm. Let $T_i : K \to K,\ i = 1, ..., r$, be a family of nonexpansive mappings with*

$$\begin{aligned} F := \bigcap_{i=1}^{r} F(T_i) &= F(T_r T_{r-1} ... T_1) \\ &= F(T_{r-1}T_{r-2}...T_1T_r) \\ &= ...\ F(T_1T_r...T_2) \neq \emptyset. \end{aligned}$$

For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm in (1.4) where $\{\alpha_n\}$ is a real sequence which satisfies the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and (iii) $\sum_{n=0}^{\infty} |\alpha_{n+r} - \alpha_n| < \infty$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, ..., T_r\}$.

Furthermore, in 2006, Zhou *et al.* [21] proved the following result in Banach spaces which has a weakly continuous normalized duality mapping J, where the mapping J from E into 2^{E^*} is defined by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 = ||f^*||^2\},$$

where $\langle ., . \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single-valued and if the norm of E is uniformly Gâteaux differentiable then the duality mapping J is norm-to-weak* uniformly continuous on bounded subsets of E (see, [5] for the details).

Theorem 1.2. *Let K be a nonempty, closed and convex subset of a reflexive Banach space E which has a uniformly Gâteaux differentiable norm, and a weakly continuous duality mapping J. Let $T_i : K \to K$, $i = 1, ..., r$, be a family of nonexpansive mappings with $F := \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$ and $\bigcap_{i=1}^{r} F(T_i) = F(T_r T_{r-1} ... T_1) = F(T_1 T_r ... T_2)$ $= ... = F(T_{r-1} T_{r-2} ... T_1 T_r)$. Assume that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm in (1.4) where $\{\alpha_n\}$ is a real sequence which satisfies the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum \alpha_n = \infty$ and $\{x_n\}$ is weakly asymptotically regular. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, ..., T_r\}$.*

Remark 1.3. *We remark that in all the above results, the operator T remains a self-mapping of a nonempty, closed and convex subset K of a Banach space E. If, however, the domain of T, $D(T)$, is a proper subset of E (and this is the case in several applications), and T maps $D(T)$ into E, then the iterative processes (1.3) and (1.4) studied by these authors may fail to be well defined.*

In order to deal with the non-self mappings, many researchers have made significant progress by employing the concept of projection or *sunny nonexpansive retraction* of the real Banach spaces E onto its closed and convex subset K of E (see, e.g. [9–11, 16, 20]).

In 2008, Matsushita and Takahashi [11] studied strong convergence theorem for nonexpansive non-self-mappings in the framework of a real uniformly convex Banach spaces. In fact, they proved the following theorem.

Theorem 1.4. *Let E be a uniformly convex Banach space whose norm is uniformly Gateâux differentiable, let K be a nonempty, closed and convex subset of E and let T be a nonexpansive mapping from K into E with $F(T) \neq \emptyset$. Suppose that K is a sunny nonexpansive retract of E. Let $\{\alpha_n\}$ be a sequence such that $0 \leq \alpha_n \leq 1$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$ and $\sum |\alpha_{n+1} - \alpha_n| < \infty$. Let u and x_0 be elements of K. Suppose that $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) Q T x_n, for\ n = 0, 1, 2, ..., \tag{1.5}$$

where Q is a sunny nonexpansive retraction from E onto K. Then $\{x_n\}$ converges strongly to $z \in F(T)$.

Several authors have also studied implicit and explicit iterative schemes of the type (1.5) (see, e.g., [7, 9–11, 13, 15, 16, 23] and the references therein) for non-self mappings. However, we observe that the method of calculating Q is generally difficult in applications, even in Hilbert spaces when Q is metric projection. It may require an approximating algorithm for itself. To avoid the necessity of using an auxiliary mapping Q, Colao and Marino [6] introduced a new search strategy for the coefficient α_n which makes Mann algorithm well defined in the Hilbert space settings. They obtained weak and strong convergence of the algorithm for nonexpansive non-self mappings in Hilbert spaces.

It is our purpose in this paper to define an algorithm and obtain a strong convergence theorems to a fixed point of a finite family of non-self nonexpansive mappings in Banach spaces more general than Hilbert spaces. The involvement of the projection or sunny nonexpansive retraction mapping in the algorithm is dispensed with. Our results improve Theorem MT and the results of Colao and Marino [6] to Banach spaces more general than Hilbert spaces and/or to a finite family of nonlinear *non-self* mappings.

2. Preliminaries

A real Banach space E with dual E^* is called *strictly convex* if for all $x, y \in E$, $x \neq y$, $||x|| = ||y|| = 1$, we have $||\lambda x + (1-\lambda)y|| < 1$, $\forall \lambda \in (0,1)$. The Banach space E is said to be *uniformly convex* if, given $\varepsilon > 0$, there exists $\delta > 0$, such that, for all $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x-y\| \geq \varepsilon$, $\left\|\frac{1}{2}(x+y)\right\| \leq 1-\delta$. It is well known that L_p, ℓ_p and Sobolev spaces W_m^p, $(1 < p < \infty)$, are uniformly convex.

The norm is said to be *uniformly Gâteaux differentiable* if for each $y \in S_1(0) := \{x \in E : ||x|| = 1\}$ the limit $\lim_{t\to 0} \frac{||x+ty||-||x||}{t}$ exists uniformly for $x \in S_1(0)$. It is well known that L_p spaces, $1 < p < \infty$, have uniformly Gâteaux differentiable norm (see e.g., [17]). Furthermore, if E has a uniformly Gâteaux differentiable norm, then the duality mapping is norm-to-weak* uniformly continuous on bounded subsets of E.

Let $K \subseteq E$ be a closed, convex and Q be a mapping of E onto K. Then Q is said to be *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $x \in E$ and $t \geq 0$. A mapping Q of E into E is said to be a *retraction* if $Q^2 = Q$. If a mapping Q is a retraction, then $Qz = z$ for every $z \in R(Q)$, *range of* Q. A subset K of E is said to be a *sunny nonexpansive retract* of E if there exists a sunny nonexpansive retraction of E onto K and it is said to be a *nonexpansive retract* of E if there exists a nonexpansive retraction of E onto K. *If $E = H$, the metric projection P_K is a sunny nonexpansive retraction from H to any closed convex subset of H.*

Let K be a nonempty subset of a Banach space E. For $x \in K$, the *inward set* of x, $I_K(x)$, is defined by $I_K(x) := \{x + \lambda(w - x) : w \in K, \lambda \geq 1\}$. A mapping $T : K \to E$ is called *inward* if $Tx \in I_K(x)$ for all $x \in K$, T is called *weakly inward* if $Tx \in cl[I_K(x)]$ for all $x \in K$, where $cl[I_K(x)]$ denotes the closure of the inward set. Every self-map is trivially inward.

In the sequel, we shall need the following lemmas.

Lemma 2.1 ([12]). *Let E be a smooth real Banach space and J be the duality mapping. Then, for each $x, y \in E$, one has*

$$||x + y||^2 \leq ||x||^2 + \langle y, J(x + y)\rangle.$$

Lemma 2.2 ([23]). *Let K be a nonempty, closed and convex subset of a real Banach space E. Let $T : K \to E$ be a nonexpansive mapping satisfying weakly inward condition with $F(T) \neq \emptyset$ and $f : K \to K$ be a contraction mapping. Then for $t \in (0,1)$, there exists a sequence $\{y_t\} \subset K$ satisfying the following condition:*

$$y_t = (1-t)f(y_t) + tT(y_t). \tag{2.1}$$

Furthermore, if E is a strictly convex and reflexive real Banach space having a uniformly Gâteaux differentiable norm, then $\{y_t\}$converges strongly to a fixed point

z of T as $t \to 1^-$ such that z is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (f-I)z, J(y-z)\rangle \leq 0, \text{ for all } y \in F(T). \tag{2.2}$$

Lemma 2.3 ([4])**.** *Let K be a nonempty, closed and convex subset of a strictly convex Banach space E. Let $T_i : K \to E,\ i = 1, 2, ..., r$ be a family of nonexpansive mappings such that $\bigcap_{i=1}^r F(T_i) \neq \emptyset$. Let $\alpha_0, \alpha_1, \alpha_2, ..., \alpha_r$ be real numbers in $(0, 1)$ such that $\sum_{i=0}^r \alpha_i = 1$ and $S := \alpha_0 I + \alpha_1 T_1 + ... + \alpha_r T_r$. Then S is nonexpansive and $F(S) = \bigcap_{i=1}^r F(T_i)$.*

Lemma 2.4 ([22])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n, n \geq n_0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions: (i) $\sum_{n=0}^\infty \alpha_n = \infty$, and (ii) $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=0}^\infty \delta_n < \infty$. Then, $\lim_{n\to\infty} a_n = 0$.

Let μ be a continuous, linear functional on l^∞ and let $(a_0, a_1, ...) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu((a_0, a_1, ...))$. We call μ a Banach limit [1] when μ satisfies $||\mu|| = \mu_n(1) = 1$ and $\mu(a_{n+1}) = \mu(a_n)$ for all $(a_0, a_1, ...) \in l^\infty$. We shall use the following lemma.

Lemma 2.5. *Let a be a real number and let $(a_0, a_1, ...) \in l^\infty$ such that $\mu_n(a_n) \leq a$ for all Banach limit μ and $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$. Then $\limsup_{n\to\infty} a_n \leq a$.*

The proof of the following lemma basically uses the method of proof of Lemma 1 of Colao and Marino [6].

Lemma 2.6. *Let K be a nonempty, closed and convex subset of E. Let $T : K \to E$ be a mapping. Define $h : K \to \mathbb{R}$ by $h(x) = \sup\{\lambda \geq 0 : (1-\lambda)x + \lambda Tx \in K\}$. Then for any $x \in K$ the following hold:*

(1) *$h(x) \in [0, 1]$ and $h(x) = 1$ if and only if $Tx \in K$;*
(2) *if $\beta \in [0, h(x)]$, then $(1-\beta)x + \beta Tx \in K$;*
(3) *if T is an inward mapping, then $h(x) > 0$;*
(4) *whenever, $Tx \notin K$, $(1-h(x))x + h(x)Tx \in \partial K$.*

Proof. We note that the proofs of (1) and (2) follow directly from the definition of $h(x)$. Now, we prove (3). Suppose that T is an inward mapping. Then for any arbitrary fixed element v we have $Tx = x + c(v - x)$ for some $c \geq 1$, $x \in K$. This implies that

$$\frac{1}{c}Tx = \Big(\frac{1}{c} - 1\Big)x + v,$$

and hence

$$\frac{1}{c}Tx + \Big(1 - \frac{1}{c}\Big)x \in K, \text{ for some } c \geq 1. \tag{2.3}$$

Therefore,

$$h(x) = \sup\{\lambda \geq 0 : (1-\lambda)x + \lambda Tx \in K\} \geq \frac{1}{c} > 0. \tag{2.4}$$

(4) Note that $h(x) > 0$ by (3) and that $(1-h(x))x + h(x)Tx \in K$. Now, let $\{w_n\} \subset (h(x), 1)$ be a sequence of real numbers converging to $h(x)$ and note that by the definition of $h(x)$, we have $z_n := (1-w_n)x + w_nTx \notin K$, for any $n \in \mathbb{N}$. Since $w_n \to h(x)$ and

$$\begin{aligned} ||z_n - ((1-h(x))x + h(x)T(x))|| &= ||(1-w_n)x + w_nTx - ((1-h(x))x \\ &\quad +h(x)Tx)|| \\ &= |w_n - h(x)|||x - Tx||, \end{aligned}$$

it follows that $z_n \to (1-h(x))x + h(x)Tx \in K$. Therefore,

$$(1-h(x))x + h(x)Tx \in \partial K.$$

□

3. Main Results

Theorem 3.1. *Let K be a nonempty, closed and convex subset of a strictly convex and reflexive real Banach space E which has a uniformly Gâteaux differentiable norm. Let $f : K \to K$ be a contraction with constant β. Let $T_i : K \to E,\ i = 1, ..., r$ be a finite family of inward nonexpansive mappings with $\mathcal{F} = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be generated from arbitrary initial point $x_0 \in K$ by*

$$\begin{cases} \lambda_0 = \min\{\frac{1}{2}, h(x_0)\}; \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)(\lambda_n Sx_n + (1-\lambda_n)x_n); \\ \lambda_{n+1} \in (0, \min\{\lambda_n, h(x_{n+1})\}], n \geq 0, \end{cases} \tag{3.1}$$

where $S = a_0I + a_1T_1 + a_2T_2 + ... + a_rT_r$, for $0 < a_i < 1, i = 0, 1, ..., r$, $\sum_{i=0}^{r} a_i = 1$, and $\lambda_n := \sup\{\lambda \geq 0 : (1-\lambda)x_n + \lambda Sx_n \in K\}$. If the real sequence $\{\alpha_n\}$ satisfies the following conditions: (i) $\lim_{n\to\infty}\ \alpha_n = 0$; (ii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; (iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, and there exists $\epsilon > 0$ such that $\lambda_n \geq \epsilon$, for all $n \geq 0$, then $\{x_n\}$ converges strongly to some common fixed point z of the family T_i $(i = 1, 2, ..., r)$ such that z is the unique solution in $\mathcal{F}$ to the following variational inequality:

$$\langle (f-I)z, J(y-z)\rangle \leq 0, \ \text{for all } y \in \mathcal{F}. \tag{3.2}$$

Proof. By Proposition 3.4 of [4] we obtain that S is weakly inward nonexpansive mapping. Let $y_n = (1-\lambda_n)x_n + \lambda_n Sx_n, n \geq 0$. Then we have $x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, n \geq 0$. Now, for $p \in F(S)$, one easily shows by induction that $||x_n - p|| \leq \max\{||x_0 - p||, \frac{1}{1-\beta}||f(p) - p||\}$, for all integers $n \geq 0$, and hence $\{x_n\}$, $\{f(x_n)\}$ and $\{Sx_n\}$ are bounded. In addition, from (3.1) we obtain

$$\begin{aligned} ||x_{n+1} - x_n|| &= ||(1-\alpha_n)(y_n - y_{n-1}) + (\alpha_n - \alpha_{n-1})(f(x_{n-1}) - y_{n-1}) \\ &\quad +\alpha_n(f(x_n) - f(x_{n-1}))||, \\ &\leq (1-\alpha_n)||y_n - y_{n-1}|| + |\alpha_n - \alpha_{n-1}|||f(x_{n-1}) - y_{n-1}|| \\ &\quad +\alpha_n\beta||x_n - x_{n-1}||, \\ &\leq (1-\alpha_n)||y_n - y_{n-1}|| + |\alpha_n - \alpha_{n-1}|M + \alpha_n\beta||x_n - x_{n-1}||, \end{aligned} \tag{3.3}$$

where $M := \sup ||f(x_{n-1}) - y_{n-1}|| < \infty$.

$$\begin{aligned} ||y_n - y_{n-1}|| &= ||(1-\lambda_n)x_n + \lambda_n Sx_n - [(1-\lambda_{n-1})x_{n-1} + \lambda_{n-1}Sx_{n-1}]|| \\ &= ||\lambda_n(Sx_n - Sx_{n-1}) + (\lambda_n - \lambda_{n-1})Sx_{n-1} + (1-\lambda_n)(x_n - x_{n-1}) \\ &\quad -(\lambda_n - \lambda_{n-1})x_{n-1}|| \\ &\leq \lambda_n||x_n - x_{n-1}|| + |\lambda_n - \lambda_{n-1}|||Sx_{n-1}|| + (1-\lambda_n)||x_n - x_{n-1}|| \\ &\quad +|\lambda_n - \lambda_{n-1}|||x_{n-1}|| \\ &\leq ||x_n - x_{n-1}|| + |\lambda_n - \lambda_{n-1}|(||Sx_{n-1}|| + ||x_{n-1}||). \end{aligned} \tag{3.4}$$

Thus, from (3.3) and (3.4) we obtain

$$\begin{aligned} ||x_{n+1} - x_n|| &\leq (1-(1-\beta)\alpha_n)||x_n - x_{n-1}|| + |\lambda_n - \lambda_{n-1}|M' \\ &\quad +|\alpha_n - \alpha_{n-1}|M', \end{aligned}$$

where $M' := \sup_n\{M + ||Sx_{n-1}|| + ||x_{n-1}||\} < \infty$. Note that $\sum_{n=0}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$ (since $\{\lambda_n\} \subset [0,1]$ is monotone decreasing) and conditions $(i)-(ii)$ of the hypotheses are satisfied. Thus, with the use of Lemma 2.4 we obtain that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| \to 0. \tag{3.5}$$

Furthermore, from (3.1) we have $||x_{n+1} - y_n|| = \alpha_n||f(x_n) - y_n|| \to 0$ as $n \to \infty$, and

$$||x_n - y_n|| \leq ||x_n - x_{n+1}|| + ||x_{n+1} - y_n|| \to 0 \text{ as } n \to \infty. \tag{3.6}$$

Thus, from the definition of y_n and the fact that $\inf_n\{\lambda_n\} > 0$, we obtain

$$||x_n - Sx_n|| = \frac{1}{\lambda_n}||y_n - x_n|| \to 0 \text{ as } n \to \infty. \tag{3.7}$$

Now, for each $t \in (0,1)$, let $z_t \in K$ be the unique fixed point of the contraction mapping S_t (see Lemma 2.2) given by

$$S_t x := (1-t)Sx + tf(x), \ x \in K.$$

Then,

$$z_t - x_n = t(f(z_t) - x_n) + (1-t)(Sz_t - x_n).$$

But the above implies that

$$\begin{aligned} ||z_t - x_n||^2 &= (1-t)\langle Sz_t - x_n, J(z_t - x_n)\rangle + t\langle f(z_t) - x_n, J(z_t - x_n)\rangle \\ &= (1-t)\langle Sz_t - Sx_n, J(z_t - x_n)\rangle + (1-t)\langle Sx_n - x_n, J(z_t - x_n)\rangle \\ &\quad +t\langle f(z_t) - z_t, J(z_t - x_n)\rangle + t||z_t - x_n||^2 \\ &\leq ||x_n - z_t||^2 + (1-t)||x_n - Sx_n||.||J(z_t - x_n)|| \\ &\quad +t\langle f(z_t) - z_t, J(z_t - x_n)\rangle, \end{aligned}$$

and hence,

$$\langle f(z_t) - z_t, J(x_n - z_t)\rangle \leq \frac{(1-t)||x_n - Sx_n||}{t}||z_t - x_n||.$$

Since $\{x_n\}$, $\{z_t\}$ and hence $\{Sx_n\}$ are bounded and $||x_n - Sx_n|| \to 0$ as $n \to \infty$ it follows from the last inequality that

$$\mu_n\langle f(z_t) - z_t, J(x_n - z_t)\rangle \leq 0. \tag{3.8}$$

Using the fact that E has a uniformly Gâteaux differentiable norm and tending $t \to 0$, inequality (3.8) provides

$$\mu_n\langle f(z) - z, J(x_n - z)\rangle \leq 0. \tag{3.9}$$

On the other hand, from (3.5) and the fact that J is norm to weak* uniformly continuous we have

$$\lim_{n\to\infty} \big|\langle f(z) - z, J(x_{n+1} - z)\rangle - \langle f(z) - z, J(x_n - z)\rangle\big| = 0. \tag{3.10}$$

Hence, by Lemma 2.5 we obtain that

$$\limsup_n \langle f(z) - z, J(x_n - z)\rangle \leq 0. \tag{3.11}$$

Now, from (3.1) and Lemma 2.1 we get

$$\begin{aligned} ||x_{n+1} - z||^2 &\leq ||\alpha_n(f(x_n) - f(z)) + (1-\alpha_n)(y_n - z)||^2 \\ &\quad +2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\ &\leq (\alpha_n\beta||x_n - z|| + (1-\alpha_n)||x_n - z||)^2 \\ &\quad +2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\ &\leq (1-\alpha_n(1-\beta))^2||x_n - z||^2 + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\ &\leq (1-\alpha_n(1-\beta))||x_n - z||^2 + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle, \\ &\leq (1-\alpha_n(1-\beta))||x_n - z||^2 + 2\alpha_n\langle f(z) - z, J(x_n - z)\rangle, \\ &\quad +2\alpha_n\langle f(z) - z, J(x_{n+1} - z) - J(x_n - z)\rangle \\ &\leq (1-\alpha_n(1-\beta))||z_n - z||^2 + \alpha_n\sigma_n, \end{aligned}$$

where $\sigma_n := 2\beta_n + 2\langle f(z) - z, J(x_{n+1} - z) - J(x_n - z)\rangle$, for $\beta_n := \langle f(z) - z, J(x_n - z)\rangle$. Note that from (3.10) and (3.11) we have that $\limsup \sigma_n \leq 0$. Therefore, by Lemma 2.4, $\{x_n\}$ converges strongly to a common fixed point z of $\{T_1, T_2, ..., T_r\}$. Furthermore, by Lemma 2.2, the point z satisfies the inequality (3.2). □

Remark 3.2. *We remark that, in practical applications, we may consider* $\lambda_{n+1} = \min\{\lambda_n, h(x_{n+1})\}$*, where* $\lambda_0 = \min\{\frac{1}{2}, h(x_0)\}$*; and* $h(x_n) = (1 - (\frac{9}{10})^{i_n})$ *for* $i_n := max\{i \in \mathbb{N} : (\frac{9}{10})^i x_n + (1 - (\frac{9}{10})^i)Sx_n \in K\}$.

We also note that $\lambda_0 = \min\{\frac{1}{2}, h(x_0)\}$ *could be replaced with* $\lambda_0 = \min\{a, h(x_0)\}$*; where* $a \in (0,1)$ *is a fixed and arbitrary value.*

If, in Theorem 3.1, we consider a single nonexpansive mapping, then we have the following corollary.

Corollary 3.3. *Let K be a nonempty, closed and convex subset of a strictly convex and reflexive real Banach space E which has a uniformly Gâteaux differentiable norm. Let $f : K \to K$ be a contraction. Let $T : K \to E$, be inward nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be generated from arbitrary initial point $x_0 \in K$ by*

$$\begin{cases} \lambda_0 = \min\{\frac{1}{2}, h(x_0)\}; \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)(\lambda_n T x_n + (1-\lambda_n)x_n); \\ \lambda_{n+1} \in (0, \min\{\lambda_n, h(x_{n+1})\}], n \geq 0, \end{cases}$$

where $\lambda_n := \sup\{\lambda \geq 0 : (1-\lambda)x_n + \lambda T x_n \in K\}$. *If the real sequence* $\{\alpha_n\}$ *satisfies the following conditions:* (i) $\lim_{n\to\infty}\ \alpha_n = 0$*;* (ii) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$*;* (iii) $\sum_{n=0}^{\infty}\alpha_n = \infty$*, and there exists* $\epsilon > 0$ *such that* $\lambda_n \geq \epsilon$*, for all* $n \geq 0$*, then* $\{x_n\}$ *converges strongly to* $z \in F(T)$ *such that* z *is the unique solution in* $F(T)$ *to the following variational inequality:*

$$\langle (f-I)z, J(y-z)\rangle \leq 0, \text{ for all } y \in F(T). \tag{3.12}$$

If, in Theorem 3.1, T_i $(i = 1,...,r)$ are self-mappings then the requirements that T_i $(i = 1, 2, ...r)$ are inward, and the assumption that there exists $\epsilon > 0$ such that $\lambda_n \geq \epsilon$, for all $n \geq 0$, may not be needed. In fact, Theorem 3.1 reduces to the following corollary which is one of the main results in [25].

Corollary 3.4. *Let* K *be a nonempty, closed and convex subset of a strictly convex and reflexive real Banach space* E *which has a uniformly Gâteaux differentiable norm. Let* $T_i : K \to K$*,* $i = 1,...,r$ *be a family of nonexpansive mappings with* $\mathcal{F} = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$*. Let* $f : K \to K$ *be contraction. For arbitrary* $x_0 \in K$*, let* $\{x_n\}$ *be generated by the algorithm*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\frac{1}{2}(Sx_n + x_n);$$

where $S = a_0 I + a_1 T_1 + a_2 T_2 + ... + a_r T_r$*, for* $0 < a_i < 1, i = 0, 1, ..., r$*,* $\sum_{i=0}^{r} a_i = 1$*. If the real sequence* $\{\alpha_n\}$ *satisfies the following conditions:* (i) $\lim_{n\to\infty}\ \alpha_n = 0$*;* (ii) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$*;* (iii) $\sum_{n=0}^{\infty}\alpha_n = \infty$*, then* $\{x_n\}$ *converges strongly to some common fixed point* z *of the family* T_i $(i = 1, 2, ..., r)$ *such that* z *is the unique solution in* $\mathcal{F}$ *to the following variational inequality:*

$$\langle (f-I)z, J(y-z)\rangle \leq 0, \text{ for all } y \in \mathcal{F}. \tag{3.13}$$

Proof. Note that $h(x_n) = 1$, for all $n \geq 0$ and hence the result follows from Theorem 3.1 with $\lambda_n = \frac{1}{2}$. □

We observe that if $f(x) = 0 \in K$, then we have the following result for approximating the minimum-norm point of fixed points of a finite family of nonexpansive non-self mappings.

Theorem 3.5. *Let* K *be a nonempty, closed and convex subset of a strictly convex and reflexive real Banach space* E*, containing zero, which has a uniformly Gâteaux differentiable norm. Let* $T_i : K \to E$*,* $i = 1,...,r$ *be a finite family of inward nonexpansive mappings with* $\mathcal{F} = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$*. Let* $\{x_n\}$ *be generated from arbitrary initial point* $x_0 \in K$ *by (3.1) with* $f(x) = 0$*. If the real sequence* $\{\alpha_n\}$ *satisfies the following conditions:* (i) $\lim_{n\to\infty}\ \alpha_n = 0$*;* (ii) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$*;* (iii) $\sum_{n=0}^{\infty}\alpha_n = \infty$*, and there exists* $\epsilon > 0$ *such that* $\lambda_n \geq \epsilon$*, for all* $n \geq 0$*, then* $\{x_n\}$ *converges strongly to some common fixed point* z *of the family* T_i $(i = 1, 2, ..., r)$ *such that* z *is the unique minimum norm point of* $\mathcal{F}$*.*

Remark 3.6. *If, in all the above theorems and corollaries, we have* $F(T)$ *is a subset of interior of* K*, then the assumption that there exists* $\epsilon > 0$ *such that* $\lambda_n \geq \epsilon\ \forall n \geq 0$*, may not be required.*

4. Numerical example

In this section, we consider two non-self nonexpansive mappings with conditions of Theorem 3.1 and present some numerical experiment result to explain the conclusion of the theorem.

Example 4.1. Let $E = \mathbb{R}^2$ and let $K = B_1 \cap B_2$, where $B_1 = \{(x, y) : (x-2)^2 + y^2 \leq (2.5)^2\}$ and $B_2 = \{(x, y) : x^2 + y^2 \leq 1\}$. Then, K is nonempty, convex and closed subset of E. Now, let $T_1, T_2 : K \to E$ be given by $T_1(x, y) = (-x, y)$ and $T_2(x, y) = (x, \frac{1}{2})$. Let $f : K \to K$ be defined by $f(x, y) = \frac{1}{2}(x, -y)$. Then, T_1 and T_2 are inward non-self nonexpansive mappings with common fixed point $\mathcal{F} := F(T_1) \cap F(T_2) = \{(0, \frac{1}{2})\}$. Now, taking $a_0 = a_1 = a_2 = \frac{1}{3}$, the scheme in (3.1) reduces to $x_n = (z_n, y_n)$ given by

$$(4.1) \qquad \begin{cases} (z_0, y_0) = (1/2, 1/3) \in K, \\ \alpha_n = \frac{1}{n+1}, \lambda_n = 0.1; \\ z_{n+1} = \frac{z_n}{2}\alpha_n + (1-\alpha_n)\big[\frac{\lambda_n}{3} z_n + (1-\lambda_n) z_n\big], \\ y_{n+1} = \frac{-y_n}{2}\alpha_n + (1-\alpha_n)\big[\frac{2}{3}\lambda_n y_n + \frac{\lambda_n}{6} + (1-\lambda_n) y_n\big], n \geq 1. \end{cases}$$

Now, if we consider $(z_0, y_0) = (\frac{1}{2}, \frac{1}{3})$, using MATLAB version 7.5.0.342(R2007b) we obtain the following numerical data for particular values of n which indicates that the sequence $x_n = (z_n, y_n)$ goes to the common fixed point $(0, \frac{1}{2})$ (see, the table below).

Table 1. Values of z_n and y_n for some values of n.

n	1	2	5	10	100	200	300	400	500
z_n	0.5000	0.3583	0.1974	0.1026	0.0001	0.0000	0.0000	0.0000	0.0000
y_n	0.3333	0.0861	0.0429	0.0619	0.3167	0.3968	0.4290	0.4459	0.4569

Remark 4.2. *Theorem 3.1 provides convergence sequence to a common fixed point of a finite family of non-self nonexpansive mappings. The algorithm does not involve projection or sunny nonexpansive mappings.*

Remark 4.3. *Theorem 3.1 improves all the results on the approximation of fixed points of self-map nonexpansive or non-self nonexpansive with projection mapping schemes (see, for example, [10, 16, 28] and the references therein) in the sense that our convergence is to a common fixed point of a finite family of non-self nonexpansive mappings without involving projections or sunny nonexpansive retraction mappings. In particular, our results extend and improve Theorem MT and the results of Colao and Marino [6] to more general Banach spaces and/or to the class of non-self mappings.*

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