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AN ITERATION SCHEME FINDING A COMMON FIXED POINT OF COMMUTING TWO NONEXPANSIVE MAPPINGS IN GENERAL BANACH SPACES

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Abstract. We present a new iteration scheme finding a common fixed point of two nonexpansive mappings in a general Banach space.

1. Introduction

In 1979, Ishikawa [6] presented an excellent and complicated method finding a common fixed point of a finite family $\{T_1, T_2, \dots, T_k\}$ of commuting nonexpansive self-mappings on D, where D is a compact convex subset of a general Banach space. It is not easy to read [6] and a long process is necessary to have his result. Then, Kubota and Takeuchi [9] surveyed the article. They clarify details of his argument and rewrite Ishikawa's results by using a double sequence of mappings which is generated by $\{T_1, T_2, \dots, T_k\}$.

In this article, we deal with common fixed points of commutative two nonexpansive mappings in a general Banach space. In this setting, that is, in the case of $k = 2$, Ishikawa's method in [6] is simple as below; see [9].

Theorem 1.1. *Let $a \in (0, 1)$. Let D be a compact convex subset of a Banach space E. Let T_1, T_2 be nonexpansive self-mappings on D with $T_1T_2 = T_2T_1$. For $i = 1, 2$, let S_i be a mapping on D defined by $S_i = aT_i + (1 - a)I$. Let $x_1 \in D$ and define a sequence $\{x_n\}$ in D by*

$$x_{n+1} = S_2 S_1^n x_n \qquad \text{for} \quad n \in N.$$

Then $\{x_n\}$ converges strongly to a common fixed point z of T_1 and T_2.

In 1998, Atsushiba and Takahashi [1] proved Theorem 1.2. Motivated by [6] and [1], in 2002, Suzuki [15] proved Theorem 1.3 by using Atushiba–Takahashi type iteration. Under the setting in Theorem 1.3, the iteration is not simpler than Ishikawa's. However, it is interesting in theory. In 2005, Suzuki [16] also presented another interesting result related to this problem.

Theorem 1.2. *Let $a \in (0, 1)$ and $\{a_n\}$ be a sequence in $[0, a]$. Let E be a uniformly convex Banach space which satisfies Opial's condition or whose norm is Fréchet differentiable. Let C be a closed convex subset of E. Let S and T be nonexpansive*

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self–mappings on C such that $ST = TS$ and $F(T) \cap F(S) \neq \emptyset$. Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by

$$x_{n+1} = \frac{a_n}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} S^i T^j x_n + (1 - a_n)x_n \qquad \text{for} \quad n \in N.$$

Then $\{x_n\}$ converges weakly to a common fixed point z of S and T.

Theorem 1.3. *Let $\{a_n\}$ be a sequence in $[0, 1]$ with*

$$0 < \liminf_n a_n \leq \limsup_n a_n < 1.$$

Let C be a compact convex subset of a Banach space E. Let S, T be nonexpansive self–mappings on C with $ST = TS$. Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by

$$x_{n+1} = \frac{a_n}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} S^i T^j x_n + (1 - a_n)x_n \qquad \text{for} \quad n \in N.$$

Then $\{x_n\}$ converges strongly to a common fixed point z of S and T.

Motivated by these works as above, we are interested in having more simple iteration in Suzuki's direction. Then, we introduce a new iteration scheme and prove a strong convergence theorem. Our arguments are essentially based on ideas and techniques prepared in Suzuki [15].

2. Preliminaries

In this article, N denotes the set of positive integers and N^2 denotes the product $N \times N$. For $k, l \in N$, N_l, $N_{k \leq l}$, N_l^2 and $N_{k \leq l}^2$ denote the following:

$$\begin{aligned} &N_l = \{i \in N : l \leq i\}, \quad N_{k \leq l} = \{i \in N : k \leq i \leq l\}, \\ &N_l^2 = \{(i, j) \in N^2 : i \in N_l, \quad j \in N_{i \leq i+1}\}, \\ &N_{k \leq l}^2 = \{(i, j) \in N^2 : i \in N_{k \leq l}, \quad j \in N_{i \leq i+1}\}. \end{aligned}$$

For a set B, $\#B$ denotes the cardinal number of B.

We denote by E a real Banach space with norm $\| \cdot \|$. Let C be a subset of a Banach space E and T be a mapping of C into E. $F(T)$ denotes the set of fixed points of T, that is, $F(T) = \{x \in C : x = Tx\}$. T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in C$.

Let C be a subset of E and let S and T be nonexpansive self–mappings on C. For each $n \in N$, we define a mapping $M(n)$ of C into E by

$$\text{(M)} \qquad \begin{aligned} M(n)x &= \frac{1}{2n} \sum_{i=1}^{n} \sum_{j=i}^{i+1} S^i T^j x \\ &= \frac{1}{2n} \sum_{i=1}^{n} S^i T^i x + \frac{1}{2n} \sum_{i=1}^{n} S^i T^{i+1} x \qquad \text{for } x \in C. \end{aligned}$$

Then, each $M(n)$ is nonexpansive. Indeed, for each $n \in N$, we have

$$\begin{aligned}\|M(n)x - M(n)y\| &\le \frac{1}{2n}\sum_{i=1}^{n}\sum_{j=i}^{i+1}\|S^iT^jx - S^iT^jy\| \\ &\le \frac{1}{2n} \times 2n\|x-y\| = \|x-y\| \qquad \text{for} \quad x,y \in C.\end{aligned}$$

To prove Lemma 2.3, we need the following lemma due to Suzuki [16].

Lemma 2.1. *Let $\{a_n\}$ be a sequence in $[0,1]$. Let $\{u_n\}$ and $\{w_n\}$ be bounded sequences in a Banach space E. Assume that*

(1) $u_{i+1} = a_iw_i + (1-a_i)u_i$ for $i \in N$,
(2) $0 < \liminf_n a_n \le \limsup_n a_n < 1$,
(3) $\limsup_n(\|w_{n+1} - w_n\| - \|u_{n+1} - u_n\|) \le 0$.

Then, $\lim_n \|w_n - u_n\| = 0$.

Lemma 2.2. *Let C be a bounded subset of a Banach space E. Let S and T be nonexpansive self–mappings on C and each $M(n)$ be the mapping defined by* (M). *Let $L = \sup\{\|x\| : x \in C\} < \infty$. Then, for each $n, k \in N$,*

$$\|M(n+k)x - M(n)x\| \le \frac{2k}{(n+k)}L \quad \text{for} \quad x \in C.$$

That is, for each $k \in N$, $\lim_n \|M(n+k)x - M(n)x\| = 0$.

Proof. We easily have the result from the following: For $x \in C$, $n, k \in N$,

$$\begin{aligned}&\|M(n+k)x - M(n)x\| \\ &= \Big\|\frac{1}{2(n+k)}\sum_{i=1}^{n+k}\sum_{j=i}^{i+1}S^iT^jx - \frac{1}{2n}\sum_{i=1}^{n}\sum_{j=i}^{i+1}S^iT^jx\Big\| \\ &\le \Big(\frac{1}{2n} - \frac{1}{2(n+k)}\Big)\sum_{i=1}^{n}\sum_{j=i}^{i+1}\|S^iT^jx\| + \frac{1}{2(n+k)}\sum_{i=n+1}^{n+k}\sum_{j=i}^{i+1}\|S^iT^jx\| \\ &= \frac{k}{2n(n+k)} \times 2nL + \frac{1}{2(n+k)} \times 2kL = \frac{2k}{(n+k)}L.\end{aligned}$$

□

Lemma 2.3. *Let $\{a_n\}$ be a sequence in $[0,1]$ such that*

$$0 < \liminf_n a_n \le \limsup_n a_n < 1.$$

Let C be a compact convex subset of a Banach space E. Let S and T be nonexpansive self–mappings on C with $ST = TS$ and each $M(n)$ be the mapping defined by (M). *Let $\{x_n\}$ be a sequence in C defined by*

$$x_1 \in C, \qquad x_{n+1} = a_nM(n)x_n + (1-a_n)x_n \quad \text{for } n \in N.$$

Then, $\lim_n \|M(n)x_n - x_n\| = 0$. *Furthermore, for a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to $z \in C$,* $\lim_k \|M(n_k)z - z\| = 0$ *holds.*

Proof. Set $w_n = M(n)x_n$ for $n \in N$. Then, by Lemma 2.2, we have

$$\begin{aligned}
&\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| \\
&\quad \le \|M(n+1)x_{n+1} - M(n+1)x_n\| \\
&\quad\quad + \|M(n+1)x_n - M(n)x_n\| - \|x_{n+1} - x_n\| \\
&\quad \le \|x_{n+1} - x_n\| + \|M(n+1)x_n - M(n)x_n\| - \|x_{n+1} - x_n\| \le \frac{2}{(n+1)}L
\end{aligned}$$

for $n \in N$, where $L = \sup\{\|x\| : x \in C\} < \infty$. By Lemma 2.1, we have

$$\lim_n \|w_n - x_n\| = \lim_n \|M(n)x_n - x_n\| = 0.$$

Suppose $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ converging to some $z \in C$. Then,

$$\begin{aligned}
\|M(n_k)z - z\| &\le \|M(n_k)z - M(n_k)x_{n_k}\| + \|M(n_k)x_{n_k} - x_{n_k}\| + \|x_{n_k} - z\| \\
&\le \|M(n_k)x_{n_k} - x_{n_k}\| + 2\|x_{n_k} - z\| \qquad \text{for} \quad k \in N.
\end{aligned}$$

This implies $\lim_k \|M(n_k)z - z\| = 0$. □

3. Lemmas

To have the results in Lemma 2.3, the condition $ST = TS$ is unnecessary and we can replace compactness of C by boundedness of C. However, in the same setting as in Lemma 2.3, we are interested in seeing $z \in F(S) \cap F(T)$.

In this direction, it is so important to show

$$d = \lim_n \sup\{\|S^iT^jz - z\| : (i,j) \in N_n^2\} = 0 \quad \text{(Lemma 3.7)}.$$

To prove Lemma 3.7, we need Lemmas 3.1–3.6 below.

In Lemmas 3.1–3.7, we assume the following:

(A_1) C is a compact and convex subset of a Banach space E.
(A_2) S and T are nonexpansive self–mappings on C with $ST = TS$.
(A_3) $\{x_n\}$ is a sequence in C and $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.
(A_4) $\{x_{n_k}\}$ converges to $z \in C$ ($\{n_k\}$ is the index set of $\{x_{n_k}\}$).
(A_5) $\lim_k \|M(n_k)z - z\| = 0$.

Moreover, we use the following notations:

$$L = \sup\{\|x\| : x \in C\}, \quad d = \lim_n \sup\{\|S^iT^jz - z\| : (i,j) \in N_n^2\}.$$

It is obvious that $0 \le d < \infty$. Let $A \subset N_1^2$ and $m \in N$. Then, we set

$$A_m = N_m^2 \cap A, \quad A_{m\le n_k} = N_{m\le n_k}^2 \cap A \quad (\ A_{1\le n_k} = N_{1\le n_k}^2 \cap A\ ),$$

where $n_k \ge m$. Let $\varepsilon > 0$ and $w \in C$. Then, set

$$A(w,\varepsilon) = \{(i,j) \in N_1^2 : \|S^iT^jz - w\| \ge d - \varepsilon\}.$$

In this setting, we also use the following notations:

$$\begin{aligned}
N_m^2(w, d-\varepsilon) &= A(w,\varepsilon)_m = \{(i,j) \in N_m^2 : \|S^iT^jz - w\| \ge d - \varepsilon\}, \\
N_{m\le n_k}^2(w, d-\varepsilon) &= A(w,\varepsilon)_{m\le n_k} \\
&= \{(i,j) \in N_{m\le n_k}^2 : \|S^iT^jz - w\| \ge d - \varepsilon\}.
\end{aligned}$$

We note that, for $\varepsilon > 0$ with $d - \varepsilon \le 0$, $A(w,\varepsilon)_m$ and N_m^2 are the same.

Lemma 3.1. *Assume $d > 0$. Then, there are $u \in C$ with $\|u - z\| = d$ and a sequence $\{(i_n, j_n)\} \subset N_1^2$ which satisfy the following:*

(1) *For $n \in N$, $(i_n, j_n) \in N_n^2(z, d - 1/n)$.*
(2) *$\{S^{i_n} T^{j_n} z\}$ converges to u.*

Furthermore, for arbitrary $\delta > 0$, there is $m_\delta \in N$ such that

(3) $\|S^i T^j z - u\| \le d + \delta \qquad \text{for} \quad (i, j) \in N_{m_\delta}^2$.

Proof. By the definition of d, for $n \in N$, $N_n^2(z, d-1/n)$ contains an element (i_n, j_n). Then, we can generate a sequence $\{(i_n, j_n)\} \subset N_1^2$. Since C is compact, by passing to subsequence, we can consider $\{(i_n, j_n)\} \subset N_1^2$ as a sequence such that $\{S^{i_n} T^{j_n} z\}$ converges to $u \in C$ satisfying $\|u - z\| = d$.

We show (3). Let $\delta > 0$. By the definition of d, there is $s \in N$ such that

$$\|S^i T^j z - z\| \le d + \delta/2 \quad \text{for} \quad (i, j) \in N_s^2 \ .$$

Furthermore, there is $(i_{n_0}, j_{n_0}) \in N_s^2$ satisfying either of the following:

$$\text{(i)} \qquad j_{n_0} = i_{n_0}, \qquad \|S^{i_{n_0}} T^{i_{n_0}} z - u\| < \delta/2,$$

$$\text{(ii)} \qquad j_n = i_n + 1, \qquad \|S^{i_n} T^{i_n+1} z - u\| < \delta/2 \qquad \text{for} \ \ (i_n, j_n) \in N_{i_{n_0}}^2.$$

Let $m_\delta = 2 i_{n_0}$. Then, $i - i_{n_0} \ge i_{n_0}$ if $i \ge m_\delta$.

In case (i), we have $(i - i_{n_0}, j - i_{n_0}) \in N_{i_{n_0}}^2 \subset N_s^2$ for $(i, j) \in N_{m_\delta}^2$. Then,

$$\text{(A)} \qquad \begin{aligned} \|S^i T^j z - u\| &\le \|S^i T^j z - S^{i_{n_0}} T^{i_{n_0}} z\| + \|S^{i_{n_0}} T^{i_{n_0}} z - u\| \\ &\le \|S^{i - i_{n_0}} T^{j - i_{n_0}} z - z\| + \delta/2 \le d + \delta \qquad \text{for} \quad (i, j) \in N_{m_\delta}^2 \ . \end{aligned}$$

Thus, (3) holds. We consider case (ii). In this case, for $(i, i+1) \in N_{m_\delta}^2$,

$$\text{(B)} \qquad \begin{aligned} \|S^i T^{i+1} z - u\| &\le \|S^i T^{i+1} z - S^{i_{n_0}} T^{i_{n_0}+1} z\| + \|S^{i_{n_0}} T^{i_{n_0}+1} z - u\| \\ &\le \|S^{i - i_{n_0}} T^{i - i_{n_0}} z - z\| + \delta/2 \le d + \delta \ . \end{aligned}$$

Let $(i, i) \in N_{m_\delta}^2$, that is, $i \ge m_\delta$. Then, there is $i_{n_1} \in N$ satisfying $i_{n_1} \ge 2i$. By $i_{n_1} > i_{n_1} - i \ge i \ge m_\delta > i_{n_0}$ and $(i_{n_1} - i, i_{n_1} - i + 1) \in N_{m_\delta}^2$, we have

$$\begin{aligned} \|S^i T^i z - u\| &\le \|S^i T^i z - S^{i_{n_1}} T^{i_{n_1}+1} z\| + \|S^{i_{n_1}} T^{i_{n_1}+1} z - u\| \\ &\le \|S^{i_{n_1} - i} T^{(i_{n_1} - i)+1} z - z\| + \delta/2 \le d + \delta. \end{aligned}$$

Then, the following holds:

$$\text{(C)} \qquad \|S^i T^i z - u\| \le d + \delta \qquad \text{for} \quad (i, i) \in N_{m_\delta}^2 \ .$$

By (B) and (C), (3) also holds. Thus, we have the result. □

Lemma 3.2. *Assume $d > 0$. Let $u \in C$ satisfy $\|u - z\| = d$. Then, for $\varepsilon \in (0, d)$, the following holds:*

$$\lim_k \frac{\#\big(N_{1 \le n_k}^2 (u, d - \varepsilon)\big)}{2 n_k} = 1.$$

Proof. Fix $\varepsilon \in (0, d)$. Let $\delta > 0$ arbitrary and let $m_\delta \in N$ satisfy conditions in Lemma 3.1 (3). We deal with n_k such that $n_k > m_\delta$.

It is obvious that $\# N^2_{1\le n_k} = 2n_k$. Set, for $k \in N$ satisfying $n_k > m_\delta$,

$$b_k = \#(N^2_{m_\delta+1\le n_k}(u, d-\varepsilon))/2n_k, \quad a_k = \#(N^2_{1\le n_k}(u, d-\varepsilon))/2n_k.$$

Note that b_k depends on δ. However, a_k does not depend on δ. It is obvious that $0 \le b_k \le a_k \le 1$ if $n_k > m_\delta$. That is, the following inequalities hold:

$$\text{(i)} \qquad \liminf_k b_k \le \liminf_k a_k \le 1, \qquad \limsup_k b_k \le \limsup_k a_k \le 1.$$

By $b_k \le a_k$ and Lemma 3.1, we have

$$\begin{aligned}\|M(n_k)z - u\| &\le \frac{1}{2n_k}\sum_{i=1}^{m_\delta}\sum_{j=i}^{i+1}\|S^iT^jz-u\| + \frac{1}{2n_k}\sum_{i=m_\delta+1}^{n_k}\sum_{j=i}^{i+1}\|S^iT^jz-u\| \\ &\le \frac{m_\delta}{n_k}L + \Big(b_k(d+\delta) + \Big(\frac{1}{2n_k}\times 2(n_k-m_\delta) - b_k\Big)(d-\varepsilon)\Big) \\ &\le \frac{m_\delta}{n_k}L + \frac{n_k-m_\delta}{n_k}(d-\varepsilon) + a_k(\varepsilon+\delta).\end{aligned}$$

Then, by $d = \|u-z\|$ and $\lim_k \|M(n_k)z - z\| = 0$, it follows that

$$d = \|z-u\| = \liminf_k \|M(n_k)z-u\| \le (d-\varepsilon) + (\liminf_k a_k)(\varepsilon+\delta).$$

Since δ is arbitrary, we have a contradiction if $\liminf_k a_k < 1$. Thus, $\liminf_k a_k \ge 1$. By (i), we have $\lim_k a_k = 1$. This completes the proof. $\square$

Lemma 3.3. *Let $A \subset N^2_1$. Assume $\lim_k \frac{\#A_{1\le n_k}}{2n_k} = 1$. Then, for any $m \in N$, $\lim_k \frac{\#A_{m\le n_k}}{2n_k} = 1$, where $n_k \ge m$. Moreover, the following hold:*

(1) *For each $n \in N$, A_n contains an element (i,i).*
(2) *For each $n \in N$, A_n contains an element $(i,i+1)$.*

Proof. Let $m \in N$ and $n_k \ge m$. Set, for such $k \in N$,

$$a_k = \#A_{1\le n_k}/2n_k, \qquad c_k(m) = \#A_{m\le n_k}/2n_k.$$

By $\lim_k a_k = 1$ and $0 \le a_k - c_k(m) \le (m-1)/n_k$, we have $\lim_k c_k(m) = 1$.

Confirm $A_n = N^2_n \cap A$ for $n \in N$ and $\#\big(N^2_{1\le n_k}\big) = 2n_k$. We show that A_n contains (i,j) satisfying $i = j$. Arguing by contradiction, assume that there is $n_0 \in N$ such that A_{n_0} contains no element (i,j) satisfying $i = j$. Then, it is obvious that

$$c_k(n_0) = \frac{\#A_{n_0\le n_k}}{2n_k} \le 1/2 \qquad \text{for} \quad k.$$

However, we know $\lim_k c_k(n_0) = 1$. We have a contradiction.

In the same way, A_n contains (i,j) satisfying $j = i+1$. $\square$

Lemma 3.4. *Let A and B be subsets of N^2_1. Assume that*

$$\lim_k \frac{\#A_{1\le n_k}}{2n_k} = 1, \qquad \lim_k \frac{\#B_{1\le n_k}}{2n_k} = 1.$$

Then, $\lim_k \frac{\#\big(A_{1\le n_k}\cap B_{1\le n_k}\big)}{2n_k} = 1$.

Proof. For $k \in N$, we know $\#\big(N^2_{1\leq n_k}\big) = 2n_k$ and

$$\frac{\#\big(A_{1\leq n_k} \cap B_{1\leq n_k}\big)}{2n_k} \leq 1, \quad \frac{\big(\#A_{1\leq n_k} - \#\big(A_{1\leq n_k} \cap B_{1\leq n_k}\big)\big) + \#B_{1\leq n_k}}{2n_k} \leq 1.$$

Then, it is easy to see that

$$\frac{\#A_{1\leq n_k}}{2n_k} + \frac{\#B_{1\leq n_k}}{2n_k} - 1 \leq \frac{\#\big(A_{1\leq n_k} \cap B_{1\leq n_k}\big)}{2n_k} \leq 1 \qquad \text{for } k \in N.$$

By $\lim_k \frac{\#A_{1\leq n_k}}{2n_k} = \lim_k \frac{\#B_{1\leq n_k}}{2n_k} = 1$, we have $\lim_k \frac{\#\big(A_{1\leq n_k}\cap B_{1\leq n_k}\big)}{2n_k} = 1$. □

Lemma 3.5. *Assume $d > 0$. Then, there are $v \in C$ and a sequence $\{(i^1_n, i^1_n)\} \subset N^2_1$ such that $\|v - z\| = d$ and $\{S^{i^1_n}T^{i^1_n}z\}$ converges to v.*

Proof. We show that, for $\delta \in (0, d)$ and $m \in N$, $N^2_m(z, d-\delta)$ contains (i, j) satisfying $i = j$. Arguing by contradiction, assume the existence of $\delta_0 \in (0, d)$ and $m_0 \in N$ such that $N^2_{m_0}(z, d - \delta_0)$ contains no element (i, j) satisfying $i = j$. By Lemma 3.1, there are $u \in C$ with $\|u - z\| = d$ and a sequence $\{(i_n, j_n)\} \subset N^2_1$ which satisfy the following:

(1) For $n \in N$, $(i_n, j_n) \in N^2_n(z, d - 1/n)$.
(2) $\{S^{i_n}T^{j_n}z\}$ converges to u.

Then, there is $(i_{n_0}, j_{n_0}) \in N^2_{n_0}(z, d - 1/n_0)$ satisfying the following:

$$\|S^{i_{n_0}}T^{j_{n_0}} - u\| < \delta_0/2, \quad 1/n_0 < \delta_0, \quad n_0 > m_0, \quad i_{n_0} > m_0.$$

By $(i_{n_0}, j_{n_0}) \in N^2_{n_0}(z, d - 1/n_0) \subset N^2_{m_0}(z, d - \delta_0)$, we have $j_{n_0} = i_{n_0} + 1$.

On the other hand, by Lemma 3.2, we know

$$\lim_k \frac{\#\big(N^2_{1\leq n_k}(u, d - \delta_0/2)\big)}{2n_k} = 1.$$

By Lemma 3.3 (2), for any $n \in N$, there is $(i, j) \in N^2_n(u, d - \delta_0/2)$ satisfying $j = i + 1$. Let $l = 2i_{n_0}$ and $(i, i + 1) \in N^2_l(u, d - \delta_0/2)$. Then, $i - i_{n_0} \geq i_{n_0} > m_0$, $(i + 1) - (i_{n_0} + 1) = i - i_{n_0}$, $\|S^iT^{i+1}z - u\| \geq d - \delta_0/2$, and

$$\begin{aligned}\|S^{i-i_{n_0}}T^{i-i_{n_0}}z - z\| &\geq \|S^iT^{i+1}z - S^{i_{n_0}}T^{i_{n_0}+1}z\| \\ &\geq \|S^iT^{i+1}z - u\| - \|S^{i_{n_0}}T^{i_{n_0}+1}z - u\| \\ &\geq d - \delta_0/2 - \delta_0/2 = d - \delta_0.\end{aligned}$$

Thus, we have $(i - i_{n_0}, i - i_{n_0}) \in N^2_{m_0}(z, d - \delta_0)$. This is a contradiction.

By taking $(i^1_n, i^1_n) \in N^2_n(z, d-1/n)$ for n, we have $\{(i^1_n, i^1_n)\} \subset N^2_1$. By passing to subsequence, we can regard $\{(i^1_n, i^1_n)\}$ as a sequence such that $\{S^{i^1_n}T^{i^1_n}z\}$ converges to some $v \in C$ satisfying $\|v - z\| = d$.

□

Lemma 3.6. *Assume $d > 0$. Then, for $\varepsilon \in (0, d)$, the following holds:*

$$\lim_k \frac{\#\big(N^2_{1\leq n_k}(z, d - \varepsilon)\big)}{2n_k} = 1.$$

Proof. By Lemma 3.5, there are $v \in C$ and $\{(i_n^1, i_n^1)\} \subset N_1^2$ such that $\|v - z\| = d$ and $\{S^{i_n^1} T^{i_n^1} z\}$ converges to v. Fix $\varepsilon \in (0, d)$ arbitrarily. Then, by $\|v - z\| = d$ and Lemma 3.2,

$$\lim_k \frac{\#\big(N^2_{1 \le n_k}(v, d - \varepsilon/2)\big)}{2n_k} = 1.$$

Let $m \in N$, $n_k \ge m$ and set, for k,

$$a_k = \frac{\#\big(N^2_{1 \le n_k}(v, d - \varepsilon/2)\big)}{2n_k}, \qquad c_k(m) = \frac{\#\big(N^2_{m \le n_k}(v, d - \varepsilon/2)\big)}{2n_k}.$$

Then, by Lemma 3.3, we know $\lim_k c_k(m) = \lim_k a_k = 1$.

It is obvious that there is $i_{n_0}^1$ satisfying $\|S^{i_{n_0}^1} T^{i_{n_0}^1} z - v\| < \varepsilon/2$. Let $m_1 = 2i_{n_0}^1$ and consider $n_k \ge m_1$. Confirm that $(i,j) \in N^2_{m_1 \le n_k}(v, d - \varepsilon/2)$ implies $(i,j) \in N^2_{m_1 \le n_k}$ and $\|S^i T^j z - v\| \ge d - \varepsilon/2$.

Then, for $(i,j) \in N^2_{m_1 \le n_k}(v, d - \varepsilon/2)$, it is easy to see that

$$\begin{aligned} \|S^{i - i_{n_0}^1} T^{j - i_{n_0}^1} z - z\| &\ge \|S^i T^j z - S^{i_{n_0}^1} T^{i_{n_0}^1} z\| \\ &\ge \|S^i T^j z - v\| - \|S^{i_{n_0}^1} T^{i_{n_0}^1} z - v\| \\ &\ge d - \varepsilon/2 - \varepsilon/2 = d - \varepsilon. \end{aligned}$$

That is, we have

$$\begin{aligned} 1 \ge \frac{\#\big(N^2_{1 \le n_k}(z, d - \varepsilon)\big)}{2n_k} &\ge \frac{\#\big(\{(i - i_{n_0}^1,\, j - i_{n_0}^1) :\ (i,j) \in N^2_{m_1 \le n_k}(v, d - \varepsilon/2)\}\big)}{2n_k} \\ &= \frac{\#\big(N^2_{m_1 \le n_k}(v, d - \varepsilon/2)\big)}{2n_k} = c_k(m_1). \end{aligned}$$

We know $\lim_k c_k(m_1) = 1$. Then, we have $\lim_k \frac{\#\left(N^2_{1 \le n_k}(z, d-\varepsilon)\right)}{2n_k} = 1$. □

Lemma 3.7. $d = 0$.

Proof. Arguing by contradiction, assume $d > 0$. Then, by Lemma 3.5, there is $v_1 \in C$ satisfying $\|v_1 - z\| = d$. Let $\varepsilon \in (0, d)$. By Lemmas 3.2, 3.6, we know the following:

$$(1) \quad \lim_k \frac{\#\big(N^2_{1 \le n_k}(v_1, d - \varepsilon)\big)}{2n_k} = 1, \qquad (2) \quad \lim_k \frac{\#\big(N^2_{1 \le n_k}(z, d - \varepsilon)\big)}{2n_k} = 1.$$

Set $A(v_1, \varepsilon) = N_1^2(v_1, d - \varepsilon)$, $A(z, \varepsilon) = N_1^2(z, d - \varepsilon)$, and $B^{(1)}(\varepsilon) = A(v_1, \varepsilon) \cap A(z, \varepsilon)$. Note that

$$B^{(1)}(1/n)_n = N_n^2(v_1, d - 1/n) \cap N_n^2(z, d - 1/n) \quad \text{for} \quad n \in N.$$

Then, (1) and (2) are rewritten to the following:

$$\lim_k \frac{\#A(v_1, \varepsilon)_{1 \le n_k}}{2n_k} = 1, \quad \lim_k \frac{\#A(z, \varepsilon)_{1 \le n_k}}{2n_k} = 1.$$

By Lemma 3.4, we have $\lim_k \frac{\#B^{(1)}(\varepsilon)_{1 \le n_k}}{2n_k} = 1$. Since ε is arbitrary, by Lemma 3.3, there is $(i_n^2, i_n^2) \in B^{(1)}(1/n)_n$ for $n \in N$. That is, we have a sequence $\{(i_n^2, i_n^2)\} \subset N_1^2$ such that $(i_n^2, i_n^2) \in B^{(1)}(1/n)_n$ for $n \in N$. By passing to subsequences, we can regard $\{(i_n^2, i_n^2)\}$ as a sequence such that $\{S^{i_n^2}T^{i_n^2}z\}$ converges to some $v_2 \in C$ and $(i_n^2, i_n^2) \in B^{(1)}(1/n)_n$ for $n \in N$. That is, $\|v_2 - v_1\| = d$ and $\|v_2 - z\| = d$. Furthermore, by Lemma 3.2,

$$\lim_k \frac{\#\big(N^2_{1\le n_k}(v_2, d-\varepsilon)\big)}{2n_k} = 1.$$

Set $A(v_2,\varepsilon) = N_1^2(v_2, d-\varepsilon)$ and $B^{(2)}(\varepsilon) = A(v_2,\varepsilon) \cap B^{(1)}(\varepsilon) = A(v_2,\varepsilon) \cap A(v_1,\varepsilon) \cap A(z,\varepsilon)$. Note that, for $n \in N$,

$$B^{(2)}(1/n)_n = N_n^2(v_2, d-1/n) \cap N_n^2(v_1, d-1/n) \cap N_n^2(z, d-1/n).$$

We already know the following:

$$\lim_k \frac{\#A(v_2,\varepsilon)_{1\le n_k}}{2n_k} = 1, \qquad \lim_k \frac{\#B^{(1)}(\varepsilon)_{1\le n_k}}{2n_k} = 1.$$

Then, by Lemma 3.4, we have $\lim_k \frac{\#B^{(2)}(\varepsilon)_{1\le n_k}}{2n_k} = 1$. Since ε is arbitrary, by Lemma 3.3, there is $(i_n^3, i_n^3) \in B^{(2)}(1/n)_n$ for $n \in N$. That is, we have $\{(i_n^3, i_n^3)\} \subset N_1^2$ such that $(i_n^3, i_n^3) \in B^{(2)}(1/n)_n$ for $n \in N$. By passing to subsequences, we can regard $\{(i_n^3, i_n^3)\}$ as a sequence such that $\{S^{i_n^3}T^{i_n^3}z\}$ converges to some $v_3 \in C$ and $(i_n^3, i_n^3) \in B^{(2)}(1/n)_n$ for $n \in N$. That is, $\|v_3 - v_2\| = d$, $\|v_3 - v_1\| = d$ and $\|v_3 - z\| = d$. Furthermore, by Lemma 3.2,

$$\lim_k \frac{\#\big(N^2_{1\le n_k}(v_3, d-\varepsilon)\big)}{2n_k} = 1.$$

By induction, we have $\{v_n\}$ in C such that $\|v_i - v_j\| = d > 0$ if $i \ne j$. That is, $\{v_n\}$ can not have a convergent subsequence. However, since C is compact, $\{v_n\}$ must have a convergent subsequence. This is a contradiction. □

4. Main result

Theorem 4.1. *Let $\{a_n\}$ be a sequence in $[0,1]$ such that*

$$0 < \liminf_n a_n \le \limsup_n a_n < 1.$$

Let C be a compact convex subset of a Banach space E. Let S and T be nonexpansive self–mappings on C with $ST = TS$ and each $M(n)$ be the mapping defined by (M). *Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = a_n M(n) x_n + (1-a_n)x_n \qquad \text{for } n \in N.$$

Then $\{x_n\}$ converges strongly to some common fixed point z of S and T.

Proof. Since C is compact, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to some $z \in C$. Under our assumptions, by Lemma 2.3, it is obvious that (A_1)–(A_5) are satisfied. Then, by Lemma 3.7,

$$d = \lim_n \sup\{\|S^iT^jz - z\| : (i,j) \in N_n^2\} = 0.$$

We know that $\{S^nT^nz\}$ has a convergent subsequence. By $d = 0$, any convergent subsequence of $\{S^nT^nz\}$ converges to z. That is, $\{S^nT^nz\}$ itself converges to z.

In the same way, we have that $\{S^n T^{n+1} z\}$ converges to z. Since S and T are continuous mappings with $ST = TS$, the following hold:

$$Sz = S(\lim_n S^n T^{n+1} z) = \lim_n (S^{n+1} T^{n+1} z) = z,$$
$$Tz = T(\lim_n S^n T^n z) = \lim_n (S^n T^{n+1} z) = z.$$

That is, $z \in F(S) \cap F(T)$. Then, we easily have $z \in \cap_n F(M(n))$ and

$$\begin{aligned}\|x_{n+1} - z\| &= \|a_n M(n) x_n + (1 - a_n) x_n - z\| \\ &\leq a_n \|M(n) x_n - z\| + (1 - a_n) \|x_n - z\| \leq \|x_n - z\|\end{aligned}$$

for $n \in N$. This implies that $\{\|x_n - z\|\}$ converges. Since $\{\|x_{n_k} - z\|\}$ converges to 0, $\{\|x_n - z\|\}$ itself converges to 0. Thus, we have the result. □

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