Linear and Nonlinear Analysis
Yokohama Publishers
ISSN 2188-8167
Volume 3, Number 2, 2017, 149–154

FIXED POINT SETS OF SUBRAHMANYAM MAPS

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Abstract. We prove that the fixed point set of a continuous Subrahmanyam map on a complete metric space is always a retract of the domain, and hence contractible whenever the domain is contractible.

1. Introduction

Banach contraction principle [1] has long been proven to be one of the most elementary, yet powerful, theorems in many areas of mathematics. It not only assures the existence of a unique fixed point of a contraction, but it also allows us to approximate the fixed point by an extremely simple process, so called the Picard iteration, starting at any initial point in the domain. Due to the simplicity of the Picard iteration, many researchers continually quest for other conditions that guarantee the convergence of this process.

The classical and elegant one was introduced by P.V. Subrahmanyam [3] back in 1974. He proved that the Picard iteration of a map satisfying his sufficient condition always converges to a fixed point no matter which initial point is chosen. Unlike Banach contraction principle, this result also includes maps having more than one fixed point. Since then, it took a couple of decades until T. Suzuki [4], in 2009, was able to improve Subrahmanyam's result, and officially ended the long quest, by giving a necessary and sufficient condition for a map whose Picard iteration converges to a fixed point regardless of an initial point in the domain.

From topological point of view, there is still something missing from the story. Even though the fixed point of a map that satisfies either Subrahmanyam's or Suzuki's condition may not be unique, no one ever describes its shape, or at least, suggests how far more complex it is from being a single point. Therefore, in this paper, by using the recently developed tool called virtual nonexpansiveness [2], we are able to attack this problem in a surprisingly elementary way to retain the simplicity from both Banach's and Subrahmanyam's original works. We start by recalling some related definitions and results in Section 2. Then, in Section 3, we first extend the original Subrahmanyam's condition to include a wider class of maps, which will be simply called Subrahmanyam maps, and prove that the fixed point set of a continuous Subrahmanyam map on a complete metric space is always a retract of the domain. In particular, when the domain is also contractible (e.g. a

2010 *Mathematics Subject Classification.* Primary 54H25; Secondary 47H09.

Key words and phrases. Subrahmanyam, Picard iteration, fixed point set, virtually nonexpansive.

closed convex subset of a Banach space), the fixed point set of such a map is always contractible. In fact, we will clearly see that the contractibility, not the convexity as usually seen in fixed point theory, arises as a natural structure of the fixed point set of a continuous Subrahmanyam map.

2. Preliminaries

Let X be a (nonempty) set and $T : X \to X$ a selfmap. For any $x \in X$ and $n \in \mathbb{N} \cup \{0\}$, we will follow the convention by writing Tx for $T(x)$, and $T^n x$ for n-th (Picard) iterate of T (where $T^0 x = x$). A point $x \in X$ is a *fixed point* of T if $Tx = x$ and the *fixed point set* of T is $F(T) = \{x : Tx = x\}$.

Recall that a selfmap T of a metric space (X, d) is called a *contraction* if there exists $c \in [0, 1)$ such that

$$d(Tx, Ty) \leq cd(x, y),$$

for all $x, y \in X$.

The followings are three well-known theorems we mentioned in the introduction.

Theorem 2.1 (Banach [1]). *Let (X, d) be a complete metric space. If $T : X \to X$ is a contraction, then T has a unique fixed point, and the iterative sequence $(T^n x)$ converges to the fixed point of T for any $x \in X$.*

Theorem 2.2 (Subrahmanyam, [3]). *Let (X, d) be a complete metric space and $T : X \to X$ a <u>continuous</u> map. If there exists $c \in [0, 1)$ such that*

$$d(T^2 x, Tx) \leq cd(Tx, x),$$

for all $x \in X$, then T has a fixed point, and the iterative sequence $(T^n x)$ converges to a fixed point of T for all $x \in X$.

Theorem 2.3 (Suzuki [4]). *Let (X, d) be a complete metric space and $T : X \to X$ a map. Then T has a fixed point, and the iterative sequence $(T^n x)$ converges to a fixed point of T for all $x \in X$ if and only if T satisfies the following two conditions.*

(1) *For $x \in X$ and $\epsilon > 0$, there exist $\delta > 0$ and $k \in \mathbb{N}$ such that*

$$d(T^i x, T^j x) < \epsilon + \delta \;\Rightarrow\; d(T^{i+k} x, T^{j+k} x) < \epsilon,$$

for all $i, j \in \mathbb{N} \cup \{0\}$.

(2) *For $x, y \in X$, there exist $k \in \mathbb{N}$ and a sequence (α_n) in $(0, \infty)$ such that*

$$d(T^i x, T^j y) < \alpha_n \;\Rightarrow\; d(T^{i+k} x, T^{j+k} y) < \frac{1}{n},$$

for all $i, j \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$.

Remark 2.4. The continuity of T is required in Theorem 2.2 to ensure the convergence to a <u>fixed point</u> of T, but not in Theorem 2.3.

We now recall the concept of virtual nonexpansiveness introduced by P. Chaoha [2] in 2007. Let (X, d) be a metric space and $B(x, r)$ denote the open ball (in X) of radius $r > 0$ centered at x. Suppose $T : X \to X$ is a continuous map with $F(T) \neq \emptyset$. The *convergence set* of T is defined to be

$$C(T) = \{x \in X : \text{ the iterative sequence } (T^n x) \text{ converges}\}.$$

Definition 2.5. We call T *virtually nonexpansive* if for each $p \in F(T)$ and $\epsilon > 0$, there exists $\delta > 0$ such that $T^n(B(p,\delta)) \subseteq B(p,\epsilon)$ for all $n \in \mathbb{N}$.

Theorem 2.6. *If T is virtually nonexpansive, then $F(T)$ is a retract of $C(T)$.*

3. Main Results

Throughout this section, we let T be a selfmap on a complete metric space (X, d).

Definition 3.1. We call T a *Subrahmanyam map* if there exists $\psi : X \to [0, 1)$ such that for each $x \in X$,

(i) $d(T^2x, Tx) \leq \psi(x)\, d(Tx, x)$,
(ii) $\psi(Tx) \leq \psi(x)$.

Notice that a Subrahmanyam map and ψ in the above definition may not be continuous. Moreover, we will see later (in Example 3.5 below) that our definition of Subrahmanyam map is strictly weaker than the original Subrahmanyam's condition in Theorem 2.2 (where $\psi(x) = c$).

Example 3.2. Let $T : [0, 1] \to [0, 1]$ be defined by

$$Tx = \begin{cases} 1, & x = 0, \\ \dfrac{x}{2}, & x \neq 0. \end{cases}$$

It is easy to see that T is a discontinuous Subrahmanyam map with $\psi(x) = \dfrac{1}{2}$. Notice also that $(T^n x)$ converges to 0, which is not a fixed point of T.

Theorem 3.3. *If T is a continuous Subrahmanyam map with respect to ψ, then T has a fixed point, and the iterative sequence $(T^n x)$ converges to a fixed point of T for all $x \in X$.*

Proof. For each $x \in X$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} d(T^{n+1}x, T^n x) &= d(T^2(T^{n-1}x), T(T^{n-1}x)) \\ &\leq \psi(T^{n-1}x)\, d(T^n x, T^{n-1}x) \\ &\leq \prod_{i=0}^{n-1} \psi(T^i x)\, d(Tx, x) \\ &\leq \psi(x)^n\, d(Tx, x). \end{aligned}$$

Since $\psi(x) < 1$, $\sum_{n=0}^{\infty} \psi(x)^n$ converges, and hence $(T^n x)$ is Cauchy. Therefore, $(T^n x)$ converges, says to $p \in X$, and by the continuity, p is a fixed point of T. □

Remark 3.4. The above theorem still holds if we replace (ii) in Definition 3.1 by a weaker condition : $\lim_{n\to\infty} \psi(T^n x) \in [0, 1)$ for all $x \in X$.

We now give an explicit example of a continuous Subrahmanyam map which does not satisfy the original Subrahmanyam's condition. Hence, the above theorem is indeed more general than Theorem 2.2.

Example 3.5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T(x,y) = \left(x, \frac{y + |x| + y\big|y - |x|\big|}{2 + \big|y - |x|\big|} \right),$$

for all $(x,y) \in \mathbb{R}^2$.

First, notice that T is continuous and $F(T) = \{(x,y) : y = |x|\}$.

For each $(x,y) \in \mathbb{R}^2$, we have $\|T(x,y) - (x,y)\| = \dfrac{|y-|x||}{2+|y-|x||}$, and hence

$$\begin{aligned}
\big\|T^2(x,y) - T(x,y)\big\| &= \left\| T\left(x, \frac{y + |x| + y\,|y - |x||}{2 + |y - |x||} \right) - \left(x, \frac{y + |x| + y\,|y - |x||}{2 + |y - |x||} \right) \right\| \\
&= \frac{\left| \dfrac{y + |x| + y\,|y - |x||}{2 + |y - |x||} - |x| \right|}{2 + \left| \dfrac{y + |x| + y\,|y - |x||}{2 + |y - |x||} - |x| \right|} \\
&= \frac{|y - |x||\,(1 + |y - |x||)}{(y - |x|)^2 + 3\,|y - |x|| + 4} \\
&= \frac{(1 + |y - |x||)(2 + |y - |x||)}{(y - |x|)^2 + 3\,|y - |x|| + 4}\,\|T(x,y) - (x,y)\| \\
&= \frac{(y - |x|)^2 + 3\,|y - |x|| + 2}{(y - |x|)^2 + 3\,|y - |x|| + 4}\,\|T(x,y) - (x,y)\|.
\end{aligned}$$

Then, by defining $\psi : \mathbb{R}^2 \to [0,1)$ by

$$\psi(x,y) = \frac{(y - |x|)^2 + 3\,|y - |x|| + 2}{(y - |x|)^2 + 3\,|y - |x|| + 4},$$

for all $(x,y) \in \mathbb{R}^2$, we immediately have for each $(x,y) \in \mathbb{R}^2$,

$$\big\|T^2(x,y) - T(x,y)\big\| \le \psi(x,y)\,\|T(x,y) - (x,y)\|,$$

and $\psi(T(x,y)) \le \psi(x,y)$ because

$$\left| \frac{y + |x| + y\,|y - |x||}{2 + |y - |x||} - |x| \right| = \left| \frac{(y - |x|)(1 + |y - |x||)}{2 + |y - |x||} \right| \le |y - |x||.$$

It follows that T is a continuous Subrahmanyam map with respect to ψ. However, T does not satisfy the original Subrahmanyam's condition in Theorem 2.2 since $\sup\left\{\psi(x,y) : (x,y) \in \mathbb{R}^2\right\} = 1$.

Remark 3.6. Recall that T is called *quasi-nonexpansive* if

$$d(Tx,p) \le d(x,p),$$

for all $x \in X$ and $p \in F(T)$. In metric fixed point theory, it is well-known that the notion of quasi-nonexpansive maps includes both contractions and nonexpansive maps, and the fixed point set of a quasi-nonexpansive selfmap on a strictly convex

Banach space is always convex. However, T in the previous example is not quasi-nonexpansive because

$$\|T(0,4)-(4,4)\| = \sqrt{16+\frac{4}{9}} > \|(0,4)-(4,4)\|,$$

and its fixed point set is not convex. Therefore, a continuous Subrahmanyam map is generally not quasi-nonexpansive, but surprisingly, it is always virtually nonexpansive according to the next theorem. As a result, the contractibility, instead of the convexity, becomes a natural structure for the fixed point set of a continuous Subrahmanyam map.

Theorem 3.7. *If T is a continuous Subrahmanyam map with respect to ψ and ψ is continuous, then T is virtually nonexpansive, and hence, $F(T)$ is a retract of X.*

Proof. Let $\epsilon > 0$, $p \in F(T)$, and $L = \sum_{i=0}^{\infty} \left(\frac{1+\psi(p)}{2}\right)^i \geq 1$.

Since T and ψ are continuous at p, there is $\delta > 0$ such that for any $x \in B(x, \delta)$, we have $d(Tx, p) < \frac{\epsilon}{3L}$ and $\psi(x) < \frac{1+\psi(p)}{2}$.

By letting $\delta' = \min\left\{\delta, \frac{\epsilon}{3L}\right\} > 0$, then for each $x \in B(p, \delta')$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} d(T^n x, p) &\leq d(T^n x, T^{n-1}x) + \ldots + d(Tx, x) + d(x, p) \\ &\leq \psi(x)^{n-1} d(Tx, x) + \ldots + d(Tx, x) + d(x, p) \\ &= \sum_{i=0}^{n-1} \psi(x)^i d(Tx, x) + d(x, p) \\ &\leq \sum_{i=0}^{n-1} \left(\frac{1+\psi(p)}{2}\right)^i d(Tx, x) + d(x, p) \\ &\leq L\left(d(Tx, p) + d(x, p)\right) + d(x, p) \\ &< L\left(\frac{\epsilon}{3L}\right) + L\left(\frac{\epsilon}{3L}\right) + \frac{\epsilon}{3L} \\ &< \epsilon. \end{aligned}$$

It follows that $T^n\left(B(p, \delta')\right) \subseteq B(p, \epsilon)$ for all $n \in \mathbb{N}$, and hence, T is virtually nonexpansive. By Theorem 2.6 and Theorem 3.3, $F(T)$ is a retract of $C(T) = X$. □

Corollary 3.8. *If X is a contractible complete metric space and T is a continuous Subrahmanyam selfmap on X with respect to a continuous function ψ, then $F(T)$ is contractible.*

Corollary 3.9. *If X is a closed convex subset of a Banach space and $T : X \to X$ is a continuous map satisfying the original Subrahmanyam's condition in Theorem 2.2, then $F(T)$ is contractible.*

The next example shows that the notion of Subrahmanyam maps is only sufficient for the contractibility of fixed point sets.

Example 3.10. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x,y) = (x,-y)$. Clearly, $F(T) = \{(x,0) : x \in \mathbb{R}\}$ is contractible, but T is not a Subrahmanyam map because $\left\|T^2(x,y) - T(x,y)\right\| = \|T(x,y) - (x,y)\|$ for all $(x,y) \in \mathbb{R}^2$.

Up this point, one might expect the contractibility of fixed point sets to hold for those maps satisfying Suzuki's condition in Theorem 2.3. Unfortunately, the following example shows that it is not the case.

Example 3.11. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T(x) = \begin{cases} 0 & \text{if } x = 0, \\ x/\|x\|, & \text{otherwise.} \end{cases}$$

Then, for each $x \in \mathbb{R}^2$, the sequence $(T^n x)$ clearly converges to a fixed point of T, and $F(T) = \{x : \|x\| = 1\} \cup \{0\}$. It follows that T satisfies Suzuki's condition in Theorem 2.3, but its fixed point set is clearly not contractible.

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Manuscript received 14 August 2017

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