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WEAK AND STRONG CONVERGENCE THEOREMS FOR COMMUTATIVE NORMALLY 2-GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we first obtain a weak convergence theorem of Mann's type iteration for two commutative normally 2-generalized hybrid mappings in a Hilbert space. Next, we obtain a strong convergence theorem of Halpern's type iteration for the mappings in a Hilbert space. We also obtain two strong convergence theorems by the hybrid method and the shrinking projection method for two commutative normally 2-generalized hybrid mappings in a Hilbert space. Using these results, we get well-known weak and strong convergence theorems in a Hilbert space.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. Let T be a mapping of C into H. Then, we denote by $A(T)$ the set of *attractive points* [26] of T, i.e., $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|,\ \forall x \in C\}$. We know from [26] that $A(T)$ is closed and convex. A mapping $T : C \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. It is well-known that if C is a bounded, closed and convex subset of H and $T : C \to C$ is nonexpansive, then $F(T)$ is nonempty, where $F(T)$ is the set of fixed points of T. Furthermore, from Baillon [3] we know the first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space: Let C be a nonempty, closed and convex subset of H and let $T : C \to C$ be a nonexpansive mapping such that $F(T)$ is nonempty. Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $\widehat{x} \in F(T)$.

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In 2010, Kocourek, Takahashi and Yao [11] defined a broad class of nonlinear mappings in a Hilbert space: Let C be a nonempty subset of H. A mapping $T : C \to H$ is called *generalized hybrid* [11] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$. Such a mapping T is called (α, β)*-generalized hybrid.* We also know the following: For $\lambda \in \mathbb{R}$, a mapping $T : C \to H$ is called λ*-hybrid* [1] if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1-\lambda)\langle x - Tx,\ y - Ty\rangle \tag{1.2}$$

for all $x, y \in C$. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a $(1, 0)$-generalized hybrid mapping is nonexpansive. It is *nonspreading* [13, 14] for $\alpha = 2$ and $\beta = 1$. It is also *hybrid* [25] for $\alpha = 3/2$ and $\beta = 1/2$. In general, nonspreading and hybrid mappings are not continuous; see [9]. We also know that λ-hybrid mappings are in the class of generalized hybrid mappings; see [8]. The nonlinear ergodic theorem by Baillon [3] for nonexpansive mappings has been extended to generalized hybrid mappings in a Hilbert space by Kocourek, Takahashi and Yao [11]. Kohsaka [12] proved the following theorem.

Theorem 1.1 ([12])**.** *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let S and T be commutative λ and μ-hybrid mappings of C into itself such that the set $F(S) \cap F(T)$ of common fixed points of S and T is nonempty. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x$$

converges weakly to a point of $F(S) \cap F(T)$.

Maruyama, Takahashi and Yao [18] also defined more broad class of nonlinear mappings which contains generalized hybrid mappings in a Hilbert space. Let C be a nonempty subset of H. A mapping $T : C \to C$ is 2*-generalized hybrid* [18] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1-\alpha_1-\alpha_2)\|x - Ty\|^2 \\ \leq \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1-\beta_1-\beta_2)\|x - y\|^2\end{aligned} \tag{1.3}$$

for all $x, y \in C$. Hojo, Takahashi and Takahashi [6] obtained attractive point and fixed point theorems for two commutative 2-generalized hybrid mappings in a Hilbert space. Using these results, they proved a nonlinear mean convergence theorem for two commutative 2-generalized hybrid mappings in a Hilbert space. These results generalize Kohsaka's fixed point theorem and mean convergence theorem [12] for two commutative λ-hybrid mappings in a Hilbert space. Very recently, Kondo and Takahashi [15] introduced the following class of nonlinear mappings which covers 2-generalized hybrid mappings in Hilbert spaces. Let C be a nonempty subset of H. A mapping $T : C \to C$ is *normally* 2*-generalized hybrid* [15] if there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that $\sum_{n=0}^{2}(\alpha_n + \beta_n) \geq 0$, $\alpha_2 + \alpha_1 + \alpha_0 > 0$ and

$$\alpha_2\|T^2x - Ty\|^2 + \alpha_1\|Tx - Ty\|^2 + \alpha_0\|x - Ty\|^2 \tag{1.4}$$

$$+\beta_2\|T^2x-y\|^2+\beta_1\|Tx-y\|^2+\beta_0\|x-y\|^2 \le 0$$

for all $x, y \in C$. This class of mappings contains *normally generalized hybrid* mappings [29] if $\alpha_2 = \beta_2 = 0$. Hojo [5] also obtained attractive point and fixed point theorem for two commutative normally 2-generalized hybrid mappings in a Hilbert space. Using these results, she proved a nonlinear mean convergence theorem for the mappings in a Hilbert space.

On the other hand, in 1953, Mann [19] introduced the following iteration process. Let C be a nonempty, closed and convex subset of a Banach space E and let $T : C \to C$ be a nonexpansive mapping. For an initial guess $x_1 \in C$, an iteration process $\{x_n\}$ is defined recursively by

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\lambda_n\}$ is a sequence in $[0,1]$. There are many investigations of Mann iterative process for finding fixed points of nonexpansive mappings. In 1967, Halpern [4] gave an iteration process as follows: Take $u, x_1 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = \lambda_n u + (1-\lambda_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\lambda_n\}$ is a sequence in $[0,1]$. There are many investigations of Halpern iterative process for finding fixed points of nonexpansive mappings. Furthermore, for proving strong convergence theorems for nonexpansive mappings, we also know the hybrid method by Nakajo and Takahashi [20] and the shrinking projection method by Takahashi, Takeuchi and Kubota [27].

In this paper, we first obtain a weak convergence theorem of Mann's type iteration for two commutative normally 2-generalized hybrid mappings in a Hilbert space. Next, we obtain a strong convergence theorem of Halpern's type iteration for the mappings in a Hilbert space. We also obtain two strong convergence theorems by the hybrid method and the shrinking projection method for two commutative normally 2-generalized hybrid mappings in a Hilbert space. Using these results, we get well-known weak and strong convergence theorems in a Hilbert space.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. In a Hilbert space, it is easily ascertained that

$$2\langle x-y,\ y\rangle \le \|x\|^2 - \|y\|^2 \le 2\langle x-y,\ x\rangle \tag{2.1}$$

for all $x, y \in H$. We know from [24] that

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$. Furthermore, we have that

$$2\langle x-y,\ z-w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2 \tag{2.3}$$

for all $x, y, z, w \in H$. Let H be a Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if

$$\|Tx - z\| \leq \|x - z\|, \quad \forall x \in C, \ z \in F(T).$$

If C is closed and convex and $T : C \to H$ with $F(T) \neq \emptyset$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [10]. For a nonempty, closed and convex subset D of H, the nearest point projection of H onto D is denoted by P_D, that is, $\|x - P_D x\| \leq \|x - y\|$ for all $x \in H$ and $y \in D$. Such a mapping P_D is called the metric projection of H onto D. We know that the metric projection P_D is firmly nonexpansive, i.e., $\|P_D x - P_D y\|^2 \leq \langle P_D x - P_D y, \, x - y\rangle$ for all $x, y \in H$. Furthermore, $\langle x - P_D x, \, P_D x - y\rangle \geq 0$ holds for all $x \in H$ and $y \in D$; see [23, 24]. Using this inequality and (2.3), we have that

$$\|x - P_D x\|^2 + \|P_D x - y\|^2 \leq \|x - y\|^2, \quad \forall x \in H, \ y \in D. \tag{2.4}$$

The following result was proved by Takahashi and Toyoda [28].

Lemma 2.1 ([28])**.** *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - z\| \leq \|x_n - z\|$ for all $n \in \mathbb{N}$ and $z \in C$, then $\{P_C x_n\}$ converges strongly to some $\overline{x} \in C$, where P_C is the metric projection of H onto C.*

Let H be a Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to C$ was called *normally 2-generalized hybrid* [15] if it satisfies (1.4). We also call such a mapping $(\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2)$-*normally* 2-*generalized hybrid.* If $x = Tx$ in (1.4), then for any $y \in C$,

$$\begin{aligned} \alpha_2\|x - Ty\|^2 &+ \alpha_1\|x - Ty\|^2 + \alpha_0\|x - Ty\|^2 \\ &+ \beta_2\|x - y\|^2 + \beta_1\|x - y\|^2 + \beta_0\|x - y\|^2 \leq 0 \end{aligned}$$

and hence,

$$(\alpha_2 + \alpha_1 + \alpha_0)\|x - Ty\|^2 \leq -(\beta_2 + \beta_1 + \beta_0)\|x - y\|^2.$$

From $\sum_{n=0}^{2} (\alpha_n + \beta_n) \geq 0$, we have that

$$\begin{aligned} (\alpha_2 + \alpha_1 + \alpha_0)\|x - Ty\|^2 &\leq -(\beta_2 + \beta_1 + \beta_0)\|x - y\|^2 \\ &\leq (\alpha_2 + \alpha_1 + \alpha_0)\|x - y\|^2. \end{aligned}$$

Since $\alpha_2 + \alpha_1 + \alpha_0 > 0$, it follows that

$$\|x - Ty\| \leq \|x - y\|, \quad \forall x \in F(T), \ y \in C. \tag{2.5}$$

Thus, if T is a normally 2-generalized hybrid mapping and $F(T) \neq \emptyset$, then it is quasi-nonexpansive; see also [15]. Hojo [5] obtained the following attractive point and fixed point theorem for commutative normally 2-generalized hybrid mappings in a Hilbert space; see also [6].

Theorem 2.2 ([5])**.** *Let H be a Hilbert space, let C be a nonempty subset of H and let S and T be commutative normally 2-generalized hybrid mappings of C into itself. Suppose that there exists an element $z \in C$ such that $\{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then, $A(S) \cap A(T)$ is nonempty. Additionally, if C is closed and convex, then $F(S) \cap F(T)$ is nonempty.*

To prove one of our main results, we also need the following lemma by Aoyama, Kimura, Takahashi and Toyoda [2].

Lemma 2.3 ([2]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0,1)$ with $\sum_{n=1}^{\infty}\alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty}\beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty}\gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then, $\lim_{n\to\infty} s_n = 0$.

3. Weak convergence theorem of Mann's type iteration

In this section, we obtain a weak convergence theorem of Mann's type iteration for commutative 2-generalized hybrid mappings in a Hilbert space. Before proving the theorem, we need the following lemma.

Lemma 3.1. *Let C be a nonempty subset of a Hilbert space H and let S and T be commutative normally 2-generalized hybrid mappings of C into itself such that $A(S)\cap A(T) \neq \emptyset$. Let $\{x_n\}$ be a bounded sequence in C. Define*

$$S_n x_n = \frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x_n$$

for all $n \in \mathbb{N}\cup\{0\}$. Then, every weak cluster point of $\{S_n x_n\}$ is a point of $A(S)\cap A(T)$. Additionally, if C is closed and convex, then every weak cluster point of $\{S_n x_n\}$ is a point of $F(S)\cap F(T)$.

Proof. Let $v \in H$ be a weak cluster point of $\{S_n x_n\}$. Then, there exists a subsequence $\{S_{n_i} x_{n_i}\}$ of $\{S_n x_n\}$ such that $S_{n_i} x_{n_i} \rightharpoonup v$. We will prove that v is a point of $A(S)\cap A(T)$. First, note that since $\{x_n\}$ is bounded, the sequences

$$\left\{\frac{1}{(n+1)^2}\sum_{l=0}^{n} T^l x_n\right\}, \quad \left\{\frac{1}{(n+1)^2}\sum_{l=0}^{n} ST^l x_n\right\}, \quad \left\{\frac{1}{(n+1)^2}\sum_{l=0}^{n} S^{n+1}T^l x_n\right\}, \quad \left\{\frac{1}{(n+1)^2}\sum_{l=0}^{n} S^{n+2}T^l x_n\right\} \tag{3.1}$$

are also bounded. Indeed, let $z \in A(S)\cap A(T)$ and $M = \max\{\|x_n - z\| : n \in \mathbb{N}\}$. Then, it holds that

$$\begin{aligned} \left\|\frac{1}{(n+1)^2}\sum_{l=0}^{n} S^{n+2}T^l x_n - z\right\| &\leq \frac{1}{(n+1)^2}\sum_{l=0}^{n}\left\|S^{n+2}T^l x_n - z\right\| \\ &\leq \frac{1}{(n+1)^2}\sum_{l=0}^{n}\left\|S^{n+1}T^l x_n - z\right\| \leq \cdots \leq \frac{1}{(n+1)^2}\sum_{l=0}^{n}\left\|T^l x_n - z\right\| \\ &\leq \frac{1}{(n+1)^2}\sum_{l=0}^{n}\left\|T^{l-1} x_n - z\right\| \leq \cdots \leq \frac{1}{(n+1)^2}\sum_{l=0}^{n}\|x_n - z\| \\ &\leq \|x_n - z\| \leq M. \end{aligned} \tag{3.2}$$

This shows that the sequences (3.1) are bounded. Since S is normally 2-generalized hybrid, there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that $\alpha_2 + \alpha_1 + \alpha_0 > 0$, $\sum_{i=0}^{2}(\alpha_i + \beta_i) \geq 0$ and

$$\alpha_2\|S^2x - Sy\|^2 + \alpha_1\|Sx - Sy\|^2 + \alpha_0\|x - Sy\|^2 + \beta_2\|S^2x - y\|^2 + \beta_1\|Sx - y\|^2 + \beta_0\|x - y\|^2 \leq 0 \tag{3.3}$$

for all $x, y \in C$. Take $x = S^kT^lx_n$ in (3.3). Then, for any $y \in C$, $k, l \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$, we have that

$$\alpha_2\|S^{k+2}T^lx_n - Sy\|^2 + \alpha_1\|S^{k+1}T^lx_n - Sy\|^2 + \alpha_0\|S^kT^lx_n - Sy\|^2 + \beta_2\|S^{k+2}T^lx_n - y\|^2 + \beta_1\|S^{k+1}T^lx_n - y\|^2 + \beta_0\|S^kT^lx_n - y\|^2 \leq 0$$

and hence,

$$\begin{aligned}&\alpha_2\{\|S^{k+2}T^lx_n - y\|^2 + \|y - Sy\|^2 + 2\langle S^{k+2}T^lx_n - y,\ y - Sy\rangle\}\\ &+ \alpha_1\{\|S^{k+1}T^lx_n - y\|^2 + \|y - Sy\|^2 + 2\langle S^{k+1}T^lx_n - y,\ y - Sy\rangle\}\\ &+ \alpha_0\{\|S^kT^lx_n - y\|^2 + \|y - Sy\|^2 + 2\langle S^kT^lx_n - y,\ y - Sy\rangle\}\\ &+ \beta_2\|S^{k+2}T^lx_n - y\|^2 + \beta_1\|S^{k+1}T^lx_n - y\|^2 + \beta_0\|S^kT^lx_n - y\|^2 \leq 0.\end{aligned}$$

This implies that

$$\begin{aligned}&(\alpha_2 + \beta_2 + \alpha_1 + \beta_1 + \alpha_0 + \beta_0)\|S^{k+2}T^lx_n - y\|^2\\ &+ (\alpha_1 + \beta_1)(\|S^{k+1}T^lx_n - y\|^2 - \|S^{k+2}T^lx_n - y\|^2)\\ &+ (\alpha_0 + \beta_0)(\|S^kT^lx_n - y\|^2 - \|S^{k+2}T^lx_n - y\|^2)\\ &+ (\alpha_2 + \alpha_1 + \alpha_0)\|y - Sy\|^2\\ &+ 2\Big\langle(\alpha_2 + \alpha_1 + \alpha_0)S^kT^lx_n + \alpha_2(S^{k+2}T^lx_n - S^kT^lx_n)\\ &\quad + \alpha_1(S^{k+1}T^lx_n - S^kT^lx_n) - (\alpha_2 + \alpha_1 + \alpha_0)y,\ y - Sy\Big\rangle \leq 0.\end{aligned}$$

Since $\sum_{j=0}^{2}(\alpha_j + \beta_j) \geq 0$, we have that

$$\begin{aligned}&(\alpha_1 + \beta_1)(\|S^{k+1}T^lx_n - y\|^2 - \|S^{k+2}T^lx_n - y\|^2)\\ &+ (\alpha_0 + \beta_0)(\|S^kT^lx_n - y\|^2 - \|S^{k+2}T^lx_n - y\|^2)\\ &+ (\alpha_2 + \alpha_1 + \alpha_0)\|y - Sy\|^2\\ &+ 2\Big\langle(\alpha_2 + \alpha_1 + \alpha_0)S^kT^lx_n + \alpha_2(S^{k+2}T^lx_n - S^kT^lx_n)\\ &\quad + \alpha_1(S^{k+1}T^lx_n - S^kT^lx_n) - (\alpha_2 + \alpha_1 + \alpha_0)y,\ y - Sy\Big\rangle \leq 0.\end{aligned}$$

Summing up these inequalities with respect to $k = 0, 1, \ldots, n$, we have

$$\begin{aligned}&(\alpha_1 + \beta_1)(\|ST^lx_n - y\|^2 - \|S^{n+2}T^lx_n - y\|^2)\\ &+ (\alpha_0 + \beta_0)(\|T^lx_n - y\|^2 + \|ST^lx_n - y\|^2 - \|S^{n+1}T^lx_n - y\|^2 - \|S^{n+2}T^lx_n - y\|^2)\\ &+ (n + 1)(\alpha_2 + \alpha_1 + \alpha_0)\|y - Sy\|^2\end{aligned}$$

$$+ 2\Big\langle(\alpha_2+\alpha_1+\alpha_0)\sum_{k=0}^{n} S^kT^lx_n + \alpha_2(S^{n+2}T^lx_n + S^{n+1}T^lx_n - ST^lx_n - T^lx_n)$$
$$+ \alpha_1(S^{n+1}T^lx_n - T^lx_n) - (n+1)(\alpha_2+\alpha_1+\alpha_0)y,\ y - Sy\Big\rangle \leq 0.$$

Furthermore, summing up these inequalities with respect to $l = 0, 1, \ldots, n$, we have

$$(\alpha_1+\beta_1)\sum_{l=0}^{n}(\|ST^lx_n - y\|^2 - \|S^{n+2}T^lx_n - y\|^2)$$
$$+ (\alpha_0+\beta_0)\sum_{l=0}^{n}(\|T^lx_n - y\|^2 + \|ST^lx_n - y\|^2$$
$$- \|S^{n+1}T^lx_n - y\|^2 - \|S^{n+2}T^lx_n - y\|^2) + (n+1)^2(\alpha_2+\alpha_1+\alpha_0)\|y - Sy\|^2$$
$$+ 2\Big\langle(\alpha_2+\alpha_1+\alpha_0)\sum_{l=0}^{n}\sum_{k=0}^{n} S^kT^lx_n$$
$$+ \alpha_2\sum_{l=0}^{n}(S^{n+2}T^lx_n + S^{n+1}T^lx_n - ST^lx_n - T^lx_n)$$
$$+ \alpha_1\sum_{l=0}^{n}(S^{n+1}T^lx_n - T^lx_n) - (n+1)^2(\alpha_2+\alpha_1+\alpha_0)y,\ y - Sy\Big\rangle \leq 0.$$

Dividing by $(n+1)^2$, we have

$$(\alpha_1+\beta_1)\frac{1}{(n+1)^2}\sum_{l=0}^{n}(\|ST^lx_n - y\|^2 - \|S^{n+2}T^lx_n - y\|^2)$$
$$+ (\alpha_0+\beta_0)\frac{1}{(n+1)^2}\sum_{l=0}^{n}(\|T^lx_n - y\|^2 + \|ST^lx_n - y\|^2$$
$$- \|S^{n+1}T^lx_n - y\|^2 - \|S^{n+2}T^lx_n - y\|^2) + (\alpha_2+\alpha_1+\alpha_0)\|y - Sy\|^2$$
$$+ 2\Big\langle(\alpha_2+\alpha_1+\alpha_0)\frac{1}{(n+1)^2}\sum_{l=0}^{n}\sum_{k=0}^{n} S^kT^lx_n$$
$$+ \alpha_2\frac{1}{(n+1)^2}\sum_{l=0}^{n}(S^{n+2}T^lx_n + S^{n+1}T^lx_n - ST^lx_n - T^lx_n)$$
$$+ \alpha_1\frac{1}{(n+1)^2}\sum_{l=0}^{n}(S^{n+1}T^lx_n - T^lx_n) - (\alpha_2+\alpha_1+\alpha_0)y,\ y - Sy\Big\rangle \leq 0.$$

Replacing n by n_i and taking the limit as $n_i \to \infty$, we have that

$$(\alpha_2+\alpha_1+\alpha_0)\|y - Sy\|^2 + 2(\alpha_2+\alpha_1+\alpha_0)\langle v - y,\ y - Sy\rangle \leq 0.$$

Since $\alpha_2+\alpha_1+\alpha_0 > 0$, we have that

$$\|y - Sy\|^2 + 2\langle v - y,\ y - Sy\rangle \leq 0.$$

This implies from (2.3) that

$$\|y - Sy\|^2 + \|v - Sy\|^2 + \|y - y\|^2 - \|v - y\|^2 - \|y - Sy\|^2 \leq 0$$

and hence,

$$\|v - Sy\|^2 - \|v - y\|^2 \leq 0.$$

This means that

$$\|v - Sy\| \leq \|v - y\|, \quad \forall y \in C$$

and hence, $v \in A(S)$. Similarly, since a mapping T is normally 2-generalized hybrid, there exist $\alpha_0', \beta_0', \alpha_1', \beta_1', \alpha_2', \beta_2' \in \mathbb{R}$ such that

$$\alpha_2' \left\|T^2x - Ty\right\|^2 + \alpha_1' \|Tx - Ty\|^2 + \alpha_0' \|x - Ty\|^2$$
$$+ \beta_2' \left\|T^2x - y\right\|^2 + \beta_1' \left\|T^2x - y\right\|^2 + \beta_0' \left\|T^2x - y\right\|^2 \leq 0$$

for all $x, y \in C$. Replacing S and T by T and S for the above proof, respectively, we have that

$$\|y - Ty\|^2 + 2\langle v - y,\ y - Ty\rangle \leq 0$$

and hence, $v \in A(T)$. Therefore, $v \in A(S) \cap A(T)$.

Additionally, if C is closed and convex, then every weak cluster point of $\{S_n x_n\}$ is a point of C. Since

$$\|y - Sy\|^2 + 2\langle v - y,\ y - Sy\rangle \leq 0,$$

putting $y = v$, we have that $\|Sv - v\|^2 \leq 0$ and hence, $Sv = v$. Similarly, we have $Tv = v$. Therefore, every weak cluster point of $\{S_n x_n\}$ is a point of $F(S) \cap F(T)$. This completes the proof. □

Theorem 3.2. *Let H be a Hilbert space and let C be a nonempty and convex subset of H. Let S and T be commutative normally 2-generalized hybrid mappings of C into itself such that $A(S) \cap A(T) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of real numbers such that $0 \leq \lambda_n < 1$ and $\liminf_{n\to\infty} \lambda_n(1 - \lambda_n) > 0$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and*

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n) \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $\overline{x} = \lim_{n\to\infty} P_{A(S)\cap A(T)} x_n$, where $P_{A(S)\cap A(T)}$ is the metric projection of H onto $A(S) \cap A(T)$. Additionally, if C is closed, then $\{x_n\}$ converges weakly to $\widehat{x} = \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

Proof. Since $A(S)$ and $A(T)$ are closed and convex subset of H, $A(S) \cap A(T)$ is also closed and convex in H. Since $A(S) \cap A(T) \neq \emptyset$ is assumed, there exists the metric projection $P_{A(S)\cap A(T)}$ of H onto $A(S) \cap A(T)$. Put

$$S_n x_n = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x_n$$

for all $n \in \mathbb{N}$. Let z be a point of $A(S) \cap A(T)$. It holds that

$$\|S_n x_n - z\| \leq \|x_n - z\| \tag{3.4}$$

for all $n \in \mathbb{N}$. Indeed,

$$\begin{aligned} (3.5) \qquad \|S_n x_n - z\| &= \left\| \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x_n - z \right\| \\ &\le \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} \left\| S^k T^l x_n - z \right\| \\ &\le \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} \|x_n - z\| \\ &= \|x_n - z\| . \end{aligned}$$

Using (3.4), we have that

$$\begin{aligned} (3.6) \qquad \|x_{n+1} - z\| &= \|\lambda_n x_n + (1-\lambda_n) S_n x_n - z\| \\ &= \|\lambda_n (x_n - z) + (1-\lambda_n)(S_n x_n - z)\| \\ &\le \lambda_n \|x_n - z\| + (1-\lambda_n)\|S_n x_n - z\| \\ &\le \lambda_n \|x_n - z\| + (1-\lambda_n)\|x_n - z\| \\ &= \|x_n - z\| \end{aligned}$$

for all $z \in A(S) \cap A(T)$ and $n \in \mathbb{N}$. Thus, $\lim_{n\to\infty} \|x_n - z\|$ exists for all $z \in A(S) \cap A(T)$. Since $A(S) \cap A(T)$ is nonempty, we have from (3.6) that the sequence $\{x_n\}$ is bounded. Thus, $\{S_n x_n\}$ is also bounded. From (2.2) and (3.4), it holds that

$$\begin{aligned} &\|x_{n+1} - z\|^2 = \|\lambda_n x_n + (1-\lambda_n) S_n x_n - z\|^2 \\ &= \|\lambda_n (x_n - z) + (1-\lambda_n)(S_n x_n - z)\|^2 \\ &= \lambda_n \|x_n - z\|^2 + (1-\lambda_n)\|S_n x_n - z\|^2 - \lambda_n(1-\lambda_n)\|S_n x_n - x_n\|^2 \\ &\le \lambda_n \|x_n - z\|^2 + (1-\lambda_n)\|x_n - z\|^2 - \lambda_n(1-\lambda_n)\|S_n x_n - x_n\|^2 \\ &= \|x_n - z\|^2 - \lambda_n(1-\lambda_n)\|S_n x_n - x_n\|^2. \end{aligned}$$

We obtain that

$$\lambda_n(1-\lambda_n)\|S_n x_n - x_n\|^2 \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

From the assumption of $\{\lambda_n\}$, we have

$$(3.7) \qquad \lim_{n\to\infty} \|S_n x_n - x_n\| = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$ for some $v \in H$. Since $S_n x_n - x_n \to 0$, we have that $S_{n_i} x_{n_i} \rightharpoonup v$. We obtain from Lemma 3.1 that $v \in A(S) \cap A(T)$. We will show that $x_n \rightharpoonup v$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u$ and $x_{n_j} \rightharpoonup v$. It holds that $u, v \in A(S) \cap A(T)$. Put $a = \lim_{n\to\infty}(\|x_n - u\|^2 - \|x_n - v\|^2)$. Since

$$\|x_n - u\|^2 - \|x_n - v\|^2 = 2\langle x_n,\ v - u\rangle + \|u\|^2 - \|v\|^2,$$

we have $a = 2\langle u,\ v-u\rangle + \|u\|^2 - \|v\|^2$ and $a = 2\langle v,\ v-u\rangle + \|u\|^2 - \|v\|^2$. From these equalities, we obtain $\langle u - v,\ v - u\rangle = 0$. So, it follows that $u = v$. Therefore, $\{x_n\}$ converges weakly to an element v of $A(S) \cap A(T)$. On the other hand, we know from (3.6) and Lemma 2.1 that $\{P_{A(S)\cap A(T)} x_n\}$ converges strongly to an element

$\overline{x}$ of $A(S) \cap A(T)$. Since $v \in A(S) \cap A(T)$, we also have from the property of the metric projection $P_{A(S)\cap A(T)}$ that

$$\langle x_n - P_{A(S)\cap A(T)}x_n,\ P_{A(S)\cap A(T)}x_n - v\rangle \geq 0$$

for all $n \in \mathbb{N}$. Taking $n \to \infty$, we obtain that $\langle v - \overline{x}, \overline{x} - v\rangle \geq 0$. So, we have $-\|v - \overline{x}\|^2 \geq 0$ and hence, $\overline{x} = v$. This implies that $\{x_n\}$ converges weakly to $\overline{x} = \lim_{n\to\infty} P_{A(S)\cap A(T)}x_n \in A(S) \cap A(T)$.

Additionally, if C is closed, then C is closed and convex. Note that $\{S^kT^lz\}$ is bounded for all $z \in C$. Indeed, let $z \in C$ and $w \in A(S) \cap A(T)$. Then, we have that

$$\begin{aligned} \left\|S^kT^lz - w\right\| &\leq \left\|S^{k-1}T^lz - w\right\| \leq \cdots \leq \left\|T^lz - w\right\| \\ &\leq \left\|T^{l-1}z - w\right\| \leq \cdots \leq \|z - w\|, \end{aligned} \tag{3.8}$$

which implies that $\{S^kT^lz\}$ is bounded for all $z \in C$. From Theorem 2.2, $F(S) \cap F(T)$ is nonempty. Since S and T are quasi-nonexpansive, $F(S)\cap F(T)$ is closed and convex. Thus, there exists the metric projection $P_{F(S)\cap F(T)}$ of H onto $F(S)\cap F(T)$. From $A(S) \cap A(T) \cap C \subset F(S) \cap F(T)$, the sequence $\{x_n\}$ converges weakly to an element v of $F(S) \cap F(T)$. On the other hand, we know from $F(S) \cap F(T) \subset A(S) \cap A(T)$ and (3.6) that

$$\|x_{n+1} - z\| \leq \|x_n - z\|, \quad \forall z \in F(S) \cap F(T).$$

Then, we have from Lemma 2.1 that $\{P_{F(S)\cap F(T)}x_n\}$ converges strongly to an element $\widehat{x}$ of $F(S) \cap F(T)$. since $v \in F(S) \cap F(T)$, we also have from the property of the metric projection $P_{F(S)\cap F(T)}$ that

$$\langle x_n - P_{F(S)\cap F(T)}x_n,\ P_{F(S)\cap F(T)}x_n - v\rangle \geq 0.$$

Taking $n \to \infty$, we have $\langle v - \widehat{x}, \widehat{x} - v\rangle \geq 0$. So, we have $-\|v - \widehat{x}\|^2 \geq 0$ and hence, $\widehat{x} = v$. This implies that $\{x_n\}$ converges weakly to $\widehat{x} = \lim_{n\to\infty} P_{F(S)\cap F(T)}x_n \in F(S) \cap F(T)$. This completes the proof. □

Since 2-generalized hybrid mappings are contained in the class of normally 2-generalized hybrid mappings, we obtain the following theorem proved by Hojo and Takahashi [7].

Theorem 3.3 ([7])**.** *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let S and T be commutative 2-generalized hybrid mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $P_{F(S)\cap F(T)}$ be the metric projection of H onto $F(S) \cap F(T)$. Let $\{\lambda_n\}$ be a sequence of real numbers such that $0 \leq \lambda_n < 1$ and $\liminf_{n\to\infty} \lambda_n(1 - \lambda_n) > 0$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and*

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^kT^lx_n, \quad \forall n \in \mathbb{N}$$

converges weakly to $\widehat{x} = \lim_{n\to\infty} P_{F(S)\cap F(T)}x_n \in F(S) \cap F(T)$.

4. Strong convergence theorem of Halpern's type iteration

Using the idea of mean convergence by Shimizu and Takahashi [21, 22], and Kurokawa and Takahashi [17], we prove the following strong convergence theorem of Halpern's type iteration for commutative normally 2-generalized hybrid mappings in a Hilbert space.

Theorem 4.1. *Let H be a Hilbert space and let C be a nonempty and convex subset of H. Let S and T be commutative normally 2-generalized hybrid mappings of C into itself such that $A(S) \cap A(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. Define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \lambda_n u_n + (1-\lambda_n) \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x_n, \quad \forall n \in \mathbb{N},$$

where $0 \leq \lambda_n < 1$, $\lambda_n \to 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Then, $\{x_n\}$ converges strongly to $\overline{u} = P_{A(S)\cap A(T)} u$, where $P_{A(S)\cap A(T)}$ is the metric projection of H onto $A(S) \cap A(T)$. Additionally, if C is closed, then $\{x_n\}$ converges strongly to $\widehat{u} = P_{F(S)\cap F(T)} u$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

Proof. Put

$$S_n x_n = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x_n$$

for all $n \in \mathbb{N}$. In much the same way as (3.5), we obtain that

$$\|S_n x_n - z\| \leq \|x_n - z\| \tag{4.1}$$

for all $z \in A(S) \cap A(T)$ and $n \in \mathbb{N}$. We will show that $\{x_n\}$ is bounded. Let $z \in A(S) \cap A(T)$. Since $\{u_n\}$ is bounded, there exists $M \geq 0$ such that $\sup_{n\in\mathbb{N}} \|u_n - z\| \leq M$. Putting $K = \max\{\|x_1 - z\|, M\}$, we have that $\|x_n - z\| \leq K$ for all $n \in \mathbb{N}$. In fact, it is obvious that $\|x_1 - z\| \leq K$. Suppose that $\|x_k - z\| \leq K$ for some $k \in \mathbb{N}$. Then, we have from (4.1) that

$$\begin{aligned} \|x_{k+1} - z\| &= \|\lambda_k u_k + (1-\lambda_k) S_k x_k - z\| \\ &\leq \lambda_k \|u_k - z\| + (1-\lambda_k)\|S_k x_k - z\| \\ &\leq \lambda_k \|u_k - z\| + (1-\lambda_k)\|x_k - z\| \\ &\leq \lambda_k M + (1-\lambda_k) K \leq K. \end{aligned}$$

Hence, by induction, we obtain that $\|x_n - z\| \leq K$ for all $n \in \mathbb{N}$, which implies that $\{x_n\}$ is bounded. As a consequence, $\{S_n x_n\}$ is also bounded. Note that

$$\begin{aligned} \|x_{n+1} - S_n x_n\| &= \|\lambda_n u_n + (1-\lambda_n) S_n x_n - S_n x_n\| \\ &= \lambda_n \|u_n - S_n x_n\|. \end{aligned}$$

Since $\{S_n x_n\}$ and $\{u_n\}$ are bounded and $\lambda_n \to 0$, we have that $\lim_{n\to\infty} \|x_{n+1} - S_n x_n\| = 0$.

Define $\overline{u} \equiv P_{A(S)\cap A(T)}u$. We have from (2.1) and (4.1) that

$$
\begin{aligned}
\|x_{n+1}-\overline{u}\|^2 &= \|\lambda_n u_n + (1-\lambda_n)S_n x_n - \overline{u}\|^2 \\
&= \|\lambda_n (u_n - \overline{u}) + (1-\lambda_n)(S_n x_n - \overline{u})\|^2 \\
&\le \|(1-\lambda_n)(S_n x_n - \overline{u})\|^2 + 2\langle \lambda_n (u_n - \overline{u}),\ x_{n+1} - \overline{u}\rangle \\
&\le (1-\lambda_n)^2\|S_n x_n - \overline{u}\|^2 + 2\lambda_n \langle u_n - \overline{u},\ x_{n+1} - \overline{u}\rangle \\
&\le (1-\lambda_n)\|S_n x_n - \overline{u}\|^2 + 2\lambda_n \langle u_n - \overline{u},\ x_{n+1} - \overline{u}\rangle \\
&\le (1-\lambda_n)\|x_n - \overline{u}\|^2 + 2\lambda_n \langle u_n - u,\ x_{n+1} - \overline{u}\rangle \\
&\quad + 2\lambda_n \langle u - \overline{u},\ x_{n+1} - \overline{u}\rangle .
\end{aligned}
\tag{4.2}
$$

Since $\{x_n\}$ is bounded, $u_n \to u$ and $\lambda_n \to 0$, we have that $\lambda_n \langle u_n - u,\ x_{n+1} - \overline{u}\rangle \to 0$. We will show that $\limsup_{n\to\infty}\langle u - \overline{u},\ x_{n+1} - \overline{u}\rangle \le 0$. Since $\{x_{n+1}\}$ is bounded, we may assume, without loss of generality, that there exits a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ such that

$$\limsup_{n\to\infty}\langle u - \overline{u},\ x_{n+1} - \overline{u}\rangle = \lim_{i\to\infty}\langle u - \overline{u},\ x_{n_i+1} - \overline{u}\rangle$$

and $x_{n_i+1} \rightharpoonup v$ for some $v \in H$. Since $\lim_{n\to\infty}\|x_{n+1} - S_n x_n\| = 0$, it holds that $S_{n_i}x_{n_i} \rightharpoonup v$. We have from Lemma 3.1 that $v \in A(S)\cap A(T)$. Since $P_{A(S)\cap A(T)}$ is the metric projection of H onto $A(S)\cap A(T)$, we obtain that

$$\limsup_{n\to\infty}\langle u - \overline{u},\ x_{n+1} - \overline{u}\rangle = \lim_{i\to\infty}\langle u - \overline{u},\ x_{n_i+1} - \overline{u}\rangle = \langle u - \overline{u},\ v - \overline{u}\rangle \le 0.$$

Putting $s_n = \|x_n - \overline{u}\|^2$, $\alpha_n = \lambda_n$, $\beta_n = 0$ and

$$\gamma_n = 2\lambda_n \langle u_n - u,\ x_{n+1} - \overline{u}\rangle + 2\lambda_n \langle u - \overline{u},\ x_{n+1} - \overline{u}\rangle$$

in Lemma 2.3, we have from (4.2) that $\lim_{n\to\infty}\|x_n - \overline{u}\|^2 = 0$. That is, $x_n \to \overline{u} = P_{A(S)\cap A(T)}u$ as $n \to \infty$.

Additionally, if C is closed, then C is closed and convex. As (3.8) in the proof of Theorem 3.2, we obtain that $\{S^k T^l z\}$ is bounded for all $z \in C$. We have from Theorem 2.2 that $F(S)\cap F(T)$ is nonempty. Since S and T are quasi-nonexpansive, $F(S)\cap F(T)$ is a closed and convex subset of H. Thus, there exists the metric projection $P_{F(S)\cap F(T)}$ of H onto $F(S)\cap F(T)$. Define $\widehat{u} \equiv P_{F(S)\cap F(T)}u$. From the above proof, we have that $x_n \to \overline{u}\left(= P_{A(S)\cap A(T)}u\right) \in A(S)\cap A(T)$. From $A(S)\cap A(T)\cap C \subset F(S)\cap F(T)$, it holds that $\overline{u} \in F(S)\cap F(T)$. Thus, $\|u - \widehat{u}\| \le \|u - \overline{u}\|$. On the other hand, since $F(S)\cap F(T) \subset A(S)\cap A(T)$, we have $\|u - \overline{u}\| \le \|u - \widehat{u}\|$. Therefore, $\|u - \overline{u}\| = \|u - \widehat{u}\|$. This implies that $\overline{u} = \widehat{u}$. Thus, $\{x_n\}$ converges strongly to $\widehat{u} = P_{F(S)\cap F(T)}u$. This completes the proof. □

As direct consequences of Theorem 4.1, we have the following theorems proved by Hojo and Takahashi [7] and Kondo and Takahashi [16].

Theorem 4.2 ([7]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let S and T be commutative 2-generalized hybrid mappings of C into itself such that $F(S)\cap F(T) \ne \emptyset$. Let $u \in C$ and define a sequence $\{x_n\}$ in*

C as follows: $x_1 = x \in C$ and

$$x_{n+1} = \lambda_n u + (1-\lambda_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x_n, \quad \forall n \in \mathbb{N},$$

where $0 \le \lambda_n < 1$, $\lambda_n \to 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Then, $\{x_n\}$ converges strongly to $\widehat{u} = P_{F(S)\cap F(T)}u$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S)\cap F(T)$.

Theorem 4.3 ([16])**.** *Let H be a Hilbert space, let C be a nonempty and convex subset of H and let $T : C \to C$ be a normally 2-generalized hybrid mapping with $A(T) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0,1)$ such that $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Given $u, x_1 \in C$, define a sequence $\{x_n\}$ in C as follows:*

$$x_{n+1} = \lambda_n u + (1-\lambda_n)\,\frac{1}{n}\sum_{k=0}^{n-1} T^k x_n$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges strongly to $\overline{u} = P_{A(T)}u$, where $P_{A(T)}$ is the metric projection of H onto $A(T)$.

5. Strong convergence theorems by hybrid methods

In this section, using the hybrid method by Nakajo and Takahashi [20], we first prove a strong convergence theorem for commutative normally 2-generalized hybrid mappings in a Hilbert space.

Theorem 5.1. *Let H be a Hilbert space and let C be a nonempty, convex and closed subset of H. Let S and T be commutative normally 2-generalized hybrid mappings of C into itself such that $F(S)\cap F(T) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \lambda_n x_n + (1-\lambda_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x - x_n,\, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n\cap Q_n}$ is the metric projection of H onto $C_n\cap Q_n$ and $\{\lambda_n\} \subset [0,1]$ satisfies $0 \le \lambda_n \le \overline{\lambda} < 1$ for some $\overline{\lambda} \in \mathbb{R}$. Then, $\{x_n\}$ converges strongly to $\widehat{x} = P_{F(S)\cap F(T)}x$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S)\cap F(T)$.

Proof. Put $S_n x_n = \frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x_n$ for all $n \in \mathbb{N}$. Since S and T are quasi-nonexpansive, $F(S)\cap F(T)$ is closed and convex. So, there exists the metric projection $P_{F(S)\cap F(T)}$ of H onto $F(S)\cap F(T)$. Since

$$\begin{aligned} &\|y_n - z\|^2 \le \|x_n - z\|^2 \\ \Longleftrightarrow &\|y_n\|^2 - \|x_n\|^2 - 2\langle y_n - x_n,\ z\rangle \le 0, \end{aligned}$$

we have that C_n, Q_n and $C_n \cap Q_n$ are closed and convex for all $n \in \mathbb{N}$. We next show that $C_n \cap Q_n$ is nonempty. Let $z \in F(S) \cap F(T)$. Since S and T are quasi-nonexpansive, we have that

$$\begin{aligned} \|y_n - z\| &= \|\lambda_n x_n + (1-\lambda_n)S_n x_n - z\| \\ &\le \lambda_n \|x_n - z\| + (1-\lambda_n)\|S_n x_n - z\| \\ &\le \lambda_n \|x_n - z\| + (1-\lambda_n)\|x_n - z\| \\ &= \|x_n - z\|. \end{aligned}$$

Thus, we have $z \in C_n$ and hence, $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$. Next, we show by induction that $F(S) \cap F(T) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. From $F(S) \cap F(T) \subset Q_1 = C$, it follows that $F(S) \cap F(T) \subset C_1 \cap Q_1$. Suppose that $F(S) \cap F(T) \subset C_k \cap Q_k$ for some $k \in \mathbb{N}$. Let $z \in F(S) \cap F(T)$. Since $F(S) \cap F(T) \subset C_k \cap Q_k$, we have from $x_{k+1} = P_{C_k \cap Q_k} x$ that

$$\langle x - x_{k+1},\ x_{k+1} - z \rangle \ge 0,$$

which implies $z \in Q_{k+1}$. So, we have that $F(S) \cap F(T) \subset C_{k+1} \cap Q_{k+1}$. By induction, we have $F(S) \cap F(T) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. This means that $\{x_n\}$ is well-defined.

Since $x_n = P_{Q_n} x$ and $x_{n+1} = P_{C_n \cap Q_n} x \in Q_n$, we have from (2.3) that

$$\begin{aligned} 0 &\le 2\langle x - x_n,\ x_n - x_{n+1} \rangle \\ &= \|x - x_{n+1}\|^2 - \|x - x_n\|^2 - \|x_n - x_{n+1}\|^2 \\ &\le \|x - x_{n+1}\|^2 - \|x - x_n\|^2. \end{aligned} \tag{5.1}$$

So, we get that

$$\|x - x_n\| \le \|x - x_{n+1}\| \tag{5.2}$$

for all $n \in \mathbb{N}$. Furthermore, since $x_n = P_{Q_n} x$ and $F(S) \cap F(T) \subset Q_n$, we have

$$\|x - x_n\| \le \|x - z\|, \qquad \forall z \in F(S) \cap F(T). \tag{5.3}$$

This implies that $\{x_n\}$ is bounded. Hence, $\{S_n x_n\}$ and $\{y_n\}$ are also bounded. We have from (5.2) that $\lim_{n\to\infty} \|x - x_n\|$ exists. From (5.1), we have

$$\|x_n - x_{n+1}\|^2 \le \|x - x_{n+1}\|^2 - \|x - x_n\|^2$$

and hence,

$$\|x_n - x_{n+1}\| \to 0. \tag{5.4}$$

From $x_{n+1} = P_{C_n \cap Q_n} x \in C_n$, we have that $\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|$. It holds from (5.4) that $\|y_n - x_{n+1}\| \to 0$. So, we have

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{5.5}$$

Since $\|y_n - x_n\| = \|\lambda_n x_n + (1-\lambda_n)S_n x_n - x_n\| = (1-\lambda_n)\|S_n x_n - x_n\|$, we have from $0 \le \lambda_n \le \overline{\lambda} < 1$ that

$$\|S_n x_n - x_n\| \to 0. \tag{5.6}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup v$ for some $v \in H$. From (5.6) and Lemma 3.1, we have $v \in F(S) \cap F(T)$.

Put $\widehat{x} = P_{F(S)\cap F(T)}x$. We will show that $v = \widehat{x}$. Since $x_{n+1} = P_{C_n\cap Q_n}x$ and $\widehat{x} \in F(S) \cap F(T) \subset C_n \cap Q_n$, we have that

$$\|x - x_{n+1}\| \leq \|x - \widehat{x}\|, \quad \forall n \in \mathbb{N} \tag{5.7}$$

Since $\|\cdot\|$ is weakly lower semicontinuous, we have from $x_{n_i} \rightharpoonup v$ and (5.7) that

$$\|x - v\| \leq \liminf_{i\to\infty} \|x - x_{n_i}\| \leq \|x - \widehat{x}\|.$$

Since $v \in F(S) \cap F(T)$, we have from the definition of $\widehat{x} = P_{F(S)\cap F(T)}x$ that $\|x-\widehat{x}\| \leq \|x-v\|$. This implies that $v = \widehat{x}$. So, we obtain $x_n \rightharpoonup \widehat{x}$. We finally show that $x_n \to \widehat{x}$. We have from (5.7) that

$$\begin{aligned}
\limsup_{n\to\infty} \|x_n - \widehat{x}\|^2 &= \limsup_{n\to\infty}(\|x_n - x\|^2 + 2\langle x_n - x,\ x - \widehat{x}\rangle + \|x - \widehat{x}\|^2) \\
&\leq \limsup_{n\to\infty}(\|x - \widehat{x}\|^2 + 2\langle x_n - x,\ x - \widehat{x}\rangle + \|x - \widehat{x}\|^2) \\
&= \|x - \widehat{x}\|^2 + 2\langle \widehat{x} - x,\ x - \widehat{x}\rangle + \|x - \widehat{x}\|^2 \\
&= 2\|x - \widehat{x}\|^2 - 2\|x - \widehat{x}\|^2 = 0.
\end{aligned}$$

So, we obtain $\lim_{n\to\infty} \|x_n - \widehat{x}\|^2 = 0$. Hence, $\{x_n\}$ converges strongly to $\widehat{x} = P_{F(S)\cap F(T)}x$. This completes the proof. □

Next, we prove a strong convergence theorem by the shrinking projection method [27] for commutative normally 2-generalized hybrid mappings in a Hilbert space.

Theorem 5.2. *Let H be a Hilbert space and let C be a nonempty, convex and closed subset of H. Let S and T be commutative normally 2-generalized hybrid mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases}
y_n = \lambda_n x_n + (1 - \lambda_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x_n, \\
C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\
x_{n+1} = P_{C_{n+1}}u_{n+1}, \quad \forall n \in \mathbb{N},
\end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1} and $\{\lambda_n\} \subset [0, 1]$ is a sequence such that $\liminf_{n\to\infty} \lambda_n < 1$. Then, $\{x_n\}$ converges strongly to $\widehat{u} = P_{F(S)\cap F(T)}u$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

Proof. Put $S_n x_n = \frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x_n$ for all $n \in \mathbb{N}$. Since S and T are quasi-nonexpansive, we have that $F(S) \cap F(T)$ is closed and convex. So, there exists the metric projection $P_{F(S)\cap F(T)}$ of H onto $F(S)\cap F(T)$. We shall show that C_n is closed and convex, and $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$. Since $C_1 = C$, it is obvious that C_1 is closed and convex, and $F(S) \cap F(T) \subset C_1$. Suppose that C_k is closed and convex, and $F(S) \cap F(T) \subset C_k$ for some $k \in \mathbb{N}$. We know that for $z \in C_k$,

$$\begin{aligned}
&\|y_k - z\|^2 \leq \|x_k - z\|^2 \\
\Longleftrightarrow &\|y_k\|^2 - \|x_k\|^2 - 2\langle y_k - x_k,\ z\rangle \leq 0.
\end{aligned}$$

So, C_{k+1} is closed and convex. By induction, C_n are closed and convex for all $n \in \mathbb{N}$. Let $z \in F(S) \cap F(T)$. Since $F(S) \cap F(T) \subset C_k$ is assumed, it holds that $z \in C_k$. Furthermore, since S and T are quasi-nonexpansive, we have that

$$\begin{aligned} \|y_k - z\| &= \|\lambda_k x_k + (1-\lambda_k) S_k x_k - z\| \\ &\leq \lambda_k \|x_k - z\| + (1-\lambda_k)\|S_k x_k - z\| \\ &\leq \lambda_k \|x_k - z\| + (1-\lambda_k)\|x_k - z\| \\ &= \|x_k - z\|. \end{aligned}$$

Hence, we have $z \in C_{k+1}$. By induction, we have that $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$. Since $F(S) \cap F(T) \neq \emptyset$ is assumed, C_n is nonempty for all $n \in \mathbb{N}$. Since C_n is nonempty, closed and convex, there exists the metric projection P_{C_n} of H onto C_n. Thus, $\{x_n\}$ is well-defined.

Define $\widehat{u} = P_{F(S)\cap F(T)} u$ and $\overline{u}_n = P_{C_n} u$. Note that since $\overline{u}_n \in C_n \subset C$, $\{\overline{u}_n\}$ is a sequence in C while $\{u_n\}$ is that in H. Then, we have that

$$\|u - \overline{u}_n\| \leq \|u - z\|$$

for all $z \in C_n$ and $n \in \mathbb{N}$. Since $\widehat{u} \in F(S) \cap F(T) \subset C_n$, we have that

$$\|u - \overline{u}_n\| \leq \|u - \widehat{u}\| \tag{5.8}$$

for all $n \in \mathbb{N}$. This means that $\{\overline{u}_n\}$ is bounded. From $\overline{u}_n = P_{C_n} u$ and $\overline{u}_{n+1} = P_{C_{n+1}} u \in C_{n+1} \subset C_n$, we have that

$$\|u - \overline{u}_n\| \leq \|u - \overline{u}_{n+1}\|.$$

Thus, $\{\|u - \overline{u}_n\|\}$ is bounded and nondecreasing. As a result, there exists the limit of $\{\|u - \overline{u}_n\|\}$. Put $c = \lim_{n\to\infty} \|u - \overline{u}_n\|$. We will show that $\{\overline{u}_n\}$ is convergent in C. For any $m, n \in \mathbb{N}$ with $m \geq n$, we have $C_m \subset C_n$. From $\overline{u}_m = P_{C_m} u \in C_m \subset C_n$ and (2.4), we have that

$$\|u - \overline{u}_n\|^2 + \|\overline{u}_n - \overline{u}_m\|^2 \leq \|u - \overline{u}_m\|^2.$$

This implies that

$$\|\overline{u}_n - \overline{u}_m\|^2 \leq \|u - \overline{u}_m\|^2 - \|u - \overline{u}_n\|^2 \leq c^2 - \|u - \overline{u}_n\|^2. \tag{5.9}$$

Since $c^2 - \|\overline{u}_n - u\|^2 \to 0$ as $n \to \infty$, we have that $\{\overline{u}_n\}$ is a Cauchy sequence. By the completeness of C, there exists a point $\overline{u} \in C$ such that $\lim_{n\to\infty} \overline{u}_n = \overline{u}$. Since the metric projection P_{C_n} is nonexpansive, it follows that

$$\begin{aligned} \|x_n - \overline{u}\| &\leq \|x_n - \overline{u}_n\| + \|\overline{u}_n - \overline{u}\| \\ &= \|P_{C_n} u_n - P_{C_n} u\| + \|\overline{u}_n - \overline{u}\| \\ &\leq \|u_n - u\| + \|\overline{u}_n - \overline{u}\|. \end{aligned}$$

Since $u_n \to u$ and $\overline{u}_n \to \overline{u}$, we obtain that

$$x_n \to \overline{u}. \tag{5.10}$$

To complete the proof, it is sufficient to show that $\widehat{u}(\equiv P_{F(S)\cap F(T)} u) = \overline{u}(= \lim x_n)$. From (5.10), we have that

$$\|x_n - x_{n+1}\| \to 0.$$

From $x_{n+1} = P_{C_{n+1}} u_{n+1} \in C_{n+1}$, we also have that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. So, we get that $\|y_n - x_{n+1}\| \to 0$. Using this, we have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0.$$

From $0 \leq \liminf_{n\to\infty} \lambda_n < 1$, we have a subsequence $\{\lambda_{n_i}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_i} \to \lambda \in [0,1)$. From

$$\|y_n - x_n\| = \|\lambda_n x_n + (1-\lambda_n) S_n x_n - x_n\| = (1-\lambda_n)\|S_n x_n - x_n\|,$$

we have that

$$\|S_{n_i} x_{n_i} - x_{n_i}\| \to 0. \tag{5.11}$$

From (5.10), (5.11) and Lemma 3.1, we have that $\overline{u} \in F(S) \cap F(T)$. Since $\widehat{u} = P_{F(S)\cap F(T)} u$ and $\overline{u} \in F(S) \cap F(T)$, it holds that

$$\|u - \widehat{u}\| \leq \|u - \overline{u}\|.$$

On the other hand, since $\overline{u}_n \to \overline{u}$, we have from (5.8) that

$$\|u - \overline{u}\| \leq \|u - \widehat{u}\|.$$

Thus, $\|u - \widehat{u}\| = \|u - \overline{u}\|$, which means that $\widehat{u} = \overline{u}$. Therefore, $\{x_n\}$ converges strongly to $\widehat{u} = P_{F(S)\cap F(T)} u$. This completes the proof. □

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