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APPROXIMATION OF A FIXED POINT OF GENERALIZED FIRMLY NONEXPANSIVE MAPPINGS WITH NONSUMMABLE ERRORS

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Abstract. In this paper, we study an iterative scheme for three different types of nonlinear mappings defined on a uniformly convex Banach space. These mappings are generalization of firmly nonexpansive mappings on a Hilbert space and their nonlinear structures are different from each other. We prove strong convergence of iterative schemes generated by the shrinking projection method with errors. When generating the iterative sequence, we consider an error for obtaining the value of metric projections and we prove that the sequence still has a desirable property for approximating a fixed point of the mapping.

1. Introduction

The study of firmly nonexpansive mappings is a central topic in convex analysis since it is closely related to the theory of subdifferentials of convex functions and monotone operators defined on a Banach space. This study also has a strong connection to metric fixed point theory and it has been investigated by a large number of researchers.

Let E be a real Banach space and C a nonempty closed convex subset of E. We say a mapping $T : C \to E$ is firmly nonexpansive [3] if

$$\|t(x-y) + (1-t)(Tx - Ty)\| \geq \|Tx - Ty\|$$

for every $x, y \in C$ and $t \geq 0$. If E is a Hilbert space, then one can show that T is firmly nonexpansive if and only if

$$\langle (x - Tx) - (y - Ty), Tx - Ty \rangle \geq 0$$

for every $x, y \in C$. One of the most important examples of this class of mappings in a real Hilbert space H is a resolvent operator $K_r : H \to H$ of a monotone operator $A : H \rightrightarrows H$ for $r > 0$ defined by $K_r = (I + rA)^{-1}$. Moreover, the metric projection P_C onto a nonempty closed convex subset C of H is also an example of firmly nonexpansive mappings since P_C is a resolvent operator of the subdifferential of i_C, the indicator function with respect to C.

As a generalization of the resolvent operator defined on a Hilbert space, some different types of resolvents defined on a Banach space have been proposed and studied. These notions correspond to variations of nonlinear mappings on a Banach

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space including firmly nonexpansive mappings. Following [1], we call them mappings of type (P), (Q), and (R); see the next section.

In the metric fixed point theory, approximation method of a fixed point of a nonlinear mapping is one of the most important topics and it has been rapidly developed in the recent research. In particular, the shrinking projection method proposed by Takahashi, Takeuchi, and Kubota [15] is a remarkable result.

Theorem 1.1 (Takahashi-Takeuchi-Kubota [15]). *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) = \{z \in C : z = Tz\}$ is nonempty. Let $\{\alpha_n\}$ be a sequence in $[0, a]$, where $0 < a < 1$. For a point $x \in H$ chosen arbitrarily, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_0 = C$, and*

$$\begin{aligned} &y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ &C_n = \{z \in C : \|z - y_n\| \le \|z - x_n\|\} \cap C_{n-1}, \\ &x_{n+1} = P_{C_n} x \end{aligned}$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x \in C$, where P_K is the metric projection of H onto a nonempty closed convex subset K of H.

We note that the original result of this theorem is a convergence theorem to a common fixed point of a family of nonexpansive mappings. Since they proved this theorem, a large number of researchers have proposed various types of generalized results of this method in the setting of Banach spaces; see Kimura, Nakajo, and Takahashi [10], Kimura and Takahashi [11] and references therein.

In this paper, we study an iterative scheme for three different types of nonlinear mappings defined on a uniformly convex Banach space. Each of these mappings is a generalization of firmly nonexpansive mappings on a Hilbert space, however, their nonlienar structures are different from each other.

The approximating sequences we proposed are generated by the shrinking projection method with errors. In the original shrinking projection method, we need to obtain the exact value of the metric projection to generate a sequence in every step, and it is a task of difficulty. We consider an error for obtaining the value of metric projections and prove that the sequence still has a nice property for approximating a fixed point of the mapping. Namely, even if the error sequence does not converges to 0, it is possible to estimate an upper bound of the approximate distance between the point in the sequence and its image by the mapping.

The technique we adopted in the results has been proposed in [7, 8].

2. Preliminaries

Let E be a real Banach space with its dual E^*. The normalized duality mapping $J : E \to E^*$ defined by

$$Jx = \{y^* \in E^* : \|x\|^2 = \langle x, y^* \rangle = \|y^*\|^2\}$$

for $x \in E$. If E is smooth, strictly convex and reflexive, then J is a single-valued bijection. Let C be a nonempty closed convex subset of a smooth Banach space E. A mapping $T : C \to E$ is said to be of type (P) [1] if

$$\langle Tx - Ty, J(x - Tx) - J(y - Ty) \rangle \ge 0$$

for each $x, y \in C$. A mapping $T : C \to E$ is said to be of type (Q) [1,12] if

$$\langle Tx - Ty, (Jx - JTx) - (Jy - JTy)\rangle \geq 0$$

for each $x, y \in C$. A mapping $T : C \to E$ is said to be of type (R) [1,5] if

$$\langle JTx - JTy, (x - Tx) - (y - Ty)\rangle \geq 0$$

for each $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ such that $x_n \rightharpoonup p$ and $x_n - Tx_n \to 0$. The set of all asymptotic fixed points of T is denoted by $\hat{F}(T)$. It is clear that if $T : C \to E$ is of type (P) and $F(T)$ is nonempty, then

$$\langle Tx - p, J(x - Tx)\rangle \geq 0 \tag{2.1}$$

for each $x \in C$ and $p \in F(T)$. Let E be a reflexive, smooth and strictly convex Banach space and let C be a nonempty closed convex subset of E. It is known that the metric projection P_C of E onto C is a mapping of type (P). We also know that if $T : C \to E$ is of type (Q) and $F(T)$ is nonempty, then

$$\langle Tx - p, Jx - JTx\rangle \geq 0 \tag{2.2}$$

for each $x \in C$ and $p \in F(T)$. If $T : C \to E$ is of type (R) and $F(T)$ is nonempty, then

$$\langle JTx - Jp, x - Tx\rangle \geq 0 \tag{2.3}$$

for each $x \in C$ and $p \in F(T)$.

The following results describe the relation between the set of fixed points and that of asymptotic fixed points for each type of mapping.

Lemma 2.1 (Aoyama-Kohsaka-Takahashi [2]). *Let E be a smooth Banach space, let C be a nonempty closed convex subset of E and let $T : C \to E$ be a mapping of type (P). If $F(T)$ is nonempty, then $F(T)$ is closed and convex and $F(T) = \hat{F}(T)$.*

Lemma 2.2 (Kohsaka-Takahashi [12]). *Let E be a strictly convex Banach space whose norm is uniformly Gâteaux differentiable, let C be a nonempty closed convex subset of E and let $T : C \to E$ be a mapping of type (Q). If $F(T)$ is nonempty, then $F(T)$ is closed and convex and $F(T) = \hat{F}(T)$.*

Lemma 2.3 (Takahashi-Yao [16]). *Let E be a strictly convex Banach space and E^* has a uniformly Gâteaux differentiable norm, let C be a nonempty subset of E such that JC is closed and convex and let $T : C \to E$ be a mapping of type (R). If $F(T)$ is nonempty, $F(T)$ is closed, $JF(T)$ is closed and convex and $F(T) = \check{F}(T)$, where $\check{F}(T)$ is the set of generalized asymptotic fixed points of T.*

The mappings of types (Q) and (R) are strongly related to each other; it is a kind of duality in the following sense. Let E be a reflexive, smooth and strictly convex Banach space, let C be a nonempty subset of E and, let T be a mapping form C into E. Define a mapping T^* as follows:

$$T^* x^* := JTJ^{-1}x^* \tag{2.4}$$

for each $x^* \in JC$, where J is the duality mapping on E and J^{-1} is the duality mapping on E^*. We know that $JF(T) = F(T^*)$; see [16]. Further, we have the following result.

Lemma 2.4 (Aoyama-Kohsaka-Takahashi [1]). *Let E be a reflexive, smooth and strictly convex Banach space, let C be a nonempty subset of E and let $T : C \to E$ be a mapping of type (R). Let $T^* : JC \to E^*$ be a mapping defined by* (2.4)*. Then T^* is of type (Q) in E^*.*

In 1984, Tsukada [17] proved the following theorem for the metric projections in a Banach space. For the exact definition of Mosco limit $\text{M-lim}_n C_n$, see [13].

Theorem 2.5 (Tsukada [17]). *Let E be a reflexive and strictly convex Banach space and let $\{C_n\}$ be a sequence of nonempty closed convex subsets of E. If $C_0 = \text{M-lim}_n C_n$ exists and nonempty, then for each $x \in E$, $\{P_{C_n}x\}$ converges weakly to $P_{C_0}x$, where P_{C_n} is the metric projection of E onto C_n. Moreover, if E has the Kadec-Klee property, the convergence is in the strong topology.*

Let E be a smooth Banach space and consider the following function $V : E \times E \to \mathbb{R}$ defined by

$$V(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{2.5}$$

for each $x, y \in E$. We know the following properties;

(1) $(\|x\| - \|y\|)^2 \le V(x, y) \le (\|x\| + \|y\|)^2$ for each $x, y \in E$;
(2) $V(x, y) + V(y, x) = 2\langle x - y, Jx - Jy\rangle$ for each $x, y \in E$;
(3) $V(x, y) = V(x, z) + V(z, y) + 2\langle x - z, Jz - Jy\rangle$ for each $x, y, z \in E$;
(4) if E is additionally assumed to be strictly convex, then $V(x, y) = 0$ if and only if $x = y$.

Lemma 2.6 (Kamimura-Takahashi [6]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_n V(x_n, y_n) = 0$, then $\lim_n \|x_n - y_n\| = 0$.*

The following results show that the existence of mappings $\underline{g}_r$, $\overline{g}_r$, $\underline{g}_r^*$, and $\overline{g}_r^*$, related to the convex structures of a Banach space E and its dual space. These mappings play important roles in our result.

Theorem 2.7 (Xu [18]). *Let E be a Banach space, $r \in \,]0, \infty[$ and $B_r = \{x \in E : \|x\| \le r\}$. Then,*

(i) *if E is uniformly convex, then there exists a continuous, strictly increasing and convex function $\underline{g}_r : [0, 2r] \to [0, \infty[$ with $\underline{g}_r(0) = 0$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \le \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\underline{g}_r(\|x - y\|)$$

for all $x, y \in B_r$ and $\alpha \in [0, 1]$;

(ii) *if E is uniformly smooth, then there exists a continuous, strictly increasing and convex function $\overline{g}_r : [0, 2r] \to [0, \infty[$ with $\overline{g}_r(0) = 0$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \ge \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\overline{g}_r(\|x - y\|)$$

for all $x, y \in B_r$ and $\alpha \in [0, 1]$.

From this theorem, we can show the following result; For the proof, see Kimura [9].

Theorem 2.8. *Let E be a smooth Banach space and let $r > 0$. Then,*

(i) *if E is uniformly convex, then the function $\underline{g}_r$ in Theorem 2.7 (i) satisfies*

$$\underline{g}_r(\|x-y\|) \leq V(x,y)$$

for all $x, y \in B_r$;

(ii) *if E is uniformly smooth, then the function $\overline{g}_r$ in Theorem 2.7 (ii) satisfies*

$$V(x,y) \leq \overline{g}_r(\|x-y\|)$$

for all $x, y \in B_r$.

Similar results for the mappings $\underline{g}_r^*$ and $\overline{g}_r^*$ also hold as follows:

Theorem 2.9. *Let E be a reflexive, smooth and strictly convex Banach space and let $r > 0$. Then,*

(i) *if E is uniformly smooth, then there exists a continuous, strictly increasing and convex function $\underline{g}_r^* : [0, 2r] \to [0, \infty[$ with $\underline{g}_r^*(0) = 0$ such that*

$$\underline{g}_r^*(\|Jx - Jy\|) \leq V(x,y)$$

for all $x, y \in B_r$;

(ii) *if E is uniformly convex, then there exists a continuous, strictly increasing and convex function $\overline{g}_r^* : [0, 2r] \to [0, \infty[$ with $\overline{g}_r^*(0) = 0$ such that*

$$V(x,y) \leq \overline{g}_r^*(\|Jx - Jy\|)$$

for all $x, y \in B_r$.

Proof. (i) Since E is uniformly smooth, we have that E^* is uniformly convex. From Theorem 2.7 (i), we have that for any $x^*, y^* \in B_r^*$ and $\alpha \in]0,1[$, there exists a continuous, strictly increasing and convex function $\underline{g}_r^* : [0, 2r] \to [0, \infty[$ with $\underline{g}_r^*(0) = 0$ such that

$$\|\alpha x^* + (1-\alpha) y^*\|^2 \leq \alpha \|x^*\|^2 + (1-\alpha)\|y^*\|^2 - \alpha(1-\alpha)\underline{g}_r^*(\|x^* - y^*\|). \tag{2.6}$$

For any $x, y \in B_r$, it is clear that $Jx, Jy \in B_r^*$. From (2.6) we obtain that

$$(1-\alpha)\underline{g}_r^*(\|Jy - Jx\|) \leq \|Jy\|^2 - \|Jx\|^2 - \frac{\|Jx + \alpha(Jy - Jx)\|^2 - \|Jx\|^2}{\alpha}.$$

Tending $\alpha \to 0$, we obtain that

$$\begin{aligned} \underline{g}_r^*(\|Jy - Jx\|) &\leq \|Jy\|^2 - \|Jx\|^2 - 2\langle J^{-1}Jx, Jy - Jx \rangle \\ &= \|y\|^2 - \|x\|^2 - 2\langle x, Jy - Jx \rangle = V(x,y). \end{aligned}$$

(ii) In the same way as in the proof of (i), we have the desired result by Theorem 2.7 (ii). □

3. Approximation theorem for the mappings of type (P)

In this section, we propose an approximation theorem for a mapping of type (P), which includes the metric projections onto nonempty closed convex subset of a uniformly convex Banach space.

Theorem 3.1. *Let E be a smooth and uniformly convex Banach space, let C be a nonempty bounded closed convex subset of E, and let $r \in]0, \infty[$ such that $C \subset B_r$. Let $T : C \to E$ be a mapping of type (P) such that $F(T)$ is nonempty. Let $\{\delta_n\}$ be a nonnegative real sequence and let $\delta_0 = \limsup_n \delta_n$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by $x_1 = x \in C$, $C_1 = C$, and*

$$C_{n+1} = \{z \in C : \langle Tx_n - z, J(x_n - Tx_n)\rangle \geq 0\} \cap C_n,$$
$$x_{n+1} \in \{z \in C : \|u - z\|^2 \leq d(u, C_{n+1})^2 + \delta_{n+1}\} \cap C_{n+1},$$

for all $n \in \mathbb{N}$. Then,

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \leq \underline{g}_r^{-1}(\delta_0).$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $P_{F(T)}u$.

Proof. Since C_n includes $F(T) \neq \emptyset$ for all $n \in \mathbb{N}$, $\{C_n\}$ is a seqence of nonempty closed convex subsets and, by definition, it is decreasing with respect to inclusion. Let $p_n = P_{C_n}u$ for all $n \in \mathbb{N}$. Then, by Theorem 2.5, we have that $\{p_n\}$ converges strongly to $p_0 = P_{C_0}u$, where $C_0 = \bigcap_{n=1}^{\infty} C_n$. Since $x_n \in C_n$ and $d(u, C_n) = \|u - p_n\|$, we have that

$$\|u - x_n\|^2 \leq \|u - p_n\|^2 + \delta_n$$

for every $n \in \mathbb{N} \setminus \{1\}$. From Theorem 2.7 (i), we have that for $\alpha \in]0, 1[$,

$$\begin{aligned}\|p_n - u\|^2 &\leq \|\alpha p_n + (1 - \alpha)x_n - u\|^2 \\ &\leq \alpha\|p_n - u\|^2 + (1 - \alpha)\|x_n - u\|^2 - \alpha(1 - \alpha)\underline{g}_r(\|p_n - x_n\|)\end{aligned}$$

and thus

$$\alpha \underline{g}_r(\|p_n - x_n\|) \leq \|x_n - u\|^2 - \|p_n - u\|^2 \leq \delta_n.$$

Tending $\alpha \to 1$, we have that $\underline{g}_r(\|p_n - x_n\|) \leq \delta_n$ and thus $\|p_n - x_n\| \leq \underline{g}_r^{-1}(\delta_n)$. Using the definition of p_n, we have that $p_{n+1} \in C_{n+1}$ and thus

$$\langle Tx_n - p_{n+1}, J(x_n - Tx_n)\rangle \geq 0,$$

or equivalently,

$$\langle x_n - p_{n+1}, J(x_n - Tx_n)\rangle \geq \|x_n - Tx_n\|^2.$$

Hence we obtain that

$$\|x_n - Tx_n\| \leq \|x_n - p_{n+1}\| \leq \|x_n - p_n\| + \|p_n - p_{n+1}\| \leq \underline{g}_r^{-1}(\delta_n) + \|p_n - p_{n+1}\|$$

for every $n \in \mathbb{N} \setminus \{1\}$. Since $\lim_n p_n = p_0$ and $\limsup_n \delta_n = \delta_0$, we have that

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \leq \underline{g}_r^{-1}(\delta_0).$$

For the latter part of the theorem, suppose that $\delta_0 = 0$. Then we have that

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \leq \underline{g}_r^{-1}(0) = 0$$

and

$$\limsup_{n\to\infty} \underline{g}_r(\|x_n - p_n\|) \leq \limsup_{n\to\infty} \delta_n = 0.$$

Therefore, we obtain that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0 \text{ and } \lim_{n\to\infty} \|x_n - p_n\| = 0.$$

Then, by Lemma 2.1 and since $p_n \to p_0$, we have that $x_n \to p_0 \in \hat{F}(T) = F(T)$. Since $F(T) \subset C_0$, we get that $p_0 = P_{C_0}u = P_{F(T)}u$, which completes the proof. $\square$

4. Approximation theorem for the mappings of type (Q)

We next consider an approximation theorem for a mapping of type (Q). This type of mappings includes the generalized projections onto nonempty closed convex subset of a uniformly convex Banach space.

Theorem 4.1. *Let E be a uniformly smooth and uniformly convex Banach space, let C be a nonempty bounded closed convex subset of E, and let $r \in \,]0, \infty[$ such that $C \subset B_r$. Let $T : C \to E$ be a mapping of type (Q) such that $F(T)$ is nonempty. Let $\{\delta_n\}$ be a nonnegative real sequence and let $\delta_0 = \limsup_n \delta_n$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by $x_1 = x \in C$, $C_1 = C$, and*

$$C_{n+1} = \{z \in C : \langle Tx_n - z, Jx_n - JTx_n \rangle \geq 0\} \cap C_n,$$
$$x_{n+1} \in \{z \in C : \|u - z\|^2 \leq d(u, C_{n+1})^2 + \delta_{n+1}\} \cap C_{n+1},$$

for all $n \in \mathbb{N}$. Then,

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \leq \underline{g}_r^{-1}(\overline{g}_r(\underline{g}_r^{-1}(\delta_0))).$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $P_{F(T)}u$.

Proof. Since C_n includes $F(T) \neq \emptyset$ for all $n \in \mathbb{N}$, $\{C_n\}$ is a seqence of nonempty closed convex subsets and, by definition, it is decreasing with respect to inclusion. Let $p_n = P_{C_n}u$ for all $n \in \mathbb{N}$. Then, by Theorem 2.5, we have that $\{p_n\}$ converges strongly to $p_0 = P_{C_0}u$, where $C_0 = \bigcap_{n=1}^{\infty} C_n$. Since $x_n \in C_n$ and $d(u, C_n) = \|u - p_n\|$, we have that

$$\|u - x_n\|^2 \leq \|u - p_n\|^2 + \delta_n$$

for every $n \in \mathbb{N} \setminus \{1\}$. From Theorem 2.7 (i), we have that for $\alpha \in]0, 1[$,

$$\begin{aligned}\|p_n - u\|^2 &\leq \|\alpha p_n + (1 - \alpha)x_n - u\|^2 \\ &\leq \alpha\|p_n - u\|^2 + (1 - \alpha)\|x_n - u\|^2 - \alpha(1 - \alpha)\underline{g}_r(\|p_n - x_n\|)\end{aligned}$$

and thus

$$\alpha\underline{g}_r(\|p_n - x_n\|) \leq \|x_n - u\|^2 - \|p_n - u\|^2 \leq \delta_n.$$

Tending $\alpha \to 1$, we have that $\underline{g}_r(\|p_n - x_n\|) \leq \delta_n$ and thus $\|p_n - x_n\| \leq \underline{g}_r^{-1}(\delta_n)$. Using the definition of p_n, we have that $p_{n+1} \in C_{n+1}$ and thus

$$\langle Tx_n - p_{n+1}, Jx_n - JTx_n \rangle \geq 0$$

From the property of the function V, we have that

$$\begin{aligned}0 &\leq 2\langle Tx_n - p_{n+1}, Jx_n - JTx_n \rangle \\ &= 2\langle p_{n+1} - Tx_n, JTx_n - Jx_n \rangle \\ &= V(p_{n+1}, x_n) - V(p_{n+1}, Tx_n) - V(Tx_n, x_n) \\ &\leq V(p_{n+1}, x_n) - V(Tx_n, x_n).\end{aligned}$$

By Theorem 2.8 (ii), we obtain that

$$V(Tx_n, x_n) \leq V(p_{n+1}, x_n)$$

$$
\begin{aligned}
&= V(p_{n+1}, p_n) + V(p_n, x_n) + 2\langle p_{n+1} - p_n, Jp_n - Jx_n\rangle \\
&\le V(p_{n+1}, p_n) + \overline{g}_r(\|p_n - x_n\|) + 2\langle p_{n+1} - p_n, Jp_n - Jx_n\rangle \\
&\le V(p_{n+1}, p_n) + \overline{g}_r(\underline{g}_r^{-1}(\delta_n)) + 2\langle p_{n+1} - p_n, Jp_n - Jx_n\rangle.
\end{aligned}
$$

Since $\limsup_n \delta_n = \delta_0$ and $p_n \to p_0$, we have that

$$\limsup_{n\to\infty} V(Tx_n, x_n) \le \overline{g}_r(\underline{g}_r^{-1}(\delta_0)).$$

Therefore, by Theorem 2.8 (i), we have that

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \le \limsup_{n\to\infty} \underline{g}_r^{-1}(V(Tx_n, x_n)) \le \underline{g}_r^{-1}(\overline{g}_r(\underline{g}_r^{-1}(\delta_0)))$$

For the latter part of the theorem, suppose that $\delta_0 = 0$. Then we have that

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \le \underline{g}_r^{-1}(\overline{g}_r(\underline{g}_r^{-1}(0))) = 0$$

and

$$\limsup_{n\to\infty} \underline{g}_r(\|x_n - p_n\|) \le \limsup_{n\to\infty} \delta_n = 0.$$

Therefore, we obtain that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0 \text{ and } \lim_{n\to\infty} \|x_n - p_n\| = 0.$$

Then, by Lemma 2.2 and $p_n \to p_0$, we have that $x_n \to p_0 \in \hat{F}(T) = F(T)$. Since $F(T) \subset C_0$, we get that $p_0 = P_{C_0}u = P_{F(T)}u$, which completes the proof. □

5. Approximation theorem for the mappings of type (R)

The mappings of type (R) is, in a sense, the dual of the mappings of type (Q). By using this fact, we obtain the following an approximation theorem for this mapping.

Theorem 5.1. *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty bounded of E with JC is closed and convex and $r \in]0, \infty[$ such that $C \subset B_r$. Let $T : C \to E$ be a mapping of type (R) such that $F(T)$ is nonempty. Let $\{\delta_n\}$ be a nonnegative real sequence and let $\delta_0 = \limsup_n \delta_n$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by $x_1 = x \in C$, $C_1 = C$, and*

$$
\begin{aligned}
&C_{n+1} = \{z \in C : \langle JTx_n - Jz, x_n - Tx_n\rangle \ge 0\} \cap C_n, \\
&x_{n+1} \in \{z \in C : \|Ju - Jz\|^2 \le d(Ju, JC_{n+1})^2 + \delta_{n+1}\} \cap C_{n+1},
\end{aligned}
$$

for all $n \in \mathbb{N}$. Then,

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \le \underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0))))).$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $J^{-1}P^*_{JF(T)}Ju$ where $P^*_{JF(T)}$ is the metric projection of E^* onto $JF(T)$.

Proof. From Lemma 2.4, we have that $T^* : JC \to E^*$ is of type (Q) in E^* with $F(T^*) \ne \emptyset$, where T^* is defined by (2.4). Put $x_n^* = Jx_n$ and $C_n^* = JC_n$ for each $n \in \mathbb{N}$. Then T^* and $\{x_n^*\}$ satisfy the conditions of Theorem 4.1 in E^*. Theorefore, we obtain that

$$\limsup_{n\to\infty} \|x_n^* - T^*x_n^*\| \le \underline{g}_r^{*-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0))) \tag{5.1}$$

where the function $\underline{g}_r^*$ and $\overline{g}_r^*$ in Theorem 2.9. Moreover, if $\delta_0 = 0$, then $\{x_n^*\}$ converge strongly to $P^*_{F(T^*)}Ju$ where $P^*_{F(T^*)}$ is the metric projection of E^* onto $F(T^*) = JF(T)$. From Theorem 2.8 (i) and 2.9 (ii), we have that

$$\underline{g}_r(\|Tx_n - x_n\|) \leq V(Tx_n, x_n) \leq \overline{g}_r^*(\|Jx_n - JTx_n\|). \tag{5.2}$$

From (5.1) and (5.2), we obtain that

$$\begin{aligned}\limsup_{n\to\infty} \|Tx_n - x_n\| &\leq \limsup_{n\to\infty} \underline{g}_r^{-1}(\overline{g}_r^*(\|Jx_n - JTx_n\|)) \\ &\leq \underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0)))))\end{aligned}$$

Finally, we show that $\{x_n\}$ converges strongly to $J^{-1}P^*_{F(T^*)}Ju$. Since E is uniformly smooth and uniformly convex, we obtain that the duality mapping J^{-1} on E^* is continuous and $x_n = J^{-1}x_n^*$ for each $n \in \mathbb{N}$. Since $x_n^* \to P^*_{F(T^*)}Ju$, we have that

$$x_n = J^{-1}x_n^* \to J^{-1}P^*_{F(T^*)}Ju$$

This completes the proof. □

6. Deduced results

In the case where E is a Hilbert space, the functions $\underline{g}_r$, $\overline{g}_r$, $\underline{g}_r^*$ and $\overline{g}_r^*$ become $\underline{g}_r = \overline{g}_r = \underline{g}_r^* = \overline{g}_r^* = |\cdot|^2$ for every $r \in]0, \infty[$. Therefore, as a direct consequence of Theorems 3.1, 4.1 and 5.1, we obtain the following result.

Corollary 6.1. *Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $T : C \to E$ be a firmly nonexpansive mapping such that $F(T)$ is nonempty. Let $\{\delta_n\}$ be a bounded nonnegative real sequence and let $\delta_0 = \limsup_n \delta_n$. For a given point $u \in H$, generate a sequence $\{x_n\}$ by $x_1 = x \in C$, $C_1 = C$, and*

$$\begin{aligned}C_{n+1} &= \{z \in C : \langle Tx_n - z, x_n - Tx_n\rangle \geq 0\} \cap C_n, \\ x_{n+1} &\in \{z \in C : \|u - z\|^2 \leq d(u, C_{n+1})^2 + \delta_{n+1}\} \cap C_{n+1},\end{aligned}$$

for all $n \in \mathbb{N}$. Then,

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \leq \sqrt{\delta_0}.$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $P_{F(T)}u$.

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