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THE VITALI TYPE THEOREM FOR THE CHOQUET INTEGRAL

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ABSTRACT. The Vitali theorem for uniformly integrable functions is fundamental in Lebesgue integration theory and contains other important convergence theorems for the abstract Lebesgue integral. The purpose of the paper is to prove Vitali type theorems for the Choquet integral and its symmetric and asymmetric extensions with respect to a nonadditive measure. The bounded convergence theorem and the dominated convergence theorem for Choquet integrals are obtained as their applications.

1. INTRODUCTION

The Vitali theorem for the abstract Lebesgue integral states that, for any finite measure μ on a measurable space $(X, \mathcal{A})$ and any sequence $\{f_n\}_{n\in\mathbb{N}}$ of $\mathcal{A}$-measurable functions converging in μ-measure to an $\mathcal{A}$-measurable function f, if $\{f_n\}_{n\in\mathbb{N}}$ is uniformly μ-integrable, then f is μ-integrable and $\int_X f_n d\mu \to \int_X f d\mu$. The Vitali theorem is fundamental in Lebesgue integration theory and contains other important convergence theorems such as the bounded convergence theorem and the dominated convergence theorem.

Recently, nonadditive measure theory has been extensively studied with applications to decision theory under uncertainty, game theory, data mining, some economic topics under Knightian uncertainty and others [3, 5, 11, 12, 16, 18, 19]. For a nonadditive measure, several types of nonlinear integrals have been proposed. Among them, the Choquet integral [1, 14] is typical and widely used in nonadditive measure theory as well as its applications. The purpose of the paper is to formulate Vitali type theorems for the Choquet integral and its symmetric and asymmetric extensions with respect to a nonadditive measure and obtain the bounded convergence theorem and the dominated convergence theorem as their applications.

The paper is organized as follows. In Section 2, we recall some basic properties of nonadditive measures and Choquet integrals. In Section 3, we introduce the notion of uniform integrability of functions that will be used when formulating our Vitali type theorems for the Choquet integral. In Section 4, we give a primitive form of the Vitali type theorem for the Choquet integral and its dual measure form. They

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are extended in Section 5 to symmetric and asymmetric Choquet integrals. In Section 6, we obtain the bounded convergence theorem and the dominated convergence theorem for Choquet integrals as applications of our Vitali type theorems.

2. Preliminaries

In this paper, unless stated otherwise, X is a non-empty set and $\mathcal{A}$ is a field of subsets of X. Let 2^X denote the family of all subsets of X.

Let $\mathbb{R}$ and $\mathbb{N}$ denote the set of all real numbers and the set of all natural numbers. Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ with usual total order. For any $a, b \in \overline{\mathbb{R}}$, let $a \vee b := \max(a, b)$ and $a \wedge b := \min(a, b)$. For any functions $f, g\colon X \to \overline{\mathbb{R}}$, let $(f \vee g)(x) := f(x) \vee g(x)$ and $(f \wedge g)(x) := f(x) \wedge g(x)$ for every $x \in X$. We adopt the usual conventions for algebraic operations on $\overline{\mathbb{R}}$. We also adopt the convention $(\pm\infty) \cdot 0 = 0 \cdot (\pm\infty) = 0$ and $\inf \emptyset = \infty$. If a positive number c may take ∞, we explicitly write $c \in (0, \infty]$ instead of the ambiguous expression $c > 0$. In other words, $c > 0$ always means $c \in (0, \infty)$. This notational convention will be used for similar cases.

Let χ_A denote the characteristic function of a set A, that is, $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise. A function $f\colon X \to \overline{\mathbb{R}}$ is called $\mathcal{A}$-*measurable* if $\{f \geq t\} := \{x \in X\colon f(x) \geq t\} \in \mathcal{A}$ and $\{f > t\} := \{x \in X\colon f(x) > t\} \in \mathcal{A}$ for every $t \in \overline{\mathbb{R}}$. Any constant function and the characteristic function χ_A of any set $A \in \mathcal{A}$ are $\mathcal{A}$-measurable. If f and g are $\mathcal{A}$-measurable and $c \in \mathbb{R}$, then so are $f^+ := f \vee 0$, $f^- := (-f) \vee 0$, $|f| := f \vee (-f)$, cf, $f + c$, $(f - c)^+$, $f \vee g$, and $f \wedge g$. Note that $f = f \wedge c + (f - c)^+$. Let $\mathcal{F}(X)$ denote the set of all $\mathcal{A}$-measurable functions $f\colon X \to \overline{\mathbb{R}}$. For every $f \in \mathcal{F}(X)$, let $\|f\| := \sup_{x \in X} |f(x)|$. Then $\|f\| < \infty$ if and only if f is bounded. Let $\mathcal{F}_b(X) := \{f \in \mathcal{F}(X)\colon \|f\| < \infty\}$, $\mathcal{F}^+(X) := \{f \in \mathcal{F}(X)\colon f \geq 0\}$, and $\mathcal{F}_b^+(X) := \{f \in \mathcal{F}_b(X)\colon f \geq 0\}$. A *simple* function is a function whose range space is a finite subset of $\mathbb{R}$.

2.1. The nonadditive measure.

A *nonadditive measure* on X is an extended real-valued set function $\mu\colon \mathcal{A} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and $\mu(A) \leq \mu(B)$ whenever $A, B \in \mathcal{A}$ and $A \subset B$. It is called *finite* if $\mu(X) < \infty$. This type of set function is also called a monotone measure [18], a capacity [1], or a fuzzy measure [16] in the literature.

Let $\mathcal{M}(X)$ denote the set of all nonadditive measures $\mu\colon \mathcal{A} \to [0, \infty]$. Let $\mathcal{M}_b(X) := \{\mu \in \mathcal{M}(X)\colon \mu(X) < \infty\}$. For any $\mu \in \mathcal{M}_b(X)$, its *dual* $\bar{\mu} \in \mathcal{M}_b(X)$ is defined by

$$\bar{\mu}(A) := \mu(X) - \mu(A^c)$$

for every $A \in \mathcal{A}$, where A^c denotes the complement of A. It is obvious that $\bar{\bar{\mu}} = \mu$. If μ is finitely additive, then $\mu = \bar{\mu}$.

Definition 2.1 ([18])**.** Let $\mu \in \mathcal{M}(X)$.

(1) μ is called *subadditive* if $\mu(A \cup B) \leq \mu(A) + \mu(B)$ whenever $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$.
(2) μ is called *autocontinuous from above* if $\mu(A \cup B_n) \to \mu(A)$ whenever $A \in \mathcal{A}$, $B_n \in \mathcal{A}$ $(n = 1, 2, \dots)$ and $\mu(B_n) \to 0$.
(3) μ is called *autocontinuous from below* if $\mu(A \setminus B_n) \to \mu(A)$ whenever $A \in \mathcal{A}$, $B_n \in \mathcal{A}$ $(n = 1, 2, \dots)$ and $\mu(B_n) \to 0$.

(4) μ is called *autocontinuous* if it is autocontinuous from above and from below.

Every subadditive nonadditive measure is obviously autocontinuous. Every nonadditive measure μ satisfying $\inf\{\mu(A)\colon A \in \mathcal{A}, A \neq \emptyset\} > 0$ is also autocontinuous [18, Theorem 6.5]. Moreover, every distorted measure μ of the form

$$\mu(A) := \varphi(m(A)), \quad A \in \mathcal{A},$$

where m is a finitely additive measure on $\mathcal{A}$ and $\varphi\colon [0, \infty] \to [0, \infty]$ is an increasing function with $\varphi(0) = 0$, is autocontinuous if φ is continuous and strictly increasing on a neighborhood of the origin (cf. [18, Theorem 6.15]).

The following example illustrates that the dual measure $\bar{\mu}$ is not autocontinuous even if μ is autocontinuous.

Example 2.2. (1) Let $X := [0, 1]$. Let $\mu\colon 2^X \to [0, 1]$ be the nonadditive measure defined by

$$\mu(A) := \begin{cases} 1 & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset. \end{cases}$$

Then μ is subadditive and hence autocontinuous. Its dual $\bar{\mu}\colon 2^X \to [0, 1]$ is given by

$$\bar{\mu}(A) = \begin{cases} 1 & \text{if } A = X, \\ 0 & \text{if } A \neq X \end{cases}$$

and neither autocontinuous from above nor from below.

(2) Let $X := [0, 1]$. Let $\mathcal{A}$ be the σ-field of all Lebesgue measurable subsets of X and λ the Lebesgue measure on $(X, \mathcal{A})$. Let $\mu\colon \mathcal{A} \to [0, 2]$ be the nonadditive measure defined by

$$\mu(A) := \begin{cases} 1 + \lambda(A) & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset. \end{cases}$$

Then μ is subadditive and hence autocontinuous. It dual $\bar{\mu}\colon \mathcal{A} \to [0, 2]$ is given by

$$\bar{\mu}(A) = \begin{cases} 2 & \text{if } A = X, \\ \lambda(A) & \text{if } A \neq X \end{cases}$$

and neither autocontinuous from above nor from below.

Nevertheless, $\bar{\mu}$ has the following form of the autocontinuity-like property.

Proposition 2.3. *Let $\mu \in \mathcal{M}_b(X)$.*

(1) *μ is autocontinuous from above if and only if $\bar{\mu}(A \setminus B_n) \to \bar{\mu}(A)$ whenever $A \in \mathcal{A}$, $B_n \in \mathcal{A}$ $(n = 1, 2, \dots)$, and $\mu(B_n) \to 0$.*

(2) *μ is autocontinuous from below if and only if $\bar{\mu}(A \cup B_n) \to \bar{\mu}(A)$ whenever $A \in \mathcal{A}$, $B_n \in \mathcal{A}$ $(n = 1, 2, \dots)$, and $\mu(B_n) \to 0$.*

Proof. The proof is elementary since $\bar{\mu}(A \setminus B) = \mu(X) - \mu(A^c \cup B)$ and $\bar{\mu}(A \cup B) = \mu(X) - \mu(A^c \setminus B)$ for every $A, B \in \mathcal{A}$. □

See [3, 12, 18] for further information on nonadditive measures.

2.2. **The Choquet integral.** The Choquet integral [1] is a typical nonlinear integral and widely used in nonadditive measure theory.

Definition 2.4. Let $(\mu, f) \in \mathcal{M}(X) \times \mathcal{F}^+(X)$. The *Choquet integral* is defined by

$$\mathrm{Ch}(\mu, f) := \int_0^\infty \mu(\{f > t\})dt,$$

where the integral of the right hand side is the Lebesgue integral.

If $f \in \mathcal{F}^+(X)$ and $\mathrm{Ch}(\mu, f) < \infty$, then f is called *μ-integrable.* If $f \in \mathcal{F}(X)$ and $\mathrm{Ch}(\mu, |f|) < \infty$, then f is called *μ-absolutely integrable*, which will be called *μ-integrable* for short without any confusion.

Remark 2.5. (1) In the definition of the Choquet integral, the μ-distribution function $\mu(\{f > t\})$ can be replaced with $\mu(\{f \geq t\})$ without any change.

(2) The Choquet integral is equal to the abstract Lebesgue integral if μ is σ-additive and $\mathcal{A}$ is a σ-field [15, Corollary 18]; see also [8, Propositions 8.1 and 8.2].

(3) In Section 5, other types of the integrability will be introduced for not necessarily non-negative functions.

The following properties of the Choquet integral can be directly proved by its definition and will be used later without mentioning explicitly.

Proposition 2.6. *Let $\mu \in \mathcal{M}(X)$, $A \in \mathcal{A}$, and $f, g, h \in \mathcal{F}^+(X)$. Let $a \geq 0$ be a constant.*

(1) $0 \leq \mathrm{Ch}(\mu, f) \leq \|f\|_\mu \cdot \mu(\{f > 0\})$, *where* $\|f\|_\mu := \inf\{c > 0 \colon \mu(\{f > c\}) = 0\}$.
(2) *If $f(x) \leq g(x)$ for every $x \in X$, then* $\mathrm{Ch}(\mu, f) \leq \mathrm{Ch}(\mu, g)$.
(3) $\mathrm{Ch}(\mu, a\chi_A) = a\mu(A)$.
(4) $\mathrm{Ch}(\mu, af) = a\,\mathrm{Ch}(\mu, f)$.
(5) $\mathrm{Ch}(\mu, f + a) = \mathrm{Ch}(\mu, f) + a\mu(X)$.
(6) *Assume that either f or g is μ-integrable. If $|f(x) - g(x)| \leq a$ for every $x \in X$, then $|\mathrm{Ch}(\mu, f) - \mathrm{Ch}(\mu, g)| \leq a\mu(X)$.*

The Choquet integral has the following useful inequality.

Proposition 2.7. *For any $\mu \in \mathcal{M}(X)$, $f \in \mathcal{F}^+(X)$, $A \in \mathcal{A}$, and $c \geq 0$, it holds that*

$$\begin{aligned} c\mu(A \cap \{f \geq c\}) &\leq \mathrm{Ch}(\mu, \chi_A f) \\ &\leq c\mu(A \cap \{f > 0\}) + \mathrm{Ch}(\mu, \chi_{A \cap \{f > c\}} f). \end{aligned}$$

Proof. The first inequality follows from

$$\begin{aligned} \mathrm{Ch}(\mu, \chi_A f) &= \int_0^\infty \mu(\{A \cap \{f \geq t\})dt \\ &\geq \int_0^c \mu(A \cap \{f \geq c\})dt = c\mu(A \cap \{f \geq c\}), \end{aligned}$$

while the second inequality follows from

$$\begin{aligned}\mathrm{Ch}(\mu,\chi_A f) &= \int_0^c \mu(A\cap\{f>t\})dt + \int_c^\infty \mu(A\cap\{f>t\})dt \\ &\le \int_0^c \mu(A\cap\{f>0\})dt + \int_c^\infty \mu(\{\chi_{A\cap\{f>c\}}f>t\})dt \\ &\le c\mu(A\cap\{f>0\}) + \mathrm{Ch}(\mu,\chi_{A\cap\{f>c\}}f)\end{aligned}$$

and the proof is complete. □

Proposition 2.8. *Let $\mu\in\mathcal{M}(X)$ and $f\in\mathcal{F}(X)$. If f is μ-integrable, then the following conditions hold.*

(1) $\lim_{c\to\infty}\mu(\{|f|>c\})=0$.
(2) $\lim_{c\to\infty}\mathrm{Ch}(\mu,\chi_{\{|f|>c\}}|f|)=0$.
(3) *f is μ-absolutely continuous, that is, for any $\varepsilon>0$, there is $\delta>0$ such that $\mathrm{Ch}(\mu,\chi_A|f|)<\varepsilon$ whenever $A\in\mathcal{A}$ and $\mu(A)<\delta$.*

Proof. (1) By Proposition 2.7, for any $c>0$,

$$\mu(\{|f|>c\}) \le \frac{1}{c}\mathrm{Ch}(\mu,|f|).$$

Since $\mathrm{Ch}(\mu,|f|)<\infty$, letting $c\to 0$ gives (1).

(2) Let $\{c_n\}_{n\in\mathbb{N}}$ be a sequence with $c_n>0$ and $c_n\to\infty$. Let $\varphi_n(t):=\mu(\{|f|>c_n\vee t\})$ $(n=1,2,\dots)$ and $\varphi(t):=\mu(\{|f|>t\})$ for every $t\in[0,\infty)$. Then, φ_n and φ are Lebesgue measurable and φ_n converges pointwise to 0. Since $0\le\varphi_n\le\varphi$ and $\int_0^\infty\varphi(t)dt=\mathrm{Ch}(\mu,|f|)<\infty$, by the dominated convergence theorem for the Lebesgue integral,

$$\lim_{n\to\infty}\mathrm{Ch}(\mu,\chi_{\{|f|>c_n\}}|f|) = \lim_{n\to\infty}\int_0^\infty \varphi_n(t)dt = 0,$$

which implies (2).

(3) Let $\varepsilon>0$. By (2) there is $c_0>0$ such that $\mathrm{Ch}(\mu,\chi_{\{|f|>c_0\}}|f|)<\varepsilon/2$. Let $\delta:=\varepsilon/(2c_0)>0$. By Proposition 2.7, if $\mu(A)<\delta$, then

$$\mathrm{Ch}(\mu,\chi_A|f|)\le c_0\mu(A)+\mathrm{Ch}(\mu,\chi_{\{|f|>c_0\}}|f|) < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Thus f is μ-absolutely continuous. □

3. Uniform integrability

The Vitali type theorem considered in this paper needs the uniform integrability of a set of functions, which takes the same form as the case of the Lebesgue integral.

Definition 3.1. Let $\mu\in\mathcal{M}(X)$ and $\mathcal{F}$ a non-empty subset of $\mathcal{F}(X)$.

(1) $\mathcal{F}$ is called *uniformly μ-integral bounded* if $\sup_{f\in\mathcal{F}}\mathrm{Ch}(\mu,|f|)<\infty$.
(2) $\mathcal{F}$ is called *uniformly μ-absolutely continuous* if, for any $\varepsilon>0$, there is $\delta>0$ such that $\sup_{f\in\mathcal{F}}\mathrm{Ch}(\mu,\chi_A|f|)<\varepsilon$ whenever $A\in\mathcal{A}$ and $\mu(A)<\delta$.
(3) $\mathcal{F}$ is called *uniformly μ-integrable* if it holds that

$$\lim_{c\to\infty}\sup_{f\in\mathcal{F}}\mathrm{Ch}(\mu,\chi_{\{|f|>c\}}|f|)=0.$$

For a non-empty $\mathcal{F} \subset \mathcal{F}(X)$ and $a \neq 0$, let $\mathcal{F}^+ := \{f^+ \colon f \in \mathcal{F}\}$, $\mathcal{F}^- := \{f^- \colon f \in \mathcal{F}\}$, $|\mathcal{F}| := \{|f| \colon f \in \mathcal{F}\}$, and $a\mathcal{F} := \{af \colon f \in \mathcal{F}\}$. The following proposition immediately follows from the definition of the uniform integrability.

Proposition 3.2. *Let $\mu \in \mathcal{M}(X)$ and $\mathcal{F}$ a non-empty subset of $\mathcal{F}(X)$. Let $a \neq 0$ be a constant. If $\mathcal{F}$ is uniformly μ-integrable, then so are $a\mathcal{F}$, $\mathcal{F}^+$, $\mathcal{F}^-$, and $|\mathcal{F}|$.*

Proposition 3.3. *Let $\mu \in \mathcal{M}(X)$. Every uniformly μ-integral bounded subset $\mathcal{F}$ of $\mathcal{F}(X)$ satisfies $\lim_{c\to\infty} \sup_{f\in\mathcal{F}} \mu(\{|f| > c\}) = 0$.*

Proof. For any $c > 0$ and $f \in \mathcal{F}$, by Proposition 2.7, $\mu(\{|f| > c\}) \leq \mathrm{Ch}(\mu, |f|)/c$, so that

$$0 \leq \sup_{f\in\mathcal{F}} \mu(\{|f| > c\}) \leq \frac{1}{c} \sup_{f\in\mathcal{F}} \mathrm{Ch}(\mu, |f|).$$

Letting $c \to \infty$ gives the conclusion. □

Proposition 3.4. *Let $\mu \in \mathcal{M}(X)$. For a non-empty subset $\mathcal{F}$ of $\mathcal{F}(X)$, consider the following two conditions*:

(i) *$\mathcal{F}$ is uniformly μ-integral bounded and uniformly μ-absolutely continuous.*
(ii) *$\mathcal{F}$ is uniformly μ-integrable.*

Then (i) *implies* (ii). *Conversely,* (ii) *implies the uniform μ-absolute continuity of $\mathcal{F}$. If μ is finite, then* (ii) *also implies the uniform μ-integral boundedness of $\mathcal{F}$.*

Proof. The proof goes along with the Lebesgue integral case and is given only for convenience of readers; see [4, Theorem 10.3.5].

(i)⇒(ii): Let $\varepsilon > 0$. Since $\mathcal{F}$ is uniformly μ-absolutely continuous, there is $\delta > 0$ such that

$$\sup_{f\in\mathcal{F}} \mathrm{Ch}(\mu, \chi_A|f|) < \varepsilon \tag{3.1}$$

for any $A \in \mathcal{A}$ with $\mu(A) < \delta$. Since $\mathcal{F}$ is uniformly μ-integral bounded, by Proposition 3.3, there is $c_0 > 0$ such that

$$\sup_{f\in\mathcal{F}} \mu(\{|f| > c_0\}) < \delta. \tag{3.2}$$

Let $c > c_0$. For each $f \in \mathcal{F}$, let $A_f := \{|f| > c\}$. Then by (3.2), $\mu(A_f) < \delta$. Thus, by (3.1), for any $f \in \mathcal{F}$,

$$\mathrm{Ch}(\mu, \chi_{\{|f|>c\}}|f|) = \mathrm{Ch}(\mu, \chi_{A_f}|f|) < \varepsilon,$$

which implies the uniform μ-integrability of $\mathcal{F}$.

Next we assume (ii). By Proposition 2.7, for any $A \in \mathcal{A}$ and $c > 0$,

$$\sup_{f\in\mathcal{F}} \mathrm{Ch}(\mu, \chi_A|f|) \leq c\mu(A) + \sup_{f\in\mathcal{F}} \mathrm{Ch}(\mu, \chi_{\{|f|>c\}}|f|). \tag{3.3}$$

Let $\varepsilon > 0$. Since $\mathcal{F}$ is uniformly μ-integrable, there is $c_0 > 0$ such that

$$\sup_{f\in\mathcal{F}} \mathrm{Ch}(\mu, \chi_{\{|f|>c_0\}}|f|) < \frac{\varepsilon}{2}. \tag{3.4}$$

Let $\delta := \varepsilon/(2c_0) > 0$ and take $A \in \mathcal{A}$ with $\mu(A) < \delta$. Then, by (3.3) and (3.4),

$$\sup_{f\in\mathcal{F}} \mathrm{Ch}(\mu, \chi_A|f|) \leq c_0\mu(A) + \sup_{f\in\mathcal{F}} \mathrm{Ch}(\mu, \chi_{\{|f|>c_0\}}|f|) < c_0\delta + \frac{\varepsilon}{2} = \varepsilon,$$

which implies the uniform μ-absolute continuity of $\mathcal{F}$.

Assume that $\mu(X) < \infty$. Since $\mathcal{F}$ is uniformly μ-integrable, there is $c_0 > 0$ such that

$$\sup_{f \in \mathcal{F}} \mathrm{Ch}(\mu, \chi_{\{|f|>c_0\}}|f|) < 1. \tag{3.5}$$

Letting $A = X$ and $c = c_0$ in (3.3), by (3.5),

$$\sup_{f \in \mathcal{F}} \mathrm{Ch}(\mu, |f|) < c_0\mu(X) + 1 < \infty.$$

Thus $\mathcal{F}$ is uniformly μ-integral bounded. □

Remark 3.5. (1) The above result gives a necessary and sufficient condition for the uniform integrability for the Choquet integral and was proved in [2, Lemma 3.2] in the case that μ is subadditive.

(2) When $\mu(X) = \infty$, the uniform μ-integral boundednes does not follow from the uniform μ-integrability; in particular, for each $f \in \mathcal{F}(X)$, $\mathrm{Ch}(\mu, |f|) < \infty$ does not follow from $\lim_{c\to\infty} \mathrm{Ch}(\mu, \chi_{\{|f|>c\}}|f|) = 0$. In fact, let $X := \mathbb{R}$, $\mathcal{A}$ the σ-field of all Lebesgue measurable subsets of $\mathbb{R}$, and λ the Lebesgue measure on $\mathbb{R}$. For each $n \in \mathbb{N}$, define the $\mathcal{A}$-measurable function $f_n(x) := 1$ for every $x \in \mathbb{R}$. Then $\{f_n\}_{n\in\mathbb{N}}$ is obviously uniformly λ-integrable, but is not uniformly λ-integral bounded since $\mathrm{Ch}(\lambda, |f_n|) = \infty$ for all $n \in \mathbb{N}$. For this reason, the concept of the uniform integrability is more interesting for finite measures.

4. The Vitali type theorems

In this section we formulate a primitive form of the Vitali type theorem and its dual measure form for the Choquet integral. Let $(X, \mathcal{A})$ be a measurable space, that is, X is a non-empty set and $\mathcal{A}$ is a σ-field of subsets of X. Let $\mathcal{F}_0(X)$ denote the set of all $\mathcal{A}$-measurable functions $f\colon X \to \mathbb{R}$ and let $\mathcal{F}_0^+(X) := \{f \in \mathcal{F}_0(X)\colon f \geq 0\}$. Let $\mu \in \mathcal{M}(X)$. We say that a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0(X)$ *converges in μ-measure* to a function $f \in \mathcal{F}_0(X)$ and write $f_n \xrightarrow{\mu} f$ if $\mu(\{|f_n - f| > \varepsilon\}) \to 0$ for every $\varepsilon > 0$. Obviously, if $f_n \xrightarrow{\mu} f$, then $|f_n| \xrightarrow{\mu} |f|$, $f_n^+ \xrightarrow{\mu} f^+$, and $f_n^- \xrightarrow{\mu} f^-$. The following proposition can be proved in a similar way to [10, Theorem 3.1].

Proposition 4.1. *Let $(X, \mathcal{A})$ be a measurable space and $\mu \in \mathcal{M}(X)$. The following conditions are equivalent.*

(i) *μ is autocontinuous.*
(ii) *$\mu(\{f_n > t\}) \to \mu(\{f > t\})$ t-a.e. for any sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0(X)$ converging in μ-measure to $f \in \mathcal{F}_0(X)$.*
(iii) *$\mu(\{f_n \leq t\}) \to \mu(\{f \leq t\})$ t-a.e. for any sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0(X)$ converging in μ-measure to $f \in \mathcal{F}_0(X)$.*

Remark 4.2. In the above proposition, "t-a.e." may be replaced with "except at most countably many values of t" or "for every continuity point of the function $\mu(\{f > t\})$ in (ii) and $\mu(\{f \leq t\})$ in (iii)."

Proposition 4.3. *Let $(X, \mathcal{A})$ be a measurable space and $\mu \in \mathcal{M}(X)$. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0(X)$ be a sequence converging in μ-measure to $f \in \mathcal{F}_0(X)$. Assume that μ is autocontinuous.*

(1) *If $\{f_n\}_{n\in\mathbb{N}}$ is uniformly μ-integral bounded, then f is μ-integrable.*
(2) *If μ is finite and $\{f_n\}_{n\in\mathbb{N}}$ is uniformly $\bar{\mu}$-integral bounded, then f is $\bar{\mu}$-integrable.*

Proof. (1) Since $|f_n| \xrightarrow{\mu} |f|$, by Proposition 4.1, $\mu(\{|f_n| > t\}) \to \mu(\{|f| > t\})$ t-a.e. Then by the Fatou lemma for the Lebesgue integral,

$$\begin{aligned}\mathrm{Ch}(\mu, |f|) &= \int_0^\infty \mu(\{|f| > t\})dt \\ &\le \liminf_{n\to\infty} \int_0^\infty \mu(\{|f_n| > t\})dt \\ &\le \sup_{n\in\mathbb{N}} \mathrm{Ch}(\mu, |f_n|) < \infty.\end{aligned}$$

(2) It can be proved in the same way as (1) since $\bar{\mu}(\{|f_n| > t\}) \to \bar{\mu}(\{|f| > t\})$ t-a.e. by Proposition 4.1. □

Remark 4.4. Example 2.2 shows that $\bar{\mu}$ is not necessarily autocontinuous even if μ is autocontinuous. Therefore, in the above proposition, (1) does not imply (2).

Now we introduce a primitive from of the Vitali type theorem for the Choquet integral.

Theorem 4.5. *Let $(X, \mathcal{A})$ be a measurable space and $\mu \in \mathcal{M}_b(X)$. The following conditions are equivalent*:

(i) *μ is autocontinuous.*
(ii) *For any sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$ converging in μ-measure to $f \in \mathcal{F}_0^+(X)$, if $\{f_n\}_{n\in\mathbb{N}}$ is uniformly μ-integrable, then f_n and f are all μ-integrable and $\mathrm{Ch}(\mu, f_n) \to \mathrm{Ch}(\mu, f)$.*

Proof. (i)⇒(ii): The uniform μ-integrability of $\{f_n\}_{n\in\mathbb{N}}$ implies

$$\lim_{c\to\infty} \sup_{n\in\mathbb{N}} \mathrm{Ch}(\mu, \chi_{\{f_n>c\}} f_n) = 0, \tag{4.1}$$

and by Proposition 3.4, $\{f_n\}_{n\in\mathbb{N}}$ is uniformly μ-integral bounded, in particular, each f_n is μ-integrable. Therefore by Proposition 4.3, f is also μ-integrable, and hence by Proposition 2.8,

$$\lim_{c\to\infty} \mathrm{Ch}(\mu, \chi_{\{f>c\}} f) = 0. \tag{4.2}$$

Let $\varepsilon > 0$. By (4.1) and (4.2), there is $c_0 > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathrm{Ch}(\mu, \chi_{\{f_n>c_0\}} f_n) < \varepsilon \tag{4.3}$$

and

$$\mathrm{Ch}(\mu, \chi_{\{f>c_0\}} f) < \varepsilon. \tag{4.4}$$

Let $g := f \wedge c_0$ and $g_n := f_n \wedge c_0$ for each $n \in \mathbb{N}$. Then, $g, g_n \in \mathcal{F}_0^+(X)$ and they are μ-integrable. Moreover,

$$\begin{aligned}\mathrm{Ch}(\mu, f) &= \int_0^{c_0} \mu(\{f > t\})dt + \int_{c_0}^{\infty} \mu(\{f > t\})dt \\ &= \int_0^{c_0} \mu(\{g > t\})dt + \int_{c_0}^{\infty} \mu(\{f > c_0\} \cap \{f > t\})dt \\ &= \int_0^{\infty} \mu(\{g > t\})dt + \int_{c_0}^{\infty} \mu(\{\chi_{\{f>c_0\}} f > t\})dt \\ &\leq \mathrm{Ch}(\mu, g) + \mathrm{Ch}(\mu, \chi_{\{f>c_0\}} f),\end{aligned}$$

and thus by (4.4),

$$|\mathrm{Ch}(\mu, f) - \mathrm{Ch}(\mu, g)| < \varepsilon. \tag{4.5}$$

In the same way, by (4.3),

$$|\mathrm{Ch}(\mu, f_n) - \mathrm{Ch}(\mu, g_n)| < \varepsilon \quad (n = 1, 2, \dots). \tag{4.6}$$

Let h be an $\mathcal{A}$-measurable simple function such that $0 \leq h(x) \leq c_0$ and $|h(x) - g(x)| < \varepsilon$ for every $x \in X$. Then h is μ-integrable and

$$|\mathrm{Ch}(\mu, g) - \mathrm{Ch}(\mu, h)| \leq \varepsilon\mu(X). \tag{4.7}$$

Let $B_n := \{|g_n - h| > 2\varepsilon\}$ for each $n \in \mathbb{N}$. Since $|g_n(x) - g(x)| \leq |f_n(x) - f(x)|$ for every $x \in X$, $f_n \xrightarrow{\mu} f$ implies $g_n \xrightarrow{\mu} g$. Moreover, since $|h(x) - g(x)| < \varepsilon$ for every $x \in X$, $\{|g_n - h| > 2\varepsilon\} \subset \{|g_n - g| > \varepsilon\}$, which yields $\mu(B_n) \to 0$. It is easy to see that the family $\{\{h > t\} \colon t \in \mathbb{R}\}$ consists of finitely many sets, say, $A_1, A_2, \dots, A_m$, the autocontinuity of μ implies $\mu(A_k \setminus B_n) \to \mu(A_k)$ for each $k = 1, 2, \dots, m$. Thus, there is $n_1 \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, if $n \geq n_1$, then

$$\mu(A_k) \leq \mu(A_k \setminus B_n) + \frac{\varepsilon}{c_0} \quad (k = 1, 2, \dots, m). \tag{4.8}$$

Take $t \in \mathbb{R}$ arbitrarily. Then $\{h > t\} = A_{k_0}$ for some k_0 $(1 \leq k_0 \leq m)$ and $\{h > t\} \setminus B_n \subset \{g_n > t - 2\varepsilon\}$ for every $n \in \mathbb{N}$. Therefore, if $n \geq n_1$, then by (4.8), for any $t \in \mathbb{R}$,

$$\mu(\{h > t\}) \leq \mu(\{g_n > t - 2\varepsilon\}) + \frac{\varepsilon}{c_0}. \tag{4.9}$$

Since $\mu(A_k \cup B_n) \to \mu(A_k)$ for each $k = 1, 2, \dots, m$, in the same way as the above argument, there is $n_2 \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, if $n \geq n_2$, then for any $t \in \mathbb{R}$,

$$\mu(\{g_n > t + 2\varepsilon\}) \leq \mu(\{h > t\}) + \frac{\varepsilon}{c_0}. \tag{4.10}$$

Let $n_0 := n_1 \vee n_2$ and fix $n \in \mathbb{N}$ with $n \geq n_0$. Then by (4.9),

$$\begin{aligned}\mathrm{Ch}(\mu, h) &= \int_0^{c_0} \mu(\{h > t\})dt \\ &\leq \int_0^{c_0} \left\{\mu(\{g_n > t - 2\varepsilon\}) + \frac{\varepsilon}{c_0}\right\} dt \\ &\leq \mathrm{Ch}(\mu, g_n + 2\varepsilon) + \varepsilon,\end{aligned}$$

so that

$$\text{Ch}(\mu, h) \le \text{Ch}(\mu, g_n) + 2\varepsilon\mu(X) + \varepsilon. \tag{4.11}$$

In the same way, by (4.10),

$$\text{Ch}(\mu, h) \ge \text{Ch}(\mu, g_n) - 2\varepsilon\mu(X) - \varepsilon. \tag{4.12}$$

Therefore, by (4.11) and (4.12),

$$|\text{Ch}(\mu, g_n) - \text{Ch}(\mu, h)| \le 2\varepsilon\mu(X) + \varepsilon. \tag{4.13}$$

Moreover, by (4.5) and (4.7),

$$|\text{Ch}(\mu, f) - \text{Ch}(\mu, h)| < \varepsilon + \varepsilon\mu(X) \tag{4.14}$$

and by (4.6) and (4.13),

$$|\text{Ch}(\mu, f_n) - \text{Ch}(\mu, h)| < 2\varepsilon + 2\varepsilon\mu(X). \tag{4.15}$$

Eventually, by (4.14) and (4.15),

$$|\text{Ch}(\mu, f_n) - \text{Ch}(\mu, f)| < 3\varepsilon + 3\varepsilon\mu(X). \tag{4.16}$$

Letting $n \to \infty$ in (4.16) gives

$$0 \le \limsup_{n\to\infty} |\text{Ch}(\mu, f_n) - \text{Ch}(\mu, f)| \le 3\varepsilon\,(1 + \mu(X)).$$

Since $\varepsilon > 0$ is arbitrary, it holds that $\text{Ch}(\mu, f_n) \to \text{Ch}(\mu, f)$.

(ii)⇒(i): Let $A, B_n \in \mathcal{A}$ $(n = 1, 2, \dots)$ and assume that $\mu(B_n) \to 0$. Let $f_n := \chi_{A\cup B_n}$ and $f := \chi_A$. Then, $f_n, f \in \mathcal{F}_0^+(X)$, $f_n \xrightarrow{\mu} f$, and $\{f_n\}_{n\in\mathbb{N}}$ is uniformly μ-integrable. Therefore, it holds that

$$\mu(A \cup B_n) = \text{Ch}(\mu, f_n) \to \text{Ch}(\mu, f) = \mu(A),$$

which implies the autocontinuity of μ from above. The autocontinuity of μ from below can be proved similarly. □

By the help of Proposition 2.3, the dual measure form of the above Vitali type theorem can be proved in a similar way to Theorem 4.5.

Theorem 4.6. *Let $(X, \mathcal{A})$ be a measurable space and $\mu \in \mathcal{M}_b(X)$. The following conditions are equivalent*:

(i) *μ is autocontinuous.*
(ii) *For any sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$ converging in μ-measure to $f \in \mathcal{F}_0^+(X)$, if $\{f_n\}_{n\in\mathbb{N}}$ is uniformly $\bar{\mu}$-integrable, then f_n and f are all $\bar{\mu}$-integrable and $\text{Ch}(\bar{\mu}, f_n) \to \text{Ch}(\bar{\mu}, f)$.*

Remark 4.7. (1) Note that in the above dual measure form the convergence $\text{Ch}(\bar{\mu}, f_n) \to \text{Ch}(\bar{\mu}, f)$ can be obtained if we assume that μ itself is autocontinuous and $f_n \xrightarrow{\mu} f$ instead of $\bar{\mu}$.

(2) By Example 2.2 the dual $\bar{\mu}$ is not necessarily autocontinuous even if μ is autocontinuous, so that Theorem 4.6 is not an immediate consequence of Theorem 4.5.

Let $k \geq 1$ be a constant. A nonadditive measure μ is called *k-subadditive* if $\mu(A \cup B) \leq \mu(A) + k\mu(B)$ whenever $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$ [12, Definition 11.9]. For instance, every subadditive nonadditive measure is 1-subadditive. This measure is a special case of k-triangular set functions that were investigated by many mathematicians in the context of the Brooks-Jewett type theorem and Dieudonné type theorems; see Section 11 of [12]. Every k-subadditive nonadditive measure is autocontinuous. Moreover, if μ is finite and k-subadditive, then $\bar{\mu} \leq k\mu$, so that the uniform $\bar{\mu}$-integrability follows from the uniform μ-integrability and, in particular, the $\bar{\mu}$-integrability follows from the μ-integrability. Consequently, we have the following corollary to Theorems 4.5 and 4.6.

Corollary 4.8. *Let $(X, \mathcal{A})$ be a measurable space and $\mu \in \mathcal{M}_b(X)$. Assume that μ is k-subadditive for some $k \geq 1$ (in particular, μ is subadditive). If $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$ is a uniformly μ-integrable sequence converging in μ-measure to $f \in \mathcal{F}_0^+(X)$, then f_n and f are all μ-integrable and $\bar{\mu}$-integrable. Moreover, it holds that $\mathrm{Ch}(\mu, f_n) \to \mathrm{Ch}(\mu, f)$ and $\mathrm{Ch}(\bar{\mu}, f_n) \to \mathrm{Ch}(\bar{\mu}, f)$.*

Example 4.9. Let $X := [0, 2]$. Let $\mathcal{A}$ be the σ-field of all Lebesgue subsets of X and λ the Lebesgue measure on $(X, \mathcal{A})$. Let $\mu\colon \mathcal{A} \to [0, 3]$ be the nonadditive measure defined by

$$\mu(A) := \begin{cases} \lambda(A) & \text{if } \lambda(A) < 1, \\ 2\lambda(A) - 1 & \text{if } \lambda(A) \geq 1. \end{cases}$$

Then, μ is 2-subadditive.

5. Extension to symmetric and asymmetric integrals

In this section, we extend the Vitali type theorem and its dual measure form to symmetric and asymmetric Choquet integrals.

Definition 5.1. The *symmetric Choquet integral* is defined by

$$\mathrm{Ch}^s(\mu, f) := \mathrm{Ch}(\mu, f^+) - \mathrm{Ch}(\mu, f^-)$$

for every $(\mu, f) \in \mathcal{D}^s$, where

$$\mathcal{D}^s := \{(\mu, f) \in \mathcal{M}(X) \times \mathcal{F}(X)\colon \mathrm{Ch}(\mu, f^+) < \infty \text{ or } \mathrm{Ch}(\mu, f^-) < \infty\}$$

and the *asymmetric Choquet integral* is defined by

$$\mathrm{Ch}^a(\mu, f) := \mathrm{Ch}(\mu, f^+) - \mathrm{Ch}(\bar{\mu}, f^-)$$

for every $(\mu, f) \in \mathcal{D}^a$, where

$$\mathcal{D}^a := \{(\mu, f) \in \mathcal{M}_b(X) \times \mathcal{F}(X)\colon \mathrm{Ch}(\mu, f^+) < \infty \text{ or } \mathrm{Ch}(\bar{\mu}, f^-) < \infty\}.$$

If $(\mu, f) \in \mathcal{D}^s$ and $|\mathrm{Ch}^s(\mu, f)| < \infty$, then f is called *symmetrically μ-integrable*. Similarly, if $(\mu, f) \in \mathcal{D}^a$ and $|\mathrm{Ch}^a(\mu, f)| < \infty$, then f is called *asymmetrically μ-integrable*. Recall that f is μ-integrable if $(\mu, f) \in \mathcal{M}(X) \times \mathcal{F}(X)$ and $\mathrm{Ch}(\mu, |f|) < \infty$. For any $(\mu, f) \in \mathcal{M}(X) \times \mathcal{F}(X)$, if f is μ-integrable, then so are f^+ and f^-, but the converse statement does not hold in general. Moreover, f is not μ-integrable even if it is symmetrically and asymmetrically μ-integrable.

Example 5.2. Let $X := (-1,0) \cup (0,1)$. Let $\mu\colon 2^X \to [0,1]$ be the nonadditive measure defined by

$$\mu(A) := \begin{cases} 1 & \text{if } A \cap (-1,0) \neq \emptyset \text{ and } A \cap (0,1) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f(x) := 1/x$ for every $x \in X$. Then $\mathrm{Ch}(\mu, f^+) = \mathrm{Ch}(\mu, f^-) = 0$, but $\mathrm{Ch}(\mu, |f|) = \infty$. Moreover, $\mathrm{Ch}^s(\mu, f) = 0$ and $\mathrm{Ch}^a(\mu, f) = -1$.

Obviously, if f is μ-integrable, then f is symmetrically μ-integrable and $|\mathrm{Ch}^s(\mu, f)| \leq \mathrm{Ch}(\mu, |f|)$, but this is not the case for the asymmetric μ-integral.

Example 5.3. Let $X := (0,1)$. Let $\mu\colon 2^X \to [0,1]$ be the nonadditive measure defined by

$$\mu(A) := \begin{cases} 1 & \text{if } A = X, \\ 0 & \text{if } A \neq X. \end{cases}$$

Let $f(x) := -1/x$ for every $x \in X$. Then $\mathrm{Ch}(\mu, |f|) = 1$ and $\mathrm{Ch}^s(\mu, f) = -1$, but $\mathrm{Ch}^a(\mu, f) = -\infty$.

By Proposition 3.2, the following Vitali type theorems for symmetric and asymmetric Choquet integrals turn out to be immediate consequences of Theorems 4.5 and 4.6.

Theorem 5.4. *Let $(X, \mathcal{A})$ be a measurable space and $\mu \in \mathcal{M}_b(X)$. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0(X)$ be a sequence converging in μ-measure to $f \in \mathcal{F}_0(X)$. Assume that μ is autocontinuous.*

1. *(1) If $\{f_n\}_{n\in\mathbb{N}}$ is uniformly μ-integrable, then f_n and f are all symmetrically μ-integrable and $\mathrm{Ch}^s(\mu, f_n) \to \mathrm{Ch}^s(\mu, f)$.*
2. *(2) If $\{f_n\}_{n\in\mathbb{N}}$ is uniformly $\bar{\mu}$-integrable, then f_n and f are all symmetrically $\bar{\mu}$-integrable and $\mathrm{Ch}^s(\bar{\mu}, f_n) \to \mathrm{Ch}^s(\bar{\mu}, f)$.*

Theorem 5.5. *Let $(X, \mathcal{A})$ be a measurable space and $\mu \in \mathcal{M}_b(X)$. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0(X)$ be a sequence converging in μ-measure to $f \in \mathcal{F}_0(X)$. Assume that μ is autocontinuous. If $\{f_n\}_{n\in\mathbb{N}}$ is simultaneously uniformly μ-integrable and uniformly $\bar{\mu}$-integrable, then f_n and f are all asymmetrically μ-integrable and asymmetrically $\bar{\mu}$-integrable. Moreover, it holds that $\mathrm{Ch}^a(\mu, f_n) \to \mathrm{Ch}^a(\mu, f)$ and $\mathrm{Ch}^a(\bar{\mu}, f_n) \to \mathrm{Ch}^a(\bar{\mu}, f)$.*

Corollary 5.6. *Let $(X, \mathcal{A})$ be a measurable space and $\mu \in \mathcal{M}_b(X)$. Assume that μ is k-subadditive for some $k \geq 1$. If $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0(X)$ is a uniformly μ-integrable sequence converging in μ-measure to $f \in \mathcal{F}_0(X)$, then f_n and f are all symmetrically μ-integrable, symmetrically $\bar{\mu}$-integrable, asymmetrically μ-integrable, and asymmetrically $\bar{\mu}$-integrable. Moreover, it holds that $\mathrm{Ch}^s(\mu, f_n) \to \mathrm{Ch}^s(\mu, f)$, $\mathrm{Ch}^s(\bar{\mu}, f_n) \to \mathrm{Ch}^s(\bar{\mu}, f)$, $\mathrm{Ch}^a(\mu, f_n) \to \mathrm{Ch}^a(\mu, f)$, and $\mathrm{Ch}^a(\bar{\mu}, f_n) \to \mathrm{Ch}^a(\bar{\mu}, f)$.*

The following example shows that the uniform μ-integrability does not imply the uniform $\bar{\mu}$-integrability even if both μ and $\bar{\mu}$ are autocontinuous.

Example 5.7. Let $X := (0,1)$. Let $\mathcal{A}$ be the σ-field of all Lebesgue measurable subsets of X and λ the Lebesgue measure on $(X, \mathcal{A})$. Let $\mu(A) := \lambda(A)^2$ for every

$A \in \mathcal{A}$. Then μ is autocontinuous and its dual $\bar{\mu}$ given by $\bar{\mu}(A) = 2\lambda(A) - \lambda(A)^2$ for every $A \in \mathcal{A}$ is also autocontinuous. For each $n \in \mathbb{N}$, let $f_n(x) := 1/(nx)$ for every $x \in X$. Then $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$ is uniformly μ-integrable and $\mathrm{Ch}(\mu, f_n) = 2/n$ for every $n \in \mathbb{N}$. However, $\mathrm{Ch}(\bar{\mu}, f_n) = \infty$ for every $n \in \mathbb{N}$, and hence by Proposition 3.4, $\{f_n\}_{n\in\mathbb{N}}$ is not uniformly $\bar{\mu}$-integrable. In particular, f_1 is μ-integrable, but is not $\bar{\mu}$-integrable.

Remark 5.8. A similar result to Theorem 5.5 was already given in [17, Theorem 7] and [18, Theorem 11.11] for a *equi-integrable* sequence $\{f_n\}_{n\in\mathbb{N}}$ under the additional assumption that μ is continuous and f is μ-integrable.

6. The bounded and the dominated convergence theorem

In this section we show that the bounded convergence theorem and the dominated convergence theorem can be obtained as applications of our Vitali type theorems established in Sections 4 and 5.

Definition 6.1. Let $\mu \in \mathcal{M}(X)$. Let $\mathcal{F}$ be a non-empty subset of $\mathcal{F}(X)$ and $f \in \mathcal{F}(X)$.

(1) The function f is called *μ-essentially bounded* if there is $c > 0$ such that $\mu(\{f \geq c\}) = 0$ and $\mu(\{f \geq -c\}) = \mu(X)$. The family $\mathcal{F}$ is called *uniformly μ-essentially bounded* if there is $c > 0$ such that $\mu(\{f \geq c\}) = 0$ and $\mu(\{f \geq -c\}) = \mu(X)$ for all $f \in \mathcal{F}$.

(2) The function f is called *μ-essentially symmetric bounded* if there is $c > 0$ such that $\mu(\{f \geq c\}) = \mu(\{f \leq -c\}) = 0$. The family $\mathcal{F}$ is called *uniformly μ-essentially symmetric bounded* if there is $c > 0$ such that $\mu(\{f \geq c\}) = \mu(\{f \leq -c\}) = 0$ for all $f \in \mathcal{F}$.

Remark 6.2. The notion of the μ-essential symmetric boundedness in this paper slightly differs from that of [7, Definition 2.1]. Both notions coincide if the functions are non-negative or if μ is weakly null-additive, that is, $\mu(A \cup B) = 0$ whenever $A, B \in \mathcal{A}$ and $\mu(A) = \mu(B) = 0$. However, from now on we will distinguish them and say that f is μ-essentially absolute bounded if there is $c > 0$ such that $\mu(\{|f| \geq c\}) = 0$, which was the definition of the μ-essential boundedness in [7].

Let $\mathcal{F}_{\mu,b}(X)$ and $\mathcal{F}_{\mu,sb}(X)$ denote the set of all $f \in \mathcal{F}(X)$ that are μ-essentially bounded and μ-essentially symmetric bounded, respectively. Obviously, the notion of the μ-essential boundedness and that of the μ-essential symmetric boundedness coincide for nonnegative functions. If μ is finitely additive and $\mu(X) < \infty$, then $\mathcal{F}_{\mu,b}(X) = \mathcal{F}_{\mu,sb}(X)$. In general, both notions are independent of each other [9, Example 2.5].

For any $f \in \mathcal{F}(X)$, let

$$\|f\|_\mu := \inf\{c > 0 \colon \mu(\{f \geq c\}) = 0 \text{ and } \mu(\{f \geq -c\}) = \mu(X)\}.$$

Then f is μ-essentially bounded if and only if $\|f\|_\mu < \infty$. It always holds that $\mathcal{F}_b(X) \subset \mathcal{F}_{\mu,b}(X) \cap \mathcal{F}_{\mu,sb}(X)$ and $\|f\|_\mu \leq \|f\|$. Let us collect some basic properties of essentially (symmetric) bounded functions, which can be proved directly from Definition 6.1.

Proposition 6.3. *Let $\mu \in \mathcal{M}(X)$. Let $\mathcal{F}$ be a non-empty subset of $\mathcal{F}(X)$ and $f \in \mathcal{F}(X)$.*

(1) *f is μ-essentially symmetric bounded if and only if f^+ and f^- are both μ-essentially bounded. Moreover, $\mathcal{F}$ is uniformly μ-essentially symmetric bounded if and only if $\mathcal{F}^+$ and $\mathcal{F}^-$ are both uniformly μ-essentially bounded.*
(2) *Assume that μ is finite. Then f is μ-essentially bounded if and only if f^+ is μ-essentially bounded and f^- is $\bar{\mu}$-essentially bounded. Moreover, $\mathcal{F}$ is uniformly μ-essentially bounded if and only if $\mathcal{F}^+$ is uniformly μ-essentially bounded and $\mathcal{F}^-$ is uniformly $\bar{\mu}$-essentially bounded.*

Proposition 6.4. *Let $\mu \in \mathcal{M}(X)$. Assume that $\mathcal{F} \subset \mathcal{F}(X)$ is uniformly μ-essentially absolute bounded.*

(1) *If $\sup_{f\in\mathcal{F}} \mu(\{|f| > 0\}) < \infty$ (in particular, $\mu(X) < \infty$), then $\mathcal{F}$ is uniformly μ-integral bounded.*
(2) *$\mathcal{F}$ is uniformly μ-absolutely continuous.*
(3) *$\mathcal{F}$ is uniformly μ-integrable.*

Proof. (1) Since $\mathcal{F}$ is uniformly μ-essentially absolute bounded, there is $c_0 > 0$ such that $\mu(\{|f| > c_0\}) = 0$ and hence

$$\mathrm{Ch}(\mu, \chi_A|f|) = \int_0^{c_0} \mu(A \cap \{|f| > t\})dt \tag{6.1}$$

for every $f \in \mathcal{F}$ and $A \in \mathcal{A}$. Therefore (1) follows from

$$\sup_{f\in\mathcal{F}} \mathrm{Ch}(\mu, |f|) = \sup_{f\in\mathcal{F}} \int_0^{c_0} \mu(\{|f| > t\})dt \le c_0 \sup_{f\in\mathcal{F}} \mu(\{|f| > 0\}) < \infty.$$

(2) For any $\varepsilon > 0$, let $\delta := \varepsilon/c_0 > 0$. If $\mu(A) < \delta$, then (6.1) gives

$$\sup_{f\in\mathcal{F}} \mathrm{Ch}(\mu, \chi_A|f|) = \sup_{f\in\mathcal{F}} \int_0^{c_0} \mu(A \cap \{|f| > t\})dt \le \int_0^{c_0} \mu(A)dt < \varepsilon,$$

which implies the uniform μ-absolute continuity of $\mathcal{F}$.

(3) For any $c > c_0$ and $f \in \mathcal{F}$, (6.1) gives $\mathrm{Ch}(\mu, \chi_{\{|f|>c\}}|f|) \le c_0\mu(\{|f| > c_0\})$. Since $\mu(\{|f| > c_0\}) = 0$, $\mathrm{Ch}(\mu, \chi_{\{|f|>c\}}|f|) = 0$ and this implies the uniform μ-integrability of $\mathcal{F}$. □

The following proposition provides a bridge from our Vitali type theorems to the bounded convergence theorem and the dominated convergence theorem.

Proposition 6.5. *Let $\mu \in \mathcal{M}(X)$. Let $\mathcal{F}$ be a non-empty subset of $\mathcal{F}(X)$.*

(1) *Assume that $\mathcal{F}$ is uniformly μ-essentially symmetric bounded. Then $\mathcal{F}^+$ and $\mathcal{F}^-$ are uniformly μ-integrable.*
(2) *Assume that μ is finite and $\mathcal{F}$ is uniformly μ-essentially bounded. Then $\mathcal{F}^+$ is uniformly μ-integrable, while $\mathcal{F}^-$ is uniformly $\bar{\mu}$-integrable.*
(3) *Assume that there is a μ-integrable function $g \in \mathcal{F}^+(X)$ such that $|f| \le g$ for every $f \in \mathcal{F}$. Then $\mathcal{F}$ is uniformly μ-integrable.*
(4) *If $\sup_{f\in\mathcal{F}} \mathrm{Ch}(\mu, |f|^p) < \infty$ for some $p > 1$, then $\mathcal{F}$ is uniformly μ-integrable.*

Proof. (1) and (2) follow from Propositions 6.3 and 6.4.

(3) For any $c > 0$, $\sup_{f\in\mathcal{F}} \mathrm{Ch}(\mu, \chi_{\{|f|>c\}}|f|) \le \mathrm{Ch}(\mu, \chi_{\{g>c\}}g)$. Since g is μ-integrable, by Proposition 2.8, $\mathcal{F}$ is uniformly μ-integrable.

(4) For any $c > 0$ and $f \in \mathcal{F}$,

$$\begin{aligned}\mathrm{Ch}(\mu, \chi_{\{|f|>c\}}|f|) &= \int_0^\infty \mu(\{|f| > c\} \cap \{|f| > t\})dt \\ &\le \int_0^\infty \mu(\{|f|^p/c^{p-1} > t\})dt \\ &= \mathrm{Ch}(\mu, |f|^p/c^{p-1}),\end{aligned}$$

so that $\sup_{f\in\mathcal{F}} \mathrm{Ch}(\mu, \chi_{\{|f|>c\}}|f|) \le \sup_{f\in\mathcal{F}} \mathrm{Ch}(\mu, |f|^p)/c^{p-1}$. Letting $c \to \infty$ gives the conclusion. □

The following bounded convergence theorem and the dominated convergence theorem for symmetric and asymmetric integrals follow from Propositions 6.5 and Theorems 4.5, 4.6, 5.4 and 5.5.

Theorem 6.6. *Let $(X, \mathcal{A})$ be a measurable space. Let $\mu \in \mathcal{M}_b(X)$. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0(X)$ be a sequence converging in μ-measure to $f \in \mathcal{F}_0(X)$. Assume that μ is autocontinuous.*

1. *If $\{f_n\}_{n\in\mathbb{N}}$ is uniformly μ-essentially symmetric bounded, then f_n and f are all symmetrically μ-integrable and $\mathrm{Ch}^s(\mu, f_n) \to \mathrm{Ch}^s(\mu, f)$.*
2. *If $\{f_n\}_{n\in\mathbb{N}}$ is uniformly $\bar{\mu}$-essentially symmetric bounded, then f_n and f are all symmetrically $\bar{\mu}$-integrable and $\mathrm{Ch}^s(\bar{\mu}, f_n) \to \mathrm{Ch}^s(\bar{\mu}, f)$.*
3. *If $\{f_n\}_{n\in\mathbb{N}}$ is uniformly μ-essentially bounded, then f_n and f are all asymmetrically μ-integrable and $\mathrm{Ch}^a(\mu, f_n) \to \mathrm{Ch}^a(\mu, f)$.*
4. *If $\{f_n\}_{n\in\mathbb{N}}$ is uniformly $\bar{\mu}$-essentially bounded, then f_n and f are all asymmetrically $\bar{\mu}$-integrable and $\mathrm{Ch}^a(\bar{\mu}, f_n) \to \mathrm{Ch}^a(\bar{\mu}, f)$.*

Theorem 6.7. *Let $(X, \mathcal{A})$ be a measurable space. Let $\mu \in \mathcal{M}_b(X)$. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0(X)$ be a sequence converging in μ-measure to $f \in \mathcal{F}_0(X)$. Assume that μ is autocontinuous.*

1. *If there is a μ-integrable function $g \in \mathcal{F}^+(X)$ such that $|f_n| \le g$ for every $n \in \mathbb{N}$, then f_n and f are all symmetrically μ-integrable and $\mathrm{Ch}^s(\mu, f_n) \to \mathrm{Ch}^s(\mu, f)$.*
2. *If there is a $\bar{\mu}$-integrable function $g \in \mathcal{F}^+(X)$ such that $|f_n| \le g$ for every $n \in \mathbb{N}$, then f_n and f are all symmetrically $\bar{\mu}$-integrable and $\mathrm{Ch}^s(\bar{\mu}, f_n) \to \mathrm{Ch}^s(\bar{\mu}, f)$.*
3. *If there is a simultaneously μ-integrable and $\bar{\mu}$-integrable function $g \in \mathcal{F}^+(X)$ such that $|f_n| \le g$ for every $n \in \mathbb{N}$ (in particular, there is a constant $c > 0$ such that $|f_n| \le c$ for every $n \in \mathbb{N}$), then f_n and f are all asymmetrically μ-integrable and asymmetrically $\bar{\mu}$-integrable. Moreover, it holds that $\mathrm{Ch}^a(\mu, f_n) \to \mathrm{Ch}^a(\mu, f)$ and $\mathrm{Ch}^a(\bar{\mu}, f_n) \to \mathrm{Ch}^a(\bar{\mu}, f)$.*

The last theorem follows from Corollary 4.8, Proposition 6.5, and Theorem 6.7.

Theorem 6.8. *Let $(X, \mathcal{A})$ be a measurable space. Let $\mu \in \mathcal{M}_b(X)$. Assume that μ is k-subadditive for some $k \ge 1$. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0(X)$ be a sequence converging in*

μ-measure to $f \in \mathcal{F}_0(X)$. If $\{f_n\}_{n\in\mathbb{N}}$ is uniformly μ-essentially symmetric bounded or there is a μ-integrable function $g \in \mathcal{F}^+(X)$ such that $|f_n| \le g$ for every $n \in \mathbb{N}$, then f_n and f are all symmetrically μ-integrable, symmetrically $\bar{\mu}$-integrable, asymmetrically μ-integrable, and asymmetrically $\bar{\mu}$-integrable. Moreover, it holds that $\mathrm{Ch}^s(\mu, f_n) \to \mathrm{Ch}^s(\mu, f)$, $\mathrm{Ch}^s(\bar{\mu}, f_n) \to \mathrm{Ch}^s(\bar{\mu}, f)$, $\mathrm{Ch}^a(\mu, f_n) \to \mathrm{Ch}^a(\mu, f)$, and $\mathrm{Ch}^a(\bar{\mu}, f_n) \to \mathrm{Ch}^a(\bar{\mu}, f)$.

Remark 6.9. (1) Theorem 6.6 (3) and its extension to Riesz space-valued nonadditive measures were already given in [10, Theorem 3.3] and [6, Theorem 3.1], respectively.

(2) Several types of convergence theorems for the asymmetric Choquet integral were given in [13] for various modes of convergence of bounded measurable functions.

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