Linear and Nonlinear Analysis
Volume 2, Number 2, 2016, 311–316
Yokohama Publishers
ISSN 2188-8167

ON THE VON NEUMANN-JORDAN CONSTANT OF GENERALIZED BANAŚ-FRĄCZEK SPACES

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Abstract. In this paper, we introduce a new class of two-dimensional normed spaces, which contains the Banaś-Frączek space $\mathbb{R}^2_\lambda$. For such a space X, the exact value of the von Neumann-Jordan constant $C_{\rm NJ}(X)$ can be calculated by using the Banach-Mazur distance $d(X, \ell_2^2)$ between X and ℓ_2^2, where ℓ_2^2 is the two-dimensional ℓ_2-space. As a consequence, the exact value of $d(X, \ell_2^2)$ is also determined.

1. Introduction

The von Neumann-Jordan (NJ-) constant of Banach spaces was introduced by Clarkson [3]. The NJ-constant, denoted by $C_{\rm NJ}(X)$, of a Banach space X is the smallest constant C for which

$$\frac{1}{C} \le \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \le C$$

holds for all $x, y \in X$ not both 0. It is known that $1 \le C_{\rm NJ}(X) \le 2$ for any Banach space X, and X is a Hilbert space if and only if $C_{\rm NJ}(X) = 1$. Clarkson calculated the NJ-constant of L_p by using the so-called Clarkson inequality. Some relations between NJ-constant and geometrical properties in general Banach space setting have been discussed in many papers, see e.g., [4–6, 9]. Moreover, the computation for concrete Banach spaces attracted the interest of several authors. In [5], Kato, Maligranda and Takahashi considered the Day-James ℓ_p-ℓ_q spaces, i.e., $\mathbb{R}^2$ with the norm $\|\cdot\|_{p,q}$ defined by

$$\|(x,y)\|_{p,q} = \begin{cases} \|(x,y)\|_p, & xy \ge 0, \\ \|(x,y)\|_q, & xy \le 0, \end{cases}$$

where $1 \le p, q \le \infty$ and $\|\cdot\|_p$ is the ℓ_p-norm on $\mathbb{R}^2$ and posed a question: How to compute NJ-constant for Day-James spaces? Yang and Wang [12] introduced a new geometrical constant $\gamma_X(t)$ and calculated $C_{\rm NJ}(X)$ for X being ℓ_2-ℓ_1 space and ℓ_∞-ℓ_1 space by using $\gamma_X(t)$ (see also [1, 8, 11, 13]). Recently, Yang [10] studied

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2010 *Mathematics Subject Classification.* 46B20.
Key words and phrases. Banach-Mazur distance, von Neumann-Jordan constant.
The second author was supported in part by Grant-in-Aid for Scientific Research (No. 15K04920), Japan Society of the Promotion of Science.

the Banaś-Frączek space $\mathbb{R}^2_\lambda$ introduced in [2], i.e., the space $\mathbb{R}^2$ with the norm $|\cdot|_\lambda$ defined by

$$|(x,y)|_\lambda = \max\left\{\lambda|x|, \|(x,y)\|_2\right\},$$

where $\lambda > 1$. Note that this space may be considered as a generalization of ℓ_2-ℓ_1 space. Indeed, considering a rotation matrix

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

we have $\|T(x,y)\|_{2,1} = \max\{\sqrt{2}|x|,\ \|(x,y)\|_2\} = |(x,y)|_{\sqrt{2}}$. Hence, $\mathbb{R}^2_{\sqrt{2}}$ is isometric to the ℓ_2-ℓ_1 space. In [10], the author showed that

$$C_{\mathrm{NJ}}(\mathbb{R}^2_\lambda) = 2 - \frac{1}{\lambda^2}, \tag{1.1}$$

which implies that $C_{\mathrm{NJ}}(\ell_2\text{-}\ell_1) = \frac{3}{2}$. The calculation method for (1.1) is similar to that of $C_{\mathrm{NJ}}(\ell_2\text{-}\ell_1)$ in [12], but the computation is somewhat complicated.

In this paper, we shall establish a formula for the calculation method of the NJ-constant $C_{\mathrm{NJ}}(X)$ by using the Banach-Mazur distance $d(X, \ell_2^2)$, where X is a two-dimensional absolute normed space. By using this formula, we obtain a simple proof of (1.1). Moreover, we introduce a new class of two-dimensional normed spaces, which contains the Banaś-Frączek space $\mathbb{R}^2_\lambda$, and calculate $C_{\mathrm{NJ}}(X)$ for such a space X by using this formula.

2. Results

For isomorphic Banach spaces X and Y, the Banach-Mazur distance between X and Y, denoted by $d(X,Y)$, is defined to be the infimum of $\|T\| \cdot \|T^{-1}\|$ taken over all bicontinuous linear operators T from X onto Y.

Lemma 2.1 ([5]). *If X and Y are isomorphic Banach spaces, then*

$$\frac{C_{\mathrm{NJ}}(X)}{d(X,Y)^2} \le C_{NJ}(Y) \le C_{\mathrm{NJ}}(X)d(X,Y)^2$$

In particular, if X and Y are isometric, then $C_{\mathrm{NJ}}(X) = C_{\mathrm{NJ}}(Y)$.

Lemma 2.2 ([5]). *Let $X = (X, \|\cdot\|)$ be a Banach space and let $X_1 = (X, \|\cdot\|_1)$, where $\|\cdot\|_1$ is an equivalent norm on X satisfying, for $\alpha, \beta > 0$,*

$$\alpha\|x\| \le \|x\|_1 \le \beta\|x\|, \quad x \in X.$$

Then

$$\frac{\alpha^2}{\beta^2} C_{\mathrm{NJ}}(X) \le C_{\mathrm{NJ}}(X_1) \le \frac{\beta^2}{\alpha^2} C_{\mathrm{NJ}}(X).$$

Note here that Lemma 2.2 follows immediately from Lemma 2.1 and the fact that $d(X, X_1) \le \beta/\alpha$. In particular, if $C_{\mathrm{NJ}}(X_1) = \frac{\beta^2}{\alpha^2} C_{\mathrm{NJ}}(X)$, then $d(X, X_1) = \beta/\alpha$.

Using Lemma 2.2 we establish a formula on NJ-constant for absolute norms on $\mathbb{R}^2$. A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(|x|, |y|)\| = \|(x,y)\|$ for any $x, y \in \mathbb{R}$, and normalized if $\|(1,0)\| = \|(0,1)\| = 1$. The ℓ_p-norms on $\mathbb{R}^2$ are such examples.

Theorem 2.3 (cf. [7]). *Let $\|\cdot\|, \|\cdot\|_H$ be absolute norms on $\mathbb{R}^2$ satisfying the following conditions:*
(i) *$(\mathbb{R}^2, \|\cdot\|_H)$ is an inner product space.*
(ii) *$\|(x,y)\| \le \|(x,y)\|_H$ for any $(x,y) \in \mathbb{R}^2$.*
(iii) *$\|(1,0)\| = \|(1,0)\|_H$ and $\|(0,1)\| = \|(0,1)\|_H$.*
Then

$$C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|)) = \beta^2, \text{ where } \beta = \max\Big\{\frac{\|(x,y)\|_H}{\|(x,y)\|} : (x,y) \in \mathbb{R}^2, (x,y) \ne (0,0)\Big\}.$$

Proof. By (i), $C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|_H)) = 1$. Since $\|(u,v)\| \ge \frac{1}{\beta}\|(u,v)\|_H$ for any $(u,v) \in \mathbb{R}^2$, it follows from Lemma 2.2 that $C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|)) \le \beta^2$. On the other hand, we take a non-zero element $x = (u', v')$ with $\|x\|_H = \beta\|x\|$ and put $y = (u', -v')$. Since the norms $\|\cdot\|$ and $\|\cdot\|_H$ are absolute, by (iii),

$$\|y\|_H = \beta\|y\|, \|x+y\| = \|x+y\|_H, \|x-y\| = \|x-y\|_H.$$

Hence, we have

$$\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} = \beta^2 \frac{\|x+y\|_H^2 + \|x-y\|_H^2}{2(\|x\|_H^2 + \|y\|_H^2)} = \beta^2,$$

which completes the proof.

□

We now introduce a new class of two-dimensional normed spaces, which is a generalization of the Banaś-Frączek space $\mathbb{R}^2_\lambda$. For $a \ge b \ge 1$ and $1 \le p < \infty$ we define the norms $\|\cdot\|$ and $\|\cdot\|_H$ on $\mathbb{R}^2$ by

$$\|(x,y)\| = \max\{a|x|, b|y|, \|(x,y)\|_p\} \tag{2.1}$$

and

$$\|(x,y)\|_H = \|(ax, by)\|_2. \tag{2.2}$$

The space $(\mathbb{R}^2, \|\cdot\|)$ is denoted by $\mathbb{R}^2_{a,b,p}$. In the case where $a = \lambda, b = 1, p = 2$, we have $\mathbb{R}^2_{\lambda,1,2} = \mathbb{R}^2_\lambda$. If $a = 1$, then $b = 1$ and $\|(x,y)\| = \|(x,y)\|_p$. Hence we have $C_{\mathrm{NJ}}(\mathbb{R}^2_{1,1,p}) = 2^{2/r-1}$, where $r = \min\{p, \frac{p}{p-1}\}$ ([3]). It is easy to see that if $\frac{1}{a^p} + \frac{1}{b^p} \le 1$, then $\|(x,y)\| = \max\{a|x|, b|y|\}$ and $C_{\mathrm{NJ}}(\mathbb{R}^2_{a,b,p}) = 2$. Hence we may consider the case where $a > 1$ and $\frac{1}{a^p} + \frac{1}{b^p} > 1$. Using Theorem 2.3 we shall calculate $C_{\mathrm{NJ}}(\mathbb{R}^2_{a,b,p})$. Note here that if $p \ge 2$, then the norms $\|\cdot\|$ and $\|\cdot\|_H$ are absolute and satisfy the conditions (i), (ii) and (iii) in Theorem 2.3, that is, $(\mathbb{R}^2, \|\cdot\|_H)$ is an inner product space, $\|(x,y)\| \le \|(x,y)\|_H$, $\|(1,0)\| = \|(1,0)\|_H$ and $\|(0,1)\| = \|(0,1)\|_H$.

Lemma 2.4. *Let $a > 1$, $a \ge b \ge 1$ and $p \ge 2$ with $\dfrac{1}{a^p} + \dfrac{1}{b^p} > 1$. Let $\|\cdot\|$ and $\|\cdot\|_H$ be the norms defined by (2.1) and (2.2), respectively, and let*

$$\beta = \max\Big\{\frac{\|x\|_H}{\|x\|} : x \in \mathbb{R}^2, x \ne 0\Big\}.$$

(i) *If* $b \le a(a^p-1)^{\frac{p-2}{2p}}$, *then*

$$\beta = \Big(1 + b^2\Big(1 - \frac{1}{a^p}\Big)^{\frac{2}{p}}\Big)^{\frac{1}{2}} \quad (= \beta_1).$$

(ii) *If* $b > a(a^p-1)^{\frac{p-2}{2p}}$, *then*

$$\beta = b\Big(1 + \Big(\frac{a}{b}\Big)^{\frac{2p}{p-2}}\Big)^{\frac{1}{2}-\frac{1}{p}} \quad (= \beta_2).$$

Proof. We show (i) and (ii). Take any $(x,y) \in \mathbb{R}^2$. We first consider the case $\|(x,y)\| = a|x|$. Since $a|x| \ge \|(x,y)\|_p$, we have $(a^p-1)^{\frac{2}{p}}x^2 \ge y^2$, from which it follows that

$$\begin{aligned}\beta_1^2\|(x,y)\|^2 - \|(x,y)\|_H^2 &= b^2\Big(\Big(1-\frac{1}{a^p}\Big)^{\frac{2}{p}}a^2x^2 - y^2\Big)\\ &= b^2\Big(\Big(a^p-1\Big)^{\frac{2}{p}}x^2 - y^2\Big) \ge 0,\end{aligned}$$

which proves $\|(x,y)\|_H \le \beta_1\|(x,y)\|$.

We next consider the case $\|(x,y)\| = b|y|$. Since $b|y| \ge \|(x,y)\|_p$, we have $(b^p - 1)^{\frac{2}{p}}y^2 \ge x^2$. From the assumption and the following identity

$$b^p\Big(1-\frac{1}{a^p}\Big) - a^p\Big(1-\frac{1}{b^p}\Big) = (a^p - b^p)\Big(\frac{1}{a^p} + \frac{1}{b^p} - 1\Big),$$

we also have

$$b\Big(1-\frac{1}{a^p}\Big)^{\frac{1}{p}} \ge a\Big(1-\frac{1}{b^p}\Big)^{\frac{1}{p}}.$$

By these two inequalities,

$$\begin{aligned}\beta_1^2\|(x,y)\|^2 - \|(x,y)\|_H^2 &= b^2\Big(1-\frac{1}{a^p}\Big)^{\frac{2}{p}}b^2y^2 - a^2x^2\\ &\ge a^2\Big(\Big(1-\frac{1}{b^p}\Big)^{\frac{2}{p}}b^2y^2 - x^2\Big)\\ &= a^2\Big((b^p-1)^{\frac{2}{p}}y^2 - x^2\Big) \ge 0,\end{aligned}$$

which proves $\|(x,y)\|_H \le \beta_1\|(x,y)\|$.

Finally we consider the case $\|(x,y)\| = \|(x,y)\|_p$. Since $\|(x,y)\|_p \ge a|x|$ and $\|(x,y)\|_p \ge b|y|$, we have $|x| \le (a^p-1)^{-\frac{1}{p}}|y|$ and $(b^p-1)^{\frac{1}{p}}|y| \le |x|$. Put $t = \dfrac{|x|}{|y|}$. Then $t_3 \le t \le t_1$, where $t_1 = (a^p-1)^{-\frac{1}{p}}$ and $t_3 = (b^p-1)^{\frac{1}{p}}$. From the assumption, we easily have $b^p < 2$, and hence $t_3 < 1$. Let us put

$$\frac{\|(x,y)\|_H}{\|(x,y)\|} = \frac{(a^2t^2+b^2)^{1/2}}{(t^p+1)^{1/p}} =: f(t).$$

We shall calculate the maximum of f on $[t_3, t_1]$. When $p = 2$, f is non-decreasing and hence has the maximum at $t = t_1$. Let $p > 2$. Since the derivative of f is

$$f'(t) = (a^2t^2+b^2)^{-1/2}(t^p+1)^{-1/p-1}t(a^2 - b^2t^{p-2}),$$

f is non-decreasing on $(0, t_2)$ and non-increasing on (t_2, ∞), where $t_2 = \left(\frac{a}{b}\right)^{\frac{2}{p-2}}$ ($\geq$ 1). If $t_1 \leq t_2$, that is, $b \leq a(a^p - 1)^{\frac{p-2}{2p}}$, then f has the maximum β_1 at $t = t_1$, and if $t_1 > t_2$ ($> t_3$), then f has the maximum β_2 at $t = t_2$. Note that $\beta_1 < \beta_2$. When $t_1 \leq t_2$, putting $x' = (t_1, 1)$ we have $\|x'\|_H = \beta_1 \|x'\|$. When $t_1 > t_2$, putting $x'' = (t_2, 1)$ we have $\|x''\|_H = \beta_2 \|x''\|$. This completes the proof. □

By Theorem 2.3 and Lemma 2.4, we obtain the main theorem.

Theorem 2.5. *Let* $a > 1, a \geq b \geq 1$ *and* $p \geq 2$ *with* $\frac{1}{a^p} + \frac{1}{b^p} > 1$.

(i) *If* $b \leq a(a^p - 1)^{\frac{p-2}{2p}}$, *then*

$$C_{\rm NJ}(\mathbb{R}^2_{a,b,p}) = 1 + b^2\Big(1 - \frac{1}{a^p}\Big)^{\frac{2}{p}}.$$

(ii) *If* $b > a(a^p - 1)^{\frac{p-2}{2p}}$, *then*

$$C_{\rm NJ}(\mathbb{R}^2_{a,b,p}) = b^2\Big(1 + \Big(\frac{a}{b}\Big)^{\frac{2p}{p-2}}\Big)^{1-\frac{2}{p}}.$$

In particular, $C_{\rm NJ}(\mathbb{R}^2_{a,b,p}) = d(\mathbb{R}^2_{a,b,p}, \ell_2^2)^2$.

Corollary 2.6. *Let* $a \geq b \geq 1$ *and* $\frac{1}{a^2} + \frac{1}{b^2} \geq 1$. *Then*

$$C_{\rm NJ}(\mathbb{R}^2_{a,b,2}) = 1 + b^2\Big(1 - \frac{1}{a^2}\Big).$$

Corollary 2.7 ([10])**.** *For* $\lambda > 1$,

$$C_{\rm NJ}(\mathbb{R}^2_\lambda) = 2 - \frac{1}{\lambda^2}.$$

Remark 2.8. (i) Let $1 \leq \lambda \leq 2^{1/p}$ and $a = b = \lambda$. Then (2.1) can be written as

$$\|(x,y)\| = \max\{\lambda\|(x,y)\|_\infty, \|(x,y)\|_p\}.$$

From Theorem 2.5,

$$C_{\rm NJ}(\mathbb{R}^2_{\lambda,\lambda,p}) = \lambda^2 2^{1-2/p}$$

(cf. Example 6 in [5]).

(ii) Takahashi [8] showed that for any Banach space X,

$$1 + \frac{\varepsilon_0(X)^2}{4} \leq C_{\rm NJ}(X),$$

where $\varepsilon_0(X) = \sup\{\varepsilon \in [0,2] : \delta_X(\varepsilon) = 0\}$ is the characteristic of convexity of X. In the case where $X = \mathbb{R}^2_{a,b,p}$, it is easy to see that

$$\varepsilon_0(\mathbb{R}^2_{a,b,p}) = 2b\Big(1 - \frac{1}{a^p}\Big)^{1/p}.$$

Thus, if $b \leq a(a^p - 1)^{\frac{p-2}{2p}}$, then

$$1 + \frac{\varepsilon_0(\mathbb{R}^2_{a,b,p})^2}{4} = C_{\rm NJ}(\mathbb{R}^2_{a,b,p}),$$

and if $b > a(a^p - 1)^{\frac{p-2}{2p}}$, then

$$1 + \frac{\varepsilon_0(\mathbb{R}^2_{a,b,p})^2}{4} < C_{\mathrm{NJ}}(\mathbb{R}^2_{a,b,p})$$

by the proof of Lemma 2.4.

Problem. Compute $C_{\mathrm{NJ}}(\mathbb{R}^2_{a,b,p})$ when $p < 2$.

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Manuscript received 29 February 2016
revised 16 June 2016

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