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WELL-POSEDNESS OF FIXED POINT PROBLEMS FOR MONOTONE NONEXPANSIVE MAPPINGS

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Abstract. We study monotone nonexpansive self-mappings of a bounded and closed subset of an ordered Banach space. Using the Baire category approach, we show, in particular, that for most such mappings, the fixed point problem is well posed. To this end, we first establish a stability result in ordered and complete metric spaces.

1. Introduction and a stability result

It is well known that the notion of well-posedness is of great significance in many areas of mathematics and its applications. In the present paper we study generic well-posedness of fixed point problems for certain classes of mappings. In this connection, we recall that since Banach's classical theorem [2], fixed point theory has been and continues to be an important part of nonlinear operator theory [3, 8–11, 15, 16, 20, 21, 23]. For example, several results regarding the existence of fixed points for general nonexpansive mappings in special Banach spaces were presented in [8, 9, 13, 14] and an existence result for contractive mappings was obtained in [12]. A number of existence and convergence results in the generic sense were obtained for general classes of nonlinear mappings by using the Baire category approach [4–6, 17–19, 21, 22, 24, 27]. More precisely, in these papers, given a certain space of mappings equipped with a complete metric, it is shown that it contains a subset, which is a countable intersection of open and everywhere dense sets, such that for each one of its elements, the corresponding fixed point problem has a unique solution and moreover, is well posed. This is true, in particular, for the class of nonexpansive self-mappings of a bounded, closed and convex set in a Banach space [4, 24]. In this paper we establish generic well-posedness of the fixed point problem for monotone nonexpansive mappings – a class of nonlinear mappings which has been the subject of a rapidly growing area of research [1, 7, 25, 26]. This generic well-posedness result is stated and proved in Section 4 (see Theorem 4.1). A preparatory stability theorem (Theorem 1.1) is stated in this section and proved in the next one. This theorem, in

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its turn, easily implies an extension, which is presented in Section 3 (see Theorem 3.1). In this connection, we remark in passing that in the present paper we study *continuous* mappings, while in [26] we deal with *uniformly continuous* ones. In [25] we study three classes of monotone *contractive* mappings on a complete metric space.

Let (X, ρ) be a complete metric space equipped with an order $\leq$ such that for all points $x, y, z \in X$,

$$x \leq x,$$

$$\text{if } x \leq y, \ y \leq x, \ \text{then } x = y,$$

and

$$\text{if } x \leq y, \ y \leq z, \ \text{then } x \leq z.$$

We assume that

$$\{(x, y) \in X \times X : \ x \leq y\}$$

is a closed subset of $X \times X$ and that for all $x, y, z \in X$ satisfying $x \leq y \leq z$, we have

$$\rho(x, y) \leq \rho(x, z). \tag{1.1}$$

Let K be a nonempty closed subset of X, which is not a singleton, and let $x_K \in K$ satisfy

$$x_K \leq x \text{ for all } x \in K. \tag{1.2}$$

For each $x \in K$ and each $r > 0$, set

$$B(x, r) := \{y \in X : \ \rho(x, y) \leq r\}.$$

Denote by $\mathcal{A}$ the set of all continuous mappings $T : K \to K$ which satisfy

$$T(x) \leq T(y) \text{ for all } x, y \in K \text{ such that } x \leq y. \tag{1.3}$$

We equip the set $\mathcal{A}$ with the uniformity determined by the base

$$\mathcal{U}(n, \epsilon) := \{(T_1, T_2) \in \mathcal{A} \times \mathcal{A} :$$

$$\rho(T_1(x), T_2(x)) \leq \epsilon \text{ for all } x \in B(x_K, n)\},$$

where $n, \epsilon > 0$. It is not difficult to see that the uniform space $\mathcal{A}$ is metrizable and complete.

Theorem 1.1. *Let $T \in \mathcal{A}$ satisfy*

$$\rho(T(x), T(y)) \leq \rho(x, y) \tag{1.4}$$

for all $x, y \in K$ such that $x \leq y$, let $\bar{x}$ and x_* belong to K, and assume that

$$T(x_*) = x_*. \tag{1.5}$$

Assume further that for each $\epsilon > 0$, there exists a natural number n_0 such that for each integer $n \geq n_0$ and each point $x \in K$ satisfying $x_K \leq x \leq \bar{x}$, we have

$$\rho(T^n(x), x_*) \leq \epsilon.$$

Then for each $\epsilon > 0$, there exist a neighborhood $\mathcal{U}$ of T in $\mathcal{A}$ and a natural number n_0 such that for each mapping $S \in \mathcal{U}$ and each point $x \in K$ satisfying $x_K \leq x \leq \bar{x}$, we have

$$\rho(S^{n_0}(x), x_*) \leq \epsilon.$$

2. Proof of Theorem 1.1

Let $\epsilon \in (0,1)$ be given. There exists a natural number n_0 such that the following property holds:

(i) for each integer $n \geq n_0$ and each point $x \in K$ satisfying $x \leq \bar{x}$, we have

$$\rho(T^n(x), x_*) \leq \epsilon/8.$$

Set $T^0(x) := x$ for all $x \in K$ and

$$\epsilon_{n_0} := \epsilon/8. \tag{2.1}$$

Next we define by induction a finite sequence of positive numbers $\{\epsilon_i\}_{i=0}^{n_0}$ such that for each integer $i \in \{0, \dots, n_0 - 1\}$, we have

$$\epsilon_i < \epsilon_{i+1}/8, \tag{2.2}$$

$$\rho(T(T^i(x_K)), T(z)) \leq \epsilon_{i+1}/8 \text{ for each } z \in K \cap B(T^i(x_K), \epsilon_i) \tag{2.3}$$

and

$$\rho(T(T^i(\bar{x})), T(z)) \leq \epsilon_{i+1}/8 \text{ for each } z \in K \cap B(T^i(\bar{x}), \epsilon_i). \tag{2.4}$$

Set

$$\mathcal{U} := \{S \in \mathcal{A} : \ \rho(T(z), S(z)) \leq \epsilon_0/4$$

$$\text{for all } z \in K \cap (\cup_{i=0}^{n_0} B(T^i(x_K), 1) \cup \cup_{i=0}^{n_0} B(T^i(\bar{x}), 1))\}. \tag{2.5}$$

Let

$$S \in \mathcal{U} \tag{2.6}$$

and

$$z \in \{\bar{x}, \ x_K\}.$$

We claim that for all $i = 0, \dots, n_0$, we have

$$\rho(S^i(z), T^i(z)) \leq \epsilon_i. \tag{2.7}$$

Clearly, (2.7) holds for $i = 0$. By (2.2), (2.5) and (2.6), inequality (2.7) also holds for $i = 1$.

Assume now that an integer i satisfies $0 \leq i < n_0$ and that (2.7) holds. It follows from (2.1), (2.2) and (2.7) that

$$S^i(z) \in K \cap (\cup_{j=0}^{n_0} B(T^j(x_K), 1) \cup \cup_{j=0}^{n_0} B(T^j(\bar{x}), 1)). \tag{2.8}$$

By (2.6) and (2.8),

$$\rho(S^{i+1}(z), T(S^i(z))) \leq \epsilon_0/4. \tag{2.9}$$

In view of (2.4) and (2.7),

$$\rho(T(S^i(z)), T(S^i(z))) \leq \epsilon_{i+1}/8. \tag{2.10}$$

It now follows from (2.2), (2.9) and (2.10) that

$$\rho(S^{i+1}(z), T^{i+1}(z)) \leq \epsilon_{i+1}/8 + \epsilon_0/4 \leq \epsilon_{i+1}.$$

Thus (2.7) indeed holds for all $n = 0, \dots, n_0$ and

$$\rho(S^{n_0}(z), T^{n_0}(z)) \leq \epsilon_{n_0} = \epsilon/8. \tag{2.11}$$

Property (i) and (2.11) imply that

$$\rho(S^{n_0}(z), x_*) \le \epsilon/4$$

and

$$\rho(S^{n_0}(x_K), x_*) \le \epsilon/4, \ \rho(S^{n_0}(\bar{x}), x_*) \le \epsilon/4. \tag{2.12}$$

Let $x \in K$ satisfy

$$x_K \le x \le \bar{x}. \tag{2.13}$$

By (1.3) and (2.13),

$$S^{n_0}(x_K) \le S^{n_0}(x) \le S^{n_0}(\bar{x}). \tag{2.14}$$

It follows from (1.1), (2.12) and (2.14) that

$$\rho(S^{n_0}(x_K), S^{n_0}(x)) \le \rho(S^{n_0}(x_K), S^{n_0}(\bar{x})) \le \epsilon/2.$$

When combined with (2.12), this implies that

$$\rho(S^{n_0}(x), x_*) \le \epsilon.$$

Theorem 1.1 is proved.

3. An extension of Theorem 1.1

Theorem 1.1 easily implies the following result.

Theorem 3.1. *Let there exist a point $\bar{x} \in K$ such that*

$$x \le \bar{x} \text{ for all } x \in K.$$

Assume that $T \in \mathcal{A}$ and that for all $x, y \in K$ such that $x \le y$, we have

$$\rho(T(x), T(y)) \le \rho(x, y).$$

Let $x_* \in K$ satisfy

$$T(x_*) = x_*.$$

Assume further that $T^n(x) \to x_*$ as $n \to \infty$, uniformly over $x \in K$. Then for each $\epsilon > 0$, there exist a neighborhood $\mathcal{U}$ of T in $\mathcal{A}$ and a natural number n_0 such that for each mapping $S \in \mathcal{U}$, each point $x \in K$ and each integer $n \ge n_0$, we have

$$\rho(S^n(x), x_*) \le \epsilon.$$

4. A generic result

Let $(X, \|\cdot\|)$ be a Banach space ordered by a closed and convex cone $X_+ \subset X$ satisfying

$$X_+ \cap (-X_+) = \{0\}.$$

Note that for $x, y \in X$,

$$x \le y \text{ if and only if } y - x \in X_+.$$

Assume that

$$\|x\| \le \|y\| \text{ for all } x, y \in X_+ \text{ satisfying } x \le y.$$

Let K be a nonempty, bounded and closed subset of X,

$$\bar{x}_K, \widehat{x}_K \in K,$$

$$\bar{x}_K \le x \le \widehat{x}_K \text{ for all } x \in K, \tag{4.1}$$

$\theta \in K$ and assume that for each $x \in K$,

$$\{\gamma x + (1-\gamma)\theta : \ \gamma \in [0,1]\} \subset K. \tag{4.2}$$

Set

$$d(K) := \sup\{\|x-y\| : \ x, y \in K\}. \tag{4.3}$$

Denote by $\mathcal{M}$ the set of all continuous mappings $T : K \to K$ such that

$$T(x) \le T(y) \text{ for all } x, y \in K \text{ satisfying } x \le y \tag{4.4}$$

and

$$\|T(x) - T(y)\| \le \|x - y\| \tag{4.5}$$

for all $x, y \in K$ satisfying $x \le y$. Elements of the set $\mathcal{M}$ are said to be *monotone nonexpansive mappings*. For each $T_1, T_2 \in \mathcal{M}$, set

$$d(T_1, T_2) := \sup\{\|T_1(x) - T_2(x)\| : \ x \in K\}. \tag{4.6}$$

It is clear that $(\mathcal{M}, d)$ is a complete metric space.

Let $T \in \mathcal{M}$. We say that the fixed point problem (FPP) is *well posed* for T if there exists a unique point $x_T \in K$ such that

$$T(x_T) = x_T$$

and the following property holds:

(WP) for each $\epsilon > 0$, there exists a natural number n_ϵ such that for each $x \in K$,

$$\|x_T - T^{n_\epsilon}(x)\| \le \epsilon.$$

We are now ready to state and prove our generic well-posedness result.

Theorem 4.1. *There exists a set $\mathcal{F} \subset \mathcal{M}$, which is a countable intersection of open and everywhere dense sets in $\mathcal{M}$, such that for each $T \in \mathcal{F}$, the FPP is well posed.*

Proof. Let $A \in \mathcal{M}$ and $\gamma \in (0,1)$ be given. Define

$$A_\gamma(x) := (1-\gamma)A(x) + \gamma\theta. \tag{4.7}$$

It is clear that $A_\gamma : K \to K$ is a continuous mapping. If $x, y \in K$ satisfy $x \le y$, then we have

$$A_\gamma(x) \le A_\gamma(y), \tag{4.8}$$

$$\begin{aligned} &\|A_\gamma(x) - A_\gamma(y)\| \\ &= \|(1-\gamma)A(x) + \gamma\theta - (1-\gamma)A(y) - \gamma\theta\| \\ &= (1-\gamma)\|A(x) - A(y)\| \le (1-\gamma)\|x-y\| \qquad (4.9) \end{aligned}$$

and

$$\begin{aligned} d(A, A_\gamma) &= \sup\{\|A(x) - A_\gamma(x)\| : \ x \in K\} \\ &= \sup\{A(x) - (1-\gamma)A(x) - \gamma\theta : \ x \in K\} \le \gamma d(K). \qquad (4.10) \end{aligned}$$

In view of (4.10),

$$\{A_\gamma : \ A \in \mathcal{M} \text{ and } \gamma \in (0,1)\}$$

in an everywhere dense subset of $\mathcal{M}$.

For $A \in \mathcal{M}$, set $A^0(x) := x$ for all $x \in K$.

Let $A \in \mathcal{M}$ and $\gamma \in (0,1)$ be given. We claim that the FPP is well posed for A_γ. By (4.1), (4.5) and (4.9), for all points $x, y \in K$ satisfying $x \le y$ and every integer $n \ge 0$, we have

$$(A_\gamma)^n(\bar{x}_K) \le (A_\gamma)^n(x) \le (A_\gamma)^n(\widehat{x}_K)$$

and

$$\|(A_\gamma)^{n+1}(x) - (A_\gamma)^{n+1}(y)\| \le (1-\gamma)\|(A_\gamma)^n(x) - (A_\gamma)^n(y)\|.$$

This implies that for all integers $n \ge 0$ and all points $x \in K$,

$$\|(A_\gamma)^n(x) - (A_\gamma)^n(\bar{x}_K)\| \le (1-\gamma)^n\|x - \bar{x}_K\| \le (1-\gamma)^n d(K). \tag{4.11}$$

Let $x \in K$. Equation (4.11) implies that $\{(A_\gamma)^n(\bar{x}_K)\}_{n=0}^\infty$ is a Cauchy sequence and so there exists

$$x_* = \lim_{n\to\infty} (A_\gamma)^n(\bar{x}_K). \tag{4.12}$$

In view of (4.11) and (4.12), the FPP is indeed well posed for A_γ.

Let $A \in \mathcal{M},\ \gamma \in (0,1)$ and let k be a natural number. There exists a unique point $x_{A,\gamma} \in K$ such that

$$A_\gamma(x_{A,\gamma}) = x_{A,\gamma}. \tag{4.13}$$

Theorem 3.1 and (4.13) imply that there exist a natural number $n(A,\gamma,k)$ and an open neighborhood $\mathcal{U}(A,\gamma,k)$ of A_γ in $\mathcal{M}$ such that the following property holds:

(P) for each mapping $S \in \mathcal{U}(A,\gamma,k)$, each point $x \in K$ and each integer $i \ge n(A,\gamma,k)$,

$$\|S^i(x) - x_{A,\gamma}\| \le 1/k.$$

Set

$$\mathcal{F} := \cap_{p=1}^\infty \cup \{\mathcal{U}(A,\gamma,k) :\ A \in \mathcal{M},\ \gamma \in (0,1),\ \text{ an integer } k \ge p\}. \tag{4.14}$$

It is clear that $\mathcal{F}$ is a countable intersection of open and everywhere dense subsets of the metric space $(\mathcal{M}, d)$.

Let

$$T \in \mathcal{F} \ \text{ and } \ \epsilon \in (0,1). \tag{4.15}$$

Choose a natural number p such that

$$4p^{-1} < \epsilon.$$

By (4.14) and (4.15), there exist

$$A \in \mathcal{M},\ \gamma \in (0,1) \text{ and an integer } k \ge p$$

such that

$$T \in \mathcal{U}(A,\gamma,k). \tag{4.16}$$

It now follows from (4.16), the choice of the natural number p and property (P) that for each point $x \in K$ and each integer $i \ge n(A,\gamma,k)$, we have

$$\|T^i(x) - x_{A,\gamma}\| \le (k)^{-1} \le p^{-1} < \epsilon/4.$$

Since $\epsilon \in (0,1)$ is arbitrary, we see that for each point $x \in K$, $\{T^i(x)\}_{i=1}^{\infty}$ is a Cauchy sequence and there exists

$$\lim_{i\to\infty} T^i(x),$$

which satisfies

$$\|\lim_{i\to\infty} T^i(x) - x_{A,\gamma}\| \leq \epsilon/4$$

for all points $x \in K$. This implies that for each $x_1, x_2 \in K$,

$$\lim_{i\to\infty} T^i(x_1) = \lim_{i\to\infty} T^i(x_2).$$

Therefore there exists a unique point $x_T \in K$ such that

$$T(x_T) = x_T,$$

$$\|x_T - x_{A,\gamma}\| \leq \epsilon/4$$

and for each point $x \in K$ and each integer $i \geq n(A, \gamma, k)$, we have

$$\|x_T - T^i(x)\| \leq \epsilon.$$

This completes the proof of Theorem 4.1. □

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