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ON ISEKI' STRICT FIXED POINT THEOREM

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Abstract. In this paper we will present an extended version of Iseki' strict fixed point theorem for multi-valued operators. We will discuss existence, uniqueness, continuous data dependence of the strict fixed point as well as some other stability properties, such as well-posednss, Ulam-Hyers stability and Ostrowski property.

1. Introduction

One of the most important metric strict fixed point theorem for multi-valued operators was proved in 1972 by Simeon Reich (see [20]), in the context of a complete metric space. The proof of this result is based on a fixed point theorem for single-valued operators given by Reich in the same paper (see also [4, 23, 24]). In 1975, K. Iseki gives (see [9]) a generalization of Reich's theorem, using the above mentioned fixed point theorem for single-valued operators of S. Reich. For related results and generalizations see [1, 3, 5, 10].

It is worth to notice that strict fixed point theorems have nice applications in mathematical economics and game theory. The strict fixed point property also appear in the context of iterative methods for finding fixed points various classes of multi-valued operators.

The aim of this paper to present an extended (by the conclusions point of view) version of Iseki' strict fixed point theorem for multi-valued operators in complete metric space. Some related stability properties will be considered: data dependence, Ulam-Hyers stability, well-posedness, Ostrovski property. A local fixed point theorem will be proved too. Our results extend a recent work related to Reich's theorem, see [14].

2. Preliminaries

Let us recall first some important preliminary concepts and results.

Let (X, d) be a metric space and $P(X)$ be the family of all nonempty subsets of X. We denote by $P_{cl}(X)$ the family of all nonempty closed subsets of X and by $P_b(X)$ the family of all nonempty bounded subsets of X. Also $P_{b,cl}(X) := P_b(X) \cap P_{cl}(X)$. For $x_0 \in X$ and $r > 0$ we will also denote by $B(x_0; r) := \{x \in X | d(x_0, x) < r\}$ the open ball, respectively by $\tilde{B}(x_0; r) := \{x \in X | d(x_0, x) \leq r\}$ the closed ball, centered in x_0 with radius r.

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The following functionals are needed in the main sections:
(a) the gap functional generated by d:

$$D_d : P(X) \times P(X) \to \mathbb{R}_+,\ D_d(A,B) := \inf\{d(a,b) \mid a \in A,\ b \in B\}.$$

(b) the excess functional of A over B generated by d:

$$e_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ e_d(A,B) := \sup\{D_d(a,B) \mid a \in A\}.$$

(c) the Hausdorff-Pompeiu functional generated by d:

$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ H_d(A,B) = \max\{e_d(A,B), e_d(B,A)\}.$$

(d) the diameter functional generated by d:

$$\delta_d : P(X) \times P(X) \to \mathbb{R}_+,\ \delta_d(A,B) := \sup\{d(a,b) \mid a \in A,\ b \in B\}.$$

The diameter of a set $A \in P(X)$ is $diam_d(A) := \delta_d(A,A)$. If the context is evident, we will avoid the subscript d.

Some useful properties of these functionals are re-called (see, for example, [2,8,12]) in the next lemmas.

Lemma 2.1. *If (X,d) is a metric space, then we have:*

- (a) *if $A \in P_{cl}(X)$ and $x \in X$ are such that $D(x,A) = 0$, then $x \in A$.*
- (b) *if $A, B \in P_b(X)$ and $q < 1$, then, for every $a \in A$ there exists $b \in B$ such that $d(a,b) \geq q\delta(A,B)$.*
- (c) *the functional δ has the following properties:*
  - (1) *$\delta(A,B) = 0$ implies that $A = B = \{x^*\}$;*
  - (2) *$\delta(A,B) \leq \delta(A,C) + \delta(C,B)$, for all $A, B, C \in P_b(X)$;*
  - (3) *$\delta(A,B) = \delta(B,A)$, for all $A, B \in P_b(X)$.*
  - (d) *if $A \in P_b(X)$ then $diam(A) = 0$ if and only if A is a singleton*

Finally, let us recall that if X is a nonempty set and $F : X \to P(X)$ is a multi-valued operator, then we denote by $Fix(F) := \{x \in X : x \in F(x)\}$ the fixed point set for F, and by $SFix(F) := \{x \in X : \{x\} = F(x)\}$ the strict fixed point set for F. In some papers, the strict fixed point is called stationary point or end-point. We also denote by $Graph(F) := \{(x,y) \in X \times X | y \in F(x)\}$ the graph of F.

Moreover, for arbitrary $(x_0, x_1) \in Graph(F)$, the sequence $(x_n)_{n\in\mathbb{N}}$ with the property $x_{n+1} \in F(x_n)$ (for $n \in \mathbb{N}$) is called the sequence of successive approximations for F staring from (x_0, x_1).

The concepts of multi-valued weakly Picard operator and multi-valued Picard operator are very important in fixed point theory for multi-valued operators.

Definition 2.2 ([15, 24, 25])**.** Let (X,d) be a metric space. Then $F : X \to P(X)$ is called a multivalued weakly Picard operator (briefly, MWP operator) if for each $x \in X$ and each $y \in F(x)$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that

- (i) $x_0 = x,\ x_1 = y$;
- (ii) $x_{n+1} \in F(x_n)$, for all $n \in \mathbb{N}$;
- (iii) the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of F.

Let us recall the following important notion.

Definition 2.3. Let (X, d) be a metric space and $F : X \to P(X)$ be an MWP operator. Then we define the multivalued operator $F^\infty : Graph(F) \to P(Fix(F))$ by the formula $F^\infty(x, y) = \{z \in Fix(F) \mid$ there exists a sequence of successive approximations of F starting from (x, y) that converges to $z\}$.

An important concept is given by the following definition.

Definition 2.4. Let (X, d) be a metric space and $F : X \to P(X)$ an MWP operator. Then F is a ψ-multi-valued weakly Picard operator (briefly ψ-MWP operator) if $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous in 0 with $\psi(0) = 0$ and there exists a selection f^∞ of F^∞ such that

$$d(x, f^\infty(x, y)) \leq \psi(d(x, y)), \text{ for all } (x, y) \in Graph(F).$$

In particular, if $\psi(t) = ct$, we say that F is a c-multi-valued weakly Picard operator (briefly c-MWP operator).

Definition 2.5 (see [9])**.** Let (X, d) be a metric space. Then, $F : X \to P_{cl}(X)$ is called a multi-valued (α, β, γ)-contraction if $\alpha, \beta, \gamma \geq 0$, $\alpha + 2\beta + 2\gamma < 1$ and

$$\begin{aligned} H(F(x_1), F(x_2)) \leq{}& \alpha d(x_1, x_2) + \beta\left(D_d(x_1, F(x_1)) + D_d(x_2, F(x_2))\right) \\ &+ \gamma\left(D_d(x_1, F(x_2)) + D_d(x_2, F(x_1))\right), \text{ for all } x_1, x_2 \in X. \end{aligned}$$

Example 2.6. An (α, β, γ)-contraction is a c-MWP operator with $c := \frac{1-\beta-\gamma}{1-(\alpha+\beta+\gamma)}$.

Definition 2.7 ([15, 16])**.** We say that $F : X \to P(X)$ is a multi-valued Picard operator if:

(i) $SFix(F) = Fix(F) = \{x^*\}$;

(ii) $F^n(x) \overset{H_d}{\to} \{x^*\}$ as $n \to \infty$, for each $x \in X$.

Several examples of Picard and weakly Picard operators and some studies of it are presented in [12, 13, 15, 16, 18].

3. Iseki's strict fixed point theorem

In 1973, Hardy and Rogers proved the following very general fixed point theorem for single-valued operators.

Theorem 3.1 (Hardy-Rogers' Theorem)**.** *Let (X, d) be a complete metric space and $f : X \to X$ be a (α, β, γ)-contraction, i.e., $\alpha, \beta, \gamma \geq 0$, $\alpha + 2\beta + 2\gamma < 1$ and*

$$\begin{aligned} d(f(x_1), f(x_2)) \leq{}& \alpha d(x_1, x_2) + \beta\left(d(x_1, f(x_1)) + d(x_2, f(x_2))\right) \\ &+ \gamma\left(d(x_1, f(x_2)) + d(x_2, f(x_1))\right), \text{ for all } x_1, x_2 \in X. \end{aligned}$$

Then f has a unique fixed point, i.e., there exists a unique $x^* \in X$ such that $x^* = f(x^*)$.

In particular, if $\gamma = 0$ in the above theorem, then we get Reich's Theorem in [20].

In 1975, Iseki extended the above result to the case of multi-valued operators, proving a strict fixed point theorem. The purpose of this section is to study the strict fixed point problem for the class of multi-valued $(\alpha, \beta, \gamma) - \delta$-contractions in the sense of Iseki.

Actually, in [9] K. Iseki proved the following strict fixed point principle.

Theorem 3.2 (Iseki's Theorem). *Let (X, d) be a complete metric space and $F : X \to P_b(X)$ be a multi-valued operator for which there exist $\alpha, \beta, \gamma \geq 0$ with $\alpha + 2\beta + 4\gamma < 1$ such that*

$$\begin{aligned}\delta(F(x), F(y)) &\leq \alpha d(x, y) + \beta\left(\delta(x, F(y)) + \delta(y, F(y))\right)\\ &\quad + \gamma\left(\delta(x, F(y)) + \delta(y, F(x))\right), \text{ for all } x, y \in X.\end{aligned}$$

Then $Fix(F) = SFix(F) = \{x^*\}$.

Proof. A. Iseki's original proof. Let $p := \sqrt{\alpha + 2\beta + 4\gamma} \in (0, 1)$. Then, by Lemma 2.1, we can define a selection $f : X \to X$ of F, by letting to each point $x \in X$ the point $f(x) \in F(x)$ which satisfies $d(x, f(x)) \geq p\delta(x, F(x))$. Then, we have

$$\begin{aligned}d(f(x), f(y)) &\leq \delta(F(x), F(y))\\ &\leq \alpha d(x, y) + \beta\left(\delta(x, F(x)) + \delta(y, F(y))\right)\\ &\quad + \gamma\left(\delta(x, F(y)) + \delta(y, F(x))\right)\\ &\leq \alpha d(x, y) + \beta\left(\delta(x, F(x)) + \delta(y, F(y))\right)\\ &\quad + \gamma\left(2d(x, y) + \delta(x, F(x)) + \delta(y, F(y))\right)\\ &\leq \alpha d(x, y) + \beta p^{-1}\left(d(x, f(x)) + d(y, f(y))\right)\\ &\quad + \gamma p^{-1}\left(2d(x, y) + d(x, f(x)) + d(y, f(y))\right)\\ &= (\alpha + 2\gamma p^{-1})d(x, y) + (\beta + \gamma)p^{-1}\left(d(x, f(x)) + d(y, f(y))\right).\end{aligned}$$

Since $\alpha + 2\beta p^{-1} + 4\gamma p^{-1} < p^{-1}(\alpha + 2\beta + 4\gamma) = p < 1$, we obtain that f satisfies all the conditions of Reich' Theorem (see Theorem 3.1 for the case $\gamma = 0$) and hence f has a unique fixed point $x^* \in X$. Thus $x^* \in Fix(F)$.

On the other hand, let us notice that, since $0 = d(x^*, f(x^*)) \geq p\delta(x^*, F(x^*))$, we get $\delta(x^*, F(x^*)) = 0$ and thus $F(x^*) = \{x^*\}$. Hence, $x^* \in SFix(F)$. We show now that $Fix(F) \subset SFix(F)$. Indeed, let $y \in Fix(F)$. If $\delta(y, F(y)) > 0$, then

$$diam(F(y)) = \delta(F(y), F(y)) \leq 2(\beta + \gamma)\delta(y, F(y)) < \delta(y, F(y)),$$

which is a contradiction. Hence $\delta(y, F(y)) = 0$ and thus $F(y) = \{y\}$.

For the uniqueness of the strict fixed point (and the fixed point too), we notice that, if $z \in X$ is another strict fixed point of F such that $x^* \neq z$, then we have

$$\begin{aligned}d(x^*, z) &= \delta(F(x^*), F(z))\\ &\leq \alpha d(x^*, z) + \beta\left(\delta(x^*, F(x^*)) + \delta(z, F(z))\right)\\ &\quad + \gamma\left(\delta(x^*, F(z)) + \delta(z, F(x^*))\right)\\ &= (\alpha + 2\gamma)d(x^*, z) < d(x^*, z),\end{aligned}$$

which gives a contradiction. Thus $z = x^*$.

B. An alternative proof. Let $q > 1$ and let $x_0 \in X$ be arbitrary. Then there exists $x_1 \in F(x_0)$ such that $\delta(x_0, F(x_0)) \leq q \cdot d(x_0, x_1)$. Thus, we have

$$\begin{aligned}\delta(x_1, F(x_1)) &\leq \delta(F(x_0), F(x_1)) \\ &\leq \alpha d(x_0, x_1) + \beta\left(\delta(x_0, F(x_0)) + \delta(x_1, F(x_1))\right) \\ &\quad + \gamma\left(\delta(x_0, F(x_1)) + \delta(x_1, F(x_0))\right) \\ &\leq \alpha d(x_0, x_1) + \beta q d(x_0, x_1) + \beta \delta(x_1, F(x_1)) \\ &\quad + \gamma\left(2d(x_0, x_1) + \delta(x_0, F(x_0)) + \delta(x_1, F(x_1))\right) \\ &\leq (\alpha + 2\gamma) d(x_0, x_1) + (\beta + \gamma) q d(x_0, x_1) + (\beta + \gamma)\delta(x_1, F(x_1)).\end{aligned}$$

Hence, we get

$$\delta(x_1, F(x_1)) \leq \frac{\alpha + 2\gamma + (\beta + \gamma)q}{1 - \beta - \gamma} d(x_0, x_1).$$

By this approach we can construct a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations for F, such that

$$d(x_n, x_{n+1}) \leq \delta(x_n, F(x_n)) \leq \left(\frac{\alpha + 2\gamma + (\beta + \gamma)q}{1 - \beta - \gamma}\right)^n d(x_0, x_1), \text{ for all } n \in \mathbb{N}.$$

Choosing $1 < q < \frac{1-(\alpha+\beta+3\gamma)}{\beta+\gamma}$ we obtain $\frac{\alpha+2\gamma+(\beta+\gamma)q}{1-\beta-\gamma} < 1$. Hence $(x_n)_{n\in\mathbb{N}}$ ia a Cauchy sequence in the complete metric space (X, d). Let us denote by $x^* \in X$ its limit. We show that x^* is a strict fixed point for F, i.e., $F(x^*) = \{x^*\}$. We have

$$\delta(x^*, F(x^*)) \leq d(x^*, x_{n+1}) + D(x_{n+1}, F(x_n)) + \delta(F(x_n), F(x^*)) \leq$$

$$\begin{aligned}&d(x^*, x_{n+1}) + \alpha d(x_n, x^*) + \beta \delta(x_n, F(x_n)) \\ &\quad + \beta \delta(x^*, F(x^*)) + \gamma\left(\delta(x_n, F(x^*)) + \delta(x^*, F(x_n))\right) \\ &\leq d(x^*, x_{n+1}) + (\alpha + 2\gamma) d(x_n, x^*) + (\beta + \gamma)\delta(x_n, F(x_n)) + (\beta + \gamma)\delta(x^*, F(x^*)) \\ &\leq d(x^*, x_{n+1}) + (\alpha + 2\gamma) d(x_n, x^*) + (\beta + \gamma)\left(\frac{\alpha + 2\gamma + (\beta + \gamma)q}{1 - \beta - \gamma}\right)^n d(x_0, x_1) \\ &\quad + (\beta + \gamma)\delta(x^*, F(x^*)).\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}\delta(x^*, F(x^*)) &\leq \frac{1}{1 - \beta - \gamma}\left(d(x^*, x_{n+1}) + (\alpha + 2\gamma) d(x_n, x^*)\right) \\ &\quad + \frac{\beta + \gamma}{1 - \beta - \gamma}\left(\frac{\alpha + 2\gamma + (\beta + \gamma)q}{1 - \beta - \gamma}\right)^n d(x_0, x_1).\end{aligned}$$

As $n \to \infty$, we obtain that $\delta(x^*, F(x^*)) = 0$ and thus $F(x^*) = \{x^*\}$. The fact that $Fix(F) = SFix(F)$ and the uniqueness of the strict fixed point follow as before. □

Remark 3.3. By the alternative proof, it also follows that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for F starting from arbitrary $x_0 \in X$, such that

$$d(x_n, x^*) \leq \frac{k^n}{1 - k} d(x_0, x_1), \text{ for all } n \in \mathbb{N},$$

where $k := \frac{\alpha+2\gamma+(\beta+\gamma)q}{1-\beta-\gamma}$, with any $q \in (1, \frac{1-(\alpha+\beta+3\gamma)}{\beta+\gamma})$.

On the other hand, it is worth to notice that by Iseki's original proof we also obtain, taking into account the proof of Reich's Theorem (see [21], [19]), that the sequence $u_n := f^n(x_0)$, for $n \in \mathbb{N}^*$ (where x_0 is arbitrary in X) converges to $x^* \in Fix(f) \subset Fix(F)$ (where f is the selection of F constructed above) and the following apriori estimation holds

$$d(u_n, x^*) \leq \frac{K^n}{1-K} d(x_0, f(x_0)), \text{ for all } n \in \mathbb{N},$$

where $K := \frac{\alpha\sqrt{\alpha+2\beta+4\gamma}+(\beta+3\gamma)}{\sqrt{\alpha+2\beta+4\gamma}-(\beta+\gamma)} < 1$. Hence, for the strict fixed point $x^* \in X$ the following estimation holds

$$d(u_n, x^*) \leq \frac{K^n}{1-K} d(x_0, x_1), \text{ for all } n \in \mathbb{N}.$$

Remark 3.4. The following open question can be pointed out in this context: prove an Iseki type theorem for multi-valued operators using Hardy-Rogers' fixed point theorem in [7], see Theorem 3.1.

We also the following interesting result.

Theorem 3.5. *Let (X, d) be a complete metric space and $F : X \to P_b(X)$ be a multi-valued operator for which there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$ with $\alpha + 2\beta + 4\gamma < 1$, such that, for all $x, y \in X$ we have*

$$\delta(F(x), F(y)) \leq \alpha d(x, y) + \beta\left(\delta(x, F(x)) + \delta(y, F(y))\right) + \gamma\left(\delta(x, F(y)) + \delta(y, F(x))\right).$$

Then F is a MP operator.

Proof. By Theorem 3.2 we know that $Fix(F) = SFix(F) = \{x^*\}$. We have to prove that $F^n(x) \overset{H_d}{\to} \{x^*\}$ as $n \to \infty$, for each $x \in X$. We have, for every $x \in X$, that

$$\begin{aligned}\delta(F(x), x^*) &= \delta(F(x), F(x^*)) \\ &\leq \alpha d(x, x^*) + \beta\left(\delta(x, F(x)) + \delta(x^*, F(x^*))\right) \\ &\quad + \gamma\left(\delta(x, F(x^*)) + \delta(x^*, F(x))\right) \\ &\leq \alpha d(x, x^*) + \beta(d(x, x^*) + \delta(x^*, F(x))) \\ &\quad + \gamma(d(x, x^*) + \delta(x^*, F(x^*)) + \delta(x^*, F(x))).\end{aligned}$$

Thus

$$\delta(F(x), x^*) \leq \frac{\alpha+\beta+\gamma}{1-\beta-\gamma} d(x, x^*), \text{ for all } x \in X.$$

Then

$$\delta(F^2(x), x^*) = \sup_{y \in F(x)} \delta(F(y), x^*) \leq$$

$$\sup_{y \in F(x)} \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right) d(y, x^*) \leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)^2 d(x, x^*).$$

By mathematical induction, we get that

$$\delta(F^n(x), x^*) \leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)^n d(x, x^*) \to 0 \text{ as } n \to +\infty, \text{ for each } x \in X.$$

The proof is now complete. □

It is important also to get a localization of the (strict) fixed point for a multi-valued operator. In the case of $(\alpha,\beta,\gamma)-\delta$-contractions we have the following local fixed point theorem.

Theorem 3.6. *Let (X,d) be a complete metric space, $x_0 \in X$ and $r>0$. Suppose that $F : \tilde{B}(x_0;r) \to P_b(X)$ is a multi-valued operator for which:*

(a) *there exist $\alpha,\beta,\gamma \in \mathbb{R}_+$ with $0<\alpha+2\beta+4\gamma<1$ such that, for all $x,y \in \tilde{B}(x_0;r) \subset X$, the following condition is satisfied*

$$\delta(F(x),F(y)) \le \alpha d(x,y)+\beta\left(\delta(x,F(x))+\delta(y,F(y))\right)+\gamma\left(\delta(x,F(y))+\delta(y,F(x))\right);$$

(b) $\delta(x_0,F(x_0)) \le \frac{1-\alpha-2\beta-4\gamma}{1+\beta+\gamma}r$.

Then there exists a unique $x^* \in \tilde{B}(x_0;r)$ such that $SFix(F)=Fix(F)=\{x^*\}$. In particular, if $\beta>0$ then $x^* \in B(x_0;r)$.

Proof. We will show that the closed ball $\tilde{B}(x_0;r)$ is invariant with respect to F, i.e., $F(\tilde{B}(x_0;r)) \subseteq \tilde{B}(x_0;r)$. For this purpose, let $x \in \tilde{B}(x_0;r)$ and $y \in F(x)$ be arbitrary chosen. Then we have $d(y,x_0) \le \delta(F(x),F(x_0))+\delta(x_0,F(x_0))$. On the other hand,

$$\begin{aligned}
\delta(F(x),F(x_0)) &\le \alpha r+\beta\left(\delta(x,F(x))+\delta(x_0,F(x_0))\right)\\
&\quad+\gamma\left(\delta(x,F(x_0))+\delta(x_0,F(x))\right)\\
&\le \alpha r+\beta\left(d(x,x_0)+\delta(x_0,F(x_0))+\delta(F(x_0),F(x))+\delta(x_0,F(x_0))\right)\\
&\quad+\gamma\left(d(x,x_0)+\delta(x_0,F(x_0))+d(x_0,x)+\delta(x,F(x))\right)\\
&\le (\alpha+\beta+3\gamma)r+(2\beta+2\gamma)\delta(x_0,F(x_0))+(\beta+\gamma)\delta(F(x_0),F(x)).
\end{aligned}$$

Hence

$$\delta(F(x),F(x_0)) \le \frac{(\alpha+\beta+3\gamma)r+2(\beta+\gamma)\delta(x_0,F(x_0))}{1-\beta-\gamma}.$$

Hence, going back to our first relation we get

$$\begin{aligned}
d(y,x_0) &\le \frac{(\alpha+\beta+3\gamma)r+2(\beta+\gamma)\delta(x_0,F(x_0))}{1-\beta-\gamma}+\delta(x_0,F(x_0))\\
&= \frac{(\alpha+\beta+3\gamma)r+(\beta+\gamma+1)\delta(x_0,F(x_0))}{1-\beta-\gamma}.
\end{aligned}$$

By (b) we obtain that $d(y,x_0) \le r$, proving that the closed ball $\tilde{B}(x_0;r)$ is invariant with respect to F. The conclusion follows now by Theorem 3.2.
If $\beta>0$, then we can show that $x^* \in B(x_0;r)$. Indeed, suppose, by contradiction, that $d(x^*,x_0)=r$. Then we have

$$\begin{aligned}
r = d(x^*,x_0) &\le \delta(F(x^*),F(x_0))+\delta(x_0,F(x_0))\\
&\le (\alpha+2\gamma)d(x^*,x_0)+(\beta+\gamma+1)\delta(x_0,F(x_0))\\
&\le (\alpha+2\gamma)r+(\beta+\gamma+1)\frac{1-\alpha-2\beta-4\gamma}{1+\beta+\gamma}r\\
&= (1-2\beta-2\gamma)r<r.
\end{aligned}$$

This is a contradiction and the proof is now complete. □

We will discuss now the well-posedness of the strict fixed point problem. For the well-posedness concept in the single-valued case see the paper Reich-Zaslavski [22], while the multi-valued case is considered in [17].

Definition 3.7. Let (X, d) be a metric space and $F : X \to P_b(X)$ be a multivalued operator. The strict fixed point problem

$$\{x\} = F(x),\ x \in X \tag{3.1}$$

is well-posed for F if:

(a_2) $SFix(F) = \{x^*\}$

(b_2) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $\delta(x_n, F(x_n)) \to 0$ as $n \to +\infty$, then $x_n \to x^*$ as $n \to +\infty$.

In this respect, we have the following result.

Theorem 3.8. *Let (X, d) be a complete metric space and $F : X \to P_b(X)$ be a multi-valued operator for which there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$ with $0 < \alpha + 2\beta + 4\gamma < 1$ such that, for all $x, y \in X$, we have*

$$\delta(F(x), F(y)) \le \alpha d(x, y) + \beta\left(\delta(x, F(x)) + \delta(y, F(y))\right) + \gamma\left(\delta(x, F(y)) + \delta(y, F(x))\right).$$

Then the strict fixed point problem is well-posed for F.

Proof. By Theorem 3.2 we know that $Fix(F) = SFix(F) = \{x^*\}$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $\delta(x_n, F(x_n)) \to 0$ as $n \to +\infty$. We will prove that $x_n \to x^*$ as $n \to +\infty$. For this purpose, we have

$$\begin{aligned} d(x_n, x^*) &\le \delta(x_n, F(x_n)) + \delta(F(x_n), F(x^*)) \\ &\le \delta(x_n, F(x_n)) + \alpha d(x_n, x^*) + \beta\left(\delta(x_n, F(x_n)) + \delta(x^*, F(x^*))\right) \\ &\quad + \gamma\left(\delta(x_n, F(x^*)) + \delta(x^*, F(x_n))\right) \\ &\le (1 + \beta + \gamma)\delta(x_n, F(x_n)) + (\alpha + 2\gamma)d(x_n, x^*). \end{aligned}$$

Letting $n \to \infty$ we obtain the desired conclusion. □

We will continue our study by presenting the concept of Ulam-Hyers stability for the strict fixed point problem. For related definitions and results see [11].

Definition 3.9. Let (X, d) be a metric space and $F : X \to P_b(X)$ be a multi-valued operator. The strict fixed point problem (3.1) is called Ulam-Hyers stable if there exists $c > 0$ such that for each $\varepsilon > 0$ and for each ε-solution $y \in X$ of the strict fixed point problem, i.e.,

$$\delta(y, F(y)) \le \varepsilon, \tag{3.2}$$

there exists a solution $x^* \in X$ of the strict fixed point inclusion (3.1) such that

$$d(y, x^*) \le c\varepsilon.$$

We have the following result concerning the Ulam-Hyers stability of the strict fixed point problem.

Theorem 3.10. *Let (X,d) be a complete metric space and $F : X \to P_b(X)$ be a multi-valued operator for which there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$ with $0 < \alpha + 2\beta + 4\gamma < 1$ such that, for all $x, y \in X$, we have*

$$\delta(F(x), F(y)) \leq \alpha d(x,y) + \beta\left(\delta(x, F(x)) + \delta(y, F(y))\right) + \gamma\left(\delta(x, F(y)) + \delta(y, F(x))\right).$$

Then the strict fixed point problem is Ulam-Hyers stable.

Proof. By Theorem 3.2 we know that $Fix(F) = SFix(F) = \{x^*\}$. Let $\varepsilon > 0$ and $y \in X$ such that $\delta(y, F(y)) \leq \varepsilon$. Then, we have

$$\begin{aligned} d(y, x^*) &\leq \delta(y, F(y)) + \delta(F(y), F(x^*)) \\ &\leq \delta(y, F(y)) + \alpha d(y, x^*) + \beta\left(\delta(y, F(y)) + \delta(x^*, F(x^*))\right) \\ &\quad + \gamma\left(\delta(y, F(x^*)) + \delta(x^*, F(y))\right) \\ &\leq (1 + \beta + \gamma)\delta(y, F(y)) + (\alpha + 2\gamma) d(y, x^*). \end{aligned}$$

Thus

$$d(y, x^*) \leq \frac{1 + \beta + \gamma}{1 - \alpha - 2\gamma}\delta(y, F(y)) \leq \frac{1 + \beta + \gamma}{1 - \alpha - 2\gamma}\varepsilon.$$

The proof is complete. □

Another stability concept is given in the next definition.

Definition 3.11. Let (X,d) be a metric space and $F : X \to P(X)$ be a multi-valued operator with $SFix(F) = \{x^*\}$. If $(y_n)_{n\in\mathbb{N}}$ is a sequence in X such that the following implication holds

$$D(y_{n+1}, F(y_n)) \to 0 \text{ as } n \to \infty \;\Rightarrow\; y_n \to x^* \text{ as } n \to \infty,$$

then we say that the strict fixed point problem (3.1) has the Ostrovski property.

Theorem 3.12. *Let (X,d) be a complete metric space and $F : X \to P_b(X)$ be a multi-valued operator for which there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$ with $0 < \alpha + 2\beta + 4\gamma < 1$ such that, for all $x, y \in X$, we have*

$$\begin{aligned} \delta(F(x), F(y)) &\leq \alpha d(x,y) + \beta\left(\delta(x, F(x)) + \delta(y, F(y))\right) \\ &\quad + \gamma\left(\delta(x, F(y)) + \delta(y, F(x))\right). \end{aligned}$$

Then the strict fixed point problem as the Ostrovski property.

Proof. By Theorem 3.2 we know that $Fix(F) = SFix(F) = \{x^*\}$. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in X such that $D(y_{n+1}, F(y_n)) \to 0$ as $n \to \infty$. Next, we have

$$d(y_{n+1}, x^*) \leq D(y_{n+1}, F(y_n)) + \delta(F(y_n), x^*).$$

On the other hand, we observe that

$$\begin{aligned} \delta(F(y_n), x^*) &= \delta(F(y_n), F(x^*)) \\ &\leq \alpha d(y_n, x^*) + \beta\delta(y_n, F(y_n)) + \gamma\left(\delta(y_n, F(x^*)) + \delta(x^*, F(y_n))\right)) \\ &\leq (\alpha + \gamma) d(y_n, x^*) + \beta\left(d(y_n, x^*) + \delta(x^*, F(y_n))\right) + \gamma\delta(x^*, F(y_n)) \\ &= (\alpha + \beta + \gamma) d(y_n, x^*) + (\beta + \gamma)\delta(x^*, F(y_n)). \end{aligned}$$

Hence

$$\delta(F(y_n), x^*) \leq \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} d(y_n, x^*) \text{ for all } n \in \mathbb{N}.$$

Denote $p := \frac{\alpha+\beta+\gamma}{1-\beta-\gamma} \in (0,1)$. As a consequence, we obtain

$$\begin{aligned} d(y_{n+1}, x^*) & \\ & \leq D(y_{n+1}, F(y_n)) + pd(y_n, x^*) \leq \cdots \\ & \leq \sum_{k=0}^{n} p^k D(y_{n-k+1}, F(y_{n-k})) + p^{n+1} d(y_0, x^*). \end{aligned}$$

By Cauchy's Lemma (see [18]), we obtain the desired conclusion. □

Finally, we will present a data dependence theorem for the strict fixed point problem.

Theorem 3.13. *Let (X,d) be a complete metric space and $F : X \to P_b(X)$ be a multi-valued operator for which there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$ with $0 < \alpha + 2\beta + 4\gamma < 1$ such that, for all $x, y \in X$, we have*

$$\delta(F(x), F(y)) \leq \alpha d(x,y) + \beta\left(\delta(x, F(x)) + \delta(y, F(y))\right) + \gamma\left(\delta(x, F(y)) + \delta(y, F(x))\right).$$

Suppose that $G : X \to P_b(X)$ is a multi-valued operator such that $SFix(G) \neq \emptyset$ and there exists $\eta > 0$ such that $\delta(F(x), G(x)) \leq \eta$, for every $x \in X$. Then

$$\delta(SFix(F), SFix(G)) \leq \frac{\eta}{1-\alpha-2\gamma}.$$

Proof. By Theorem 3.2 we know that $Fix(F) = SFix(F) = \{x^*\}$. Let $y \in SFix(G)$ be arbitrary chosen. Then, we also have

$$\begin{aligned} d(y, x^*) = \delta(G(y), F(x^*)) \leq \delta(G(y), F(y)) + \delta(F(y), F(x^*)) \leq \\ \eta + (\alpha + 2\gamma) d(y, x^*). \end{aligned}$$

Thus $d(y, x^*) \leq \frac{\eta}{1-\alpha-2\gamma}$, which gives immediately the desired conclusion. □

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