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THE MONOTONE CONVERGENCE THEOREMS FOR NONLINEAR INTEGRALS ON A TOPOLOGICAL SPACE

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Abstract. The purpose of the paper is to investigate the monotone convergence theorems for the Choquet, the Šipoš, the Sugeno, and the Shilkret integrals with respect to totally o-continuous and c-continuous nonadditive measures on a topological space. This is accomplished by the efficient use of the perturbation of integral functional and a unified approach to limit theorems for nonlinear integrals.

1. Introduction

A nonadditive measure is a set function that is monotonic and vanishes at the empty set. This type of set function, in conjunction with several types of nonlinear integrals such as the Choquet, the Šipoš, the Sugeno, and the Shilkret integrals, has been extensively studied with applications to decision theory under uncertainty, game theory, data mining, some economic topics under Knightian uncertainty and others [2–4, 12, 14–16, 20, 21, 23, 25, 26, 28].

In practical problems, we often consider nonlinear integrals of a function f with respect to a nonadditive measure μ, both of which are defined on a space X with some topology. In this case the monotone convergence theorem for the Choquet integral has its net version valid for a uniformly bounded, increasing (decreasing) net $\{f_\alpha\}_{\alpha\in\Gamma}$ of lower (upper) semicontinuous functions on X with pointwise limit f if a finite nonadditive measure μ is totally o-continuous (totally c-continuous) [6, Theorem 5]. The purpose of the paper is to investigate this type of net versions of the monotone convergence theorem for other nonlinear integrals such as the Šipoš, the Sugeno, and the Shilkret integrals. This is accomplished by an efficient use of the perturbation of integral functional and the unified approach to limit theorems for nonlinear integrals, which are developed in [8–10]. It is worth mentioning that our approach is also applicable to the abstract Lebesgue integral when the nonadditive measure μ is σ-additive.

The paper is organized as follows. In Section 2 we recall some definitions on nonadditive measures, nonlinear integrals, and the uniform μ-essential (symmetric) boundedness of a family of measurable functions for a nonadditive measure μ. In Sections 3 and 4 we introduce some classes of integral functionals, in particular, the

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class of perturbative functionals and recall that the Choquet, the Šipoš, the Sugeno, and the Shilkret integrals belong to those classes. In Section 5 we discuss some net versions of the monotone convergence theorem for nonlinear integrals with respect to totally o-continuous and c-continuous nonadditive measures. Section 6 considers their extensions to symmetric and asymmetric integrals and Section 7 gives some examples of nonadditive measures with total continuities.

2. Preliminaries

In this paper, unless stated otherwise, X is a non-empty set and $\mathcal{A}$ is a field of subsets of X. Let $\mathbb{R}$ and $\mathbb{N}$ denote the set of all real numbers and the set of all natural numbers. Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ with usual total order. For any $a, b \in \overline{\mathbb{R}}$, let $a \vee b := \max(a, b)$ and $a \wedge b := \min(a, b)$. For any functions $f, g \colon X \to \overline{\mathbb{R}}$, let $(f \vee g)(x) := f(x) \vee g(x)$ and $(f \wedge g)(x) := f(x) \wedge g(x)$ for every $x \in X$. If $A \subset \mathbb{R}$ is non-empty and not bounded from above (below) in $\mathbb{R}$, let $\sup A := \infty$ ($\inf A := -\infty$). With this convention, every non-empty subset of $\mathbb{R}$ has a supremum and an infimum in $\overline{\mathbb{R}}$. We adopt the usual conventions for algebraic operations on $\overline{\mathbb{R}}$. We also adopt the convention $(\pm\infty) \cdot 0 = 0 \cdot (\pm\infty) = 0$ and $\inf \emptyset = \infty$. If a positive number c may take ∞, we explicitly write $c \in (0, \infty]$ instead of an ambiguous expression $c > 0$. In other words, $c > 0$ always means $c \in (0, \infty)$. This notational convention will be used for similar cases.

Let χ_A denote the characteristic function of a set A, that is, $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise. A function $f \colon X \to \overline{\mathbb{R}}$ is called *$\mathcal{A}$-measurable* if $\{f \geq t\} := \{x \in X \colon f(x) \geq t\} \in \mathcal{A}$ and $\{f > t\} := \{x \in X \colon f(x) > t\} \in \mathcal{A}$ for every $t \in \overline{\mathbb{R}}$. Any constant function and the characteristic function χ_A of any set $A \in \mathcal{A}$ are $\mathcal{A}$-measurable. If f and g are $\mathcal{A}$-measurable and $c \in \mathbb{R}$, then so are $f^+ := f \vee 0$, $f^- := (-f) \vee 0$, $|f| := f \vee (-f)$, cf, $f + c$, $(f - c)^+$, $f \vee g$, and $f \wedge g$. Note that $f = f \wedge c + (f - c)^+$. Let $\mathcal{F}(X)$ denote the set of all $\mathcal{A}$-measurable functions on X. For every $f \in \mathcal{F}(X)$, let $\|f\| := \sup_{x \in X} |f(x)|$. Then $\|f\| < \infty$ if and only if f is bounded. Let $\mathcal{F}_b(X) := \{f \in \mathcal{F}(X) \colon \|f\| < \infty\}$. For any $\mathcal{F} \subset \mathcal{F}(X)$, let $\mathcal{F}^+$ always denote its positive cone, that is, $\mathcal{F}^+ := \{f \in \mathcal{F} \colon f \geq 0\}$.

A *simple* function is a function whose range space is a finite subset of $\mathbb{R}$. Let $\mathcal{S}(X)$ denote the set of all $\mathcal{A}$-measurable simple functions on X. Every $f \in \mathcal{S}^+(X)$ is represented by

$$f = \sum_{i=1}^{n} (r_i - r_{i-1}) \chi_{\{f \geq r_i\}} = \bigvee_{i=1}^{n} r_i \chi_{\{f \geq r_i\}} = \sum_{i=1}^{n-1} r_i \chi_{\{r_{i-1} \leq f < r_i\}} + r_n \chi_{\{f \geq r_n\}}$$

for some $0 = r_0 < r_1 < r_2 < \cdots < r_n < \infty$. It this case, $f(X) \setminus \{0\} = \{r_1, r_2, \ldots, r_n\}$. Every $f \in \mathcal{F}^+(X)$ is the pointwise limit of an increasing sequence $\{h_n\}_{n \in \mathbb{N}} \subset \mathcal{S}^+(X)$. Throughout this paper, all functions are supposed to be $\mathcal{A}$-measurable.

2.1. **Nonadditive measures.** A *nonadditive measure* is an extended real-valued set function $\mu \colon \mathcal{A} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and $\mu(A) \leq \mu(B)$ whenever $A, B \in \mathcal{A}$ and $A \subset B$. It is called *finite* if $\mu(X) < \infty$. This type of set function is also called a monotone measure [25], a capacity [2], or a fuzzy measure [17, 23] in the literature. Let $\mathcal{M}(X)$ denote the set of all nonadditive measures $\mu \colon \mathcal{A} \to [0, \infty]$

and let $\mathcal{M}_b(X) := \{\mu \in \mathcal{M}(X)\colon \mu(X) < \infty\}$. For any $\mu \in \mathcal{M}_b(X)$, its *dual* $\bar{\mu} \in \mathcal{M}_b(X)$ is defined by $\bar{\mu}(A) := \mu(X) - \mu(A^c)$ for every $A \in \mathcal{A}$, where A^c denotes the complement of A. It is obvious that $\bar{\bar{\mu}} = \mu$. If μ is finitely additive, then $\mu = \bar{\mu}$. See [3, 15, 25] for further information on nonadditive measures.

2.2. **Nonlinear integrals.** The Choquet integral [2, 18], the Šipoš integral [21], the Sugeno integral [17, 23], and the Shilkret integral [20, 28] are typical nonlinear integrals and widely used in nonadditive measure theory.

Definition 2.1. Let $(\mu, f) \in \mathcal{M}(X) \times \mathcal{F}^+(X)$.

(1) The *Choquet integral* is defined by

$$\mathrm{Ch}(\mu, f) := \int_0^\infty \mu(\{f \geq t\})dt,$$

where the integral of the right hand side is the Lebesgue integral.

(2) Let $\varDelta^+$ denote the directed set of all partitions of $[0, \infty]$ of the form $P = \{a_1, a_2, \ldots, a_n\}$, where $0 < a_1 < a_2 < \cdots < a_n < \infty$, with partial order given by the usual set inclusion. Let

$$S_P(\mu, f) := \sum_{i=1}^n (a_i - a_{i-1})\mu(\{f \geq a_i\})$$

for $P = \{a_1, a_2, \ldots, a_n\}$, where $a_0 := 0$. The *Šipoš integral* is defined by the limit of the net $\{S_P(\mu, f)\}_{P\in\varDelta^+}$, that is,

$$\mathrm{Si}(\mu, f) := \lim_{P\in\varDelta^+} S_P(\mu, f).$$

(3) The *Sugeno integral* is defined by

$$\mathrm{Su}(\mu, f) := \sup_{t\in[0,\infty]} \big[t \wedge \mu(\{f \geq t\})\big].$$

(4) The *Shilkret integral* is defined by

$$\mathrm{Sh}(\mu, f) := \sup_{t\in[0,\infty]} \big[t \cdot \mu(\{f \geq t\})\big].$$

Remark 2.2. (1) In the definitions of the above integrals, the μ-distribution function $\mu(\{f \geq t\})$ can be replaced with $\mu(\{f > t\})$ without any change. Furthermore, in the definitions of the Sugeno and the Shilkret integrals the closed interval $[0, \infty]$ can be also replaced with the open interval $(0, \infty)$.

(2) It is well known that $\mathrm{Ch}(\mu, f) = \mathrm{Si}(\mu, f)$ for every $(\mu, f) \in \mathcal{M}(X)\times\mathcal{F}^+(X)$ [22, Remark], and they are equal to the abstract Lebesgue integral if μ is σ-additive and $\mathcal{A}$ is a σ-field [21, Corollary 18]; see also [10, Propositions 8.1 and 8.2].

Definition 2.3. Let $\mu \in \mathcal{M}(X)$. Let $\mathcal{F}$ be a non-empty subset of $\mathcal{F}(X)$ and $f \in \mathcal{F}^+(X)$.

(1) The function f is called *μ-essentially bounded* if there is $M > 0$ such that $\mu(\{f \geq M\}) = 0$ and $\mu(\{f \geq -M\} = \mu(X)$ and the family $\mathcal{F}$ is called *uniformly μ-essentially bounded* if there is $M > 0$ such that $\mu(\{f \geq M\}) = 0$ and $\mu(\{f \geq -M\} = \mu(X)$ for all $f \in \mathcal{F}$.

(2) The function f is called *μ-essentially symmetric bounded* if there is $M > 0$ such that $\mu(\{f \geq M\}) = \mu(\{f \leq -M\} = 0$ and the family $\mathcal{F}$ is called *uniformly μ-essentially symmetric bounded* if there is $M > 0$ such that $\mu(\{f \geq M\}) = \mu(\{f \leq -M\} = 0$ for all $f \in \mathcal{F}$.

Remark 2.4. The notion of μ-essential symmetric boundedness in this paper slightly differs from that of [8, Definition 2.1]. Although both notions coincide if the functions are nonnegative or if μ is null-additive, that is, $\mu(A \cup B) = \mu(A)$ whenever $A, B \in \mathcal{A}$ and $\mu(B) = 0$, from now on we will distinguish them and say that f is μ-essentially absolute bounded if there is $M > 0$ such that $\mu(\{|f| \geq M\}) = 0$, which was the definition of the μ-essential boundedness in [8].

Let $\mathcal{F}_{\mu,b}(X)$ and $\mathcal{F}_{\mu,sb}(X)$ denote the set of all $f \in \mathcal{F}(X)$ that are μ-essentially bounded and μ-essentially symmetric bounded, respectively. Obviously, $\mathcal{F}^+_{\mu,b}(X) = \mathcal{F}^+_{\mu,sb}(X)$, that is, the notions of μ-essential boundedness and μ-essential symmetric boundedness coincide for nonnegative functions. If μ is finitely additive and $\mu(X) < \infty$, then $\mathcal{F}_{\mu,b}(X) = \mathcal{F}_{\mu,sb}(X)$. The following simple example illustrates that both notions are independent of each other in general.

Example 2.5. Let $X := [0, 1]$ and $\mathcal{A}$ the σ-field of all Lebesgue subsets of X. Let $f\colon X \to [-\infty, \infty]$ be the $\mathcal{A}$-measurable function defined by

$$f(x) := \begin{cases} -\infty & \text{if } x \in [0, 1/2], \\ 0 & \text{if } x \in (1/2, 1]. \end{cases}$$

(1) Let $\mu\colon \mathcal{A} \to [0, \infty]$ be the nonadditive measure defined by

$$\mu(A) := \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{if } A \neq \emptyset. \end{cases}$$

Then $f \in \mathcal{F}_{\mu,b}(X)$ but $f \notin \mathcal{F}_{\mu,sb}(X)$.

(2) Let $\mu\colon \mathcal{A} \to [0, \infty]$ the nonadditive measure defined by

$$\mu(A) := \begin{cases} 0 & \text{if } A \neq X, \\ 1 & \text{if } A = X. \end{cases}$$

Then $f \notin \mathcal{F}_{\mu,b}(X)$ but $f \in \mathcal{F}_{\mu,sb}(X)$.

For any $f \in \mathcal{F}(X)$, let

$$\|f\|_\mu := \inf\{M > 0\colon \mu(\{f \geq M\}) = 0 \text{ and } \mu(\{f \geq -M\} = \mu(X)\}.$$

Then f is μ-essentially bounded if and only if $\|f\|_\mu < \infty$. It always holds that $\mathcal{F}_b(X) \subset \mathcal{F}_{\mu,b}(X) \cap \mathcal{F}_{\mu,sb}(X)$ and $\|f\|_\mu \leq \|f\|$. Let us collect some basic properties of essentially (symmetric) bounded functions, which can be proved directly from Definition 2.3.

Proposition 2.6. *Let $\mu \in \mathcal{M}(X)$. Let $\mathcal{F}$ be a non-empty subset of $\mathcal{F}(X)$ and $f \in \mathcal{F}(X)$.*

(1) *The function f is μ-essentially symmetric bounded if and only if f^+ and f^- are both μ-essentially bounded. Moreover, the family $\mathcal{F}$ is uniformly μ-essentially symmetric bounded if and only if $\{f^+ \colon f \in \mathcal{F}\}$ and $\{f^- \colon f \in \mathcal{F}\}$ are both uniformly μ-essentially bounded.*

(2) *Assume that $\mu \in \mathcal{M}_b(X)$. The function f is μ-essentially bounded if and only if f^+ is μ-essentially bounded and f^- is $\bar{\mu}$-essentially bounded. Moreover, the family $\mathcal{F}$ is uniformly μ-essentially bounded if and only if $\{f^+ \colon f \in \mathcal{F}\}$ is uniformly μ-essentially bounded and $\{f^- \colon f \in \mathcal{F}\}$ is uniformly $\bar{\mu}$-essentially bounded.*

3. Integral functionals

When discussing nonlinear integrals in a unified way, two binary operations $\oplus$ and $\ominus$, which are generalizations of the usual addition and subtraction, are useful. A binary operation $\oplus : [0,\infty]^2 \to [0,\infty]$ is called a *pseudo-addition* if it is associative, increasing in both coordinates, continuous, and 0 is its neutral element [24, 25, 27]. Every pseudo-addition $\oplus$ is commutative and defines its *pseudo-difference* $\ominus \colon [0,\infty]^2 \to [0,\infty]$ by $a \ominus b := \inf\{x \in [0,\infty] : b \oplus x \geq a\}$ for every $a, b \in [0,\infty]$ [1]. For every $a, b, c \in [0,\infty]$ with $a \leq b \leq c$, it holds that $a \oplus (b \ominus a) = b$ and $(c \ominus b) \oplus (b \ominus a) = c \ominus a$. Thus, by the associativity of $\oplus$, for any $n \geq 2$, any $0 = a_0 < a_1 < \cdots < a_n \leq \infty$, and any $i \in \{1, \ldots, n-1\}$ and $j \in \{1, \ldots, n-i\}$, the following computation rule holds [11]:

- $a_{i-1} \oplus (a_i \ominus a_{i-1}) \oplus (a_{i+1} \ominus a_i) \oplus \cdots \oplus (a_{i+j} \ominus a_{i+j-1}) = a_{i+j}$,
- $(a_i \ominus a_{i-1}) \oplus (a_{i+1} \ominus a_i) \oplus \cdots \oplus (a_{i+j} \ominus a_{i+j-1}) = a_{i+j} \ominus a_{i-1}$.

For every $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in [0,\infty]$ let $\bigoplus_{i=1}^n a_i := a_1 \oplus a_2 \oplus \cdots \oplus a_n$.

Remark 3.1. Pseudo-addition has also been referred to as pan-addition in the literature. In [27], the pan-integral was introduced by pan-operations and some basic properties of the pan-integral were studied.

Example 3.2. (1) $a \oplus b := g^{-1}(g(a) + g(b))$, where $g : [0,\infty] \to [0,\infty]$ is an increasing bijection, is a pseudo-addition. Its pseudo-difference is given by

$$a \ominus b = \begin{cases} g^{-1}(g(a) - g(b)) & \text{if } a > b, \\ 0 & \text{if } a \leq b. \end{cases}$$

In particular, if $a \oplus b := a + b$ and $a > b$, then $a \ominus b = a - b$.

(2) $a \oplus b := a \vee b$ is a pseudo-addition. Its pseudo-difference is given by

$$a \ominus b = \begin{cases} a & \text{if } a > b, \\ 0 & \text{if } a \leq b. \end{cases}$$

Every $f \in \mathcal{S}^+(X)$ with $f(X) \setminus \{0\} = \{r_1, r_2, \ldots, r_n\}$ has a unique *standard $\oplus$-step representation* [1]

$$f = \bigoplus_{i=1}^n (r_i \ominus r_{i-1}) \chi_{\{f \geq r_i\}},$$

where $n \in \mathbb{N}$ and $0 = r_0 < r_1 < \cdots < r_n < \infty$. In particular, f is expressed by $f = \sum_{i=1}^n (r_i - r_{i-1}) \chi_{\{f \geq r_i\}}$ if $\oplus = +$ and by $f = \bigvee_{i=1}^n r_i \chi_{\{f \geq r_i\}}$ if $\oplus = \vee$.

Now we introduce some classes of functionals containing the Lebesgue, the Choquet, the Šipoš, the Sugeno, and the Shilkret integrals as their special cases.

Definition 3.3. Let $I\colon \mathcal{M}(X) \times \mathcal{F}^+(X) \to [0,\infty]$ be a functional.

(1) I is called an *integral* if it satisfies the following conditions:
   (i) $I(\mu, 0) = I(0, f) = 0$ for every $\mu \in \mathcal{M}(X)$ and $f \in \mathcal{F}^+(X)$.
   (ii) I is *jointly monotone*, that is, $I(\mu, f) \le I(\nu, g)$ for every $\mu, \nu \in \mathcal{M}(X)$ with $\mu \le \nu$ and $f, g \in \mathcal{F}^+(X)$ with $f \le g$.

(2) I is called *generative* if there is a function $\theta\colon [0,\infty]^2 \to [0,\infty]$ such that $I(\mu, r\chi_A) = \theta(r, \mu(A))$ for every $\mu \in \mathcal{M}(X)$, $A \in \mathcal{A}$, and $r \in [0,\infty]$. The function θ is called a *generator* of I.

(3) I is called *elementary* if it is generative with generator θ and there is a pseudo-addition $\oplus$ such that

$$I\left(\mu, \bigoplus_{i=1}^{n} (r_i \ominus r_{i-1})\chi_{A_i}\right) = \bigoplus_{i=1}^{n} \theta\left(r_i \ominus r_{i-1}, \mu(A_i)\right)$$

for every $\mu \in \mathcal{M}(X)$, $n \in \mathbb{N}$, $A_1, A_2, \dots, A_n \in \mathcal{A}$, and $r_1, r_2, \dots, r_n \in (0,\infty)$ with $A_1 \supset A_2 \supset \cdots \supset A_n$ and $0 = r_0 < r_1 < r_2 < \cdots < r_n$.

(4) I is called *measure-truncated* if $I(\mu, f) = \sup_{s>0} I(\mu \wedge s, f)$ for every $\mu \in \mathcal{M}(X)$ and $f \in \mathcal{F}^+(X)$.

If I is generative, then its generator θ has the property that $\theta(a, 0) = \theta(0, b) = 0$ for every $a, b \in [0,\infty]$. A generator $\theta\colon [0,\infty]^2 \to [0,\infty]$ is called *limit preserving* if $b_\alpha \to b$ whenever $\{b_\alpha\}_{\alpha \in \Gamma} \subset [0,\infty]$ is a net, $b \in [0,\infty]$, and $\theta(r, b_\alpha) \to \theta(r, b)$ for every $r \in (0,\infty)$. In addition, it is called *of finite type* if $\theta(a, b) < \infty$ whenever $a, b \in [0,\infty)$ and *of continuous type* if it is continuous on the set $D := [0,\infty]^2 \setminus \{(0,\infty), (\infty, 0)\}$. Among others, $\theta(a, b) := a \cdot b$ and $\theta(a, b) := a \wedge b$ are typical limit-preserving generators of finite and continuous type. The Lebesgue integral is obviously elementary with generator $\theta(a, b) := a \cdot b$ with respect to the usual addition $a \oplus b := a + b$. The following are well-known and easily follow from the definition of integral; see [8, Propositions 2.5, 2.6, and 2.7] for $I = \mathrm{Ch}, \mathrm{Su}, \mathrm{Sh}$ and [21, Theorem 5, Corollary 15] for $I = \mathrm{Si}$.

Proposition 3.4. *Let $\mu \in \mathcal{M}(X)$.*

(1) *The integral functionals* Ch *and* Si *are elementary and measure-truncated with generator $\theta(a, b) := a \cdot b$ with respect to $a \oplus b := a + b$.*
(2) *The integral functional* Su *is elementary and measure-truncated with generator $\theta(a, b) := a \wedge b$ with respect to $a \oplus b := a \vee b$.*
(3) *The integral functional* Sh *is elementary and measure-truncated with generator $\theta(a, b) := a \cdot b$ with respect to $a \oplus b := a \vee b$.*

4. Perturbation of integral functionals

In this section we introduce the notion of perturbation of functional, that is, a key tool when formulating in a unified way limit theorems for nonlinear integrals. To this end we need the following notions [8–10].

Definition 4.1. Let $\mu, \nu\colon \mathcal{A} \to [0,\infty]$ be set functions and $f, g \in \mathcal{F}(X)$. We say that the pair (μ, f) is *dominated* by (ν, g) and write $(\mu, f) \prec (\nu, g)$ if $\mu(\{f \geq t\}) \leq \nu(\{g \geq t\})$ for every $t \in \mathbb{R}$. In particular, we say that f is *μ-dominated* by g and write $f \prec_\mu g$ if $(\mu, f) \prec (\mu, g)$.

In what follows, let Φ denote the set of all functions $\varphi\colon [0,\infty) \to [0,\infty)$ satisfying $\varphi(0) = \lim_{t\to+0} \varphi(t) = 0$. A function belonging to Φ is called a *control function.* Recall that $\|f\|_\mu = \inf\{M > 0\colon \mu(\{f \geq M\}) = 0\}$ for every $f \in \mathcal{F}^+(X)$, which is nothing but the usual μ-essential norm of f.

Definition 4.2. Let $I\colon \mathcal{M}(X) \times \mathcal{F}^+(X) \to [0,\infty]$ be a functional.

(1) I is called *strongly monotone* (*s-monotone* for short) if $I(\mu, f) \leq I(\mu, g)$ whenever $\mu \in \mathcal{M}(X)$, $f, g \in \mathcal{F}^+(X)$, and $f \prec_\mu g$.

(2) I is called *perturbative* if, for each $p, q > 0$, there are control functions $\varphi_{p,q}, \psi_{p,q} \in \Phi$ satisfying the following *perturbation*: for any $\mu \in \mathcal{M}(X)$, $f, g \in \mathcal{F}^+(X)$, $\varepsilon \geq 0$, and $\delta \geq 0$, it holds that
$$I(\mu, f) \leq I(\mu, g) + \varphi_{p,q}(\delta) + \psi_{p,q}(\varepsilon)$$
whenever $\|f\|_\mu < p$, $\|g\|_\mu < p$, $\mu(X) < q$, and $(\mu, f) \prec (\mu + \delta, g + \varepsilon)$.

Every strongly monotone functional I is monotone with respect to functions, that is, $I(\mu, f) \leq I(\mu, g)$ whenever $\mu \in \mathcal{M}(X)$, $f, g \in \mathcal{F}^+(X)$ and $f \leq g$. The perturbation of functional manages not only the monotonicity of the functional but also the small change of the functional value $I(\mu, f)$ caused by adding small amounts δ and ε to the measure μ and the function f, respectively.

Remark 4.3. Assume that a functional $I\colon \mathcal{M}(X) \times \mathcal{F}^+(X) \to [0,\infty]$ is *jointly s-monotone*, that is, $I(\mu, f) \leq I(\nu, g)$ if $(\mu, f), (\nu, g) \in \mathcal{M}(X) \times \mathcal{F}^+(X)$ and $(\mu, f) \prec (\nu, g)$. Then the value $I(\mu, f)$ is uniquely determined by the decreasing μ-distribution function $\mu(\{f \geq t\})$, that is, $I(\mu, f) = I(\nu, g)$ whenever $(\mu, f), (\nu, g) \in \mathcal{M}(X) \times \mathcal{F}^+(X)$ and $\mu(\{f \geq t\}) = \nu(\{g \geq t\})$ for every $t \in \mathbb{R}$. Every jointly s-monotone functional is jointly monotone. A jointly s-monotone, generative functional is called a universal integral in [11].

The Lebesgue integral is obviously strongly monotone and perturbative and it is the case with nonlinear integrals as the following proposition shows.

Proposition 4.4. *The integral functionals* Ch, Si, Su, *and* Sh *are all strongly monotone and perturbative. In fact they are all jointly s-monotone.*

Proof. See [10, Proposition 4.4]. □

Although the perturbation of functional does not imply the strong monotonicity, every perturbative functional has the following *weak type of strong monotonicity*:

Proposition 4.5. *Let $I\colon \mathcal{M}(X) \times \mathcal{F}^+(X) \to [0,\infty]$ be a functional. If I is perturbative, then $I(\mu, f) \leq I(\mu, g)$ whenever $\mu \in \mathcal{M}_b(X)$, $f, g \in \mathcal{F}^+_{\mu,b}(X)$ and $f \prec_\mu g$.*

Proof. Let $\mu \in \mathcal{M}_b(X)$ and $f, g \in \mathcal{F}^+_{\mu,b}(X)$ with $f \prec_\mu g$. Let $p := \|f\|_\mu \vee \|g\|_\mu + 1 > 0$ and $q := \mu(X) + 1 > 0$. Since I is perturbative, there are control functions $\varphi := \varphi_{p,q}, \psi := \psi_{p,q} \in \Phi$ such that $I(\mu, f) \leq I(\mu, g) + \varphi(0) + \psi(0)$. Thus $I(\mu, f) \leq I(\mu, g)$ since $\varphi(0) = \psi(0) = 0$. □

Proposition 4.6. *Let* $I\colon \mathcal{M}(X)\times\mathcal{F}^+(X)\to[0,\infty]$ *be a generative functional with generator* θ*.*

(1) *If* I *is strongly monotone, then* $I(\mu,f)\le\theta(\|f\|_\mu,\mu(\{f>0\}))$ *for any* $(\mu,f)\in\mathcal{M}(X)\times\mathcal{F}^+(X)$*.*

(2) *If* I *is perturbative, then the same inequality as above holds for any* $\mu\in\mathcal{M}_b(X)$ *and* $f\in\mathcal{F}^+_{\mu,b}(X)$*.*

Thus $I(\mu,f)<\infty$ *for* $I=\mathrm{Ch},\mathrm{Si},\mathrm{Sh}$ *if* μ *is finite and* f *is* μ*-essentially bounded. By contrast,* $\mathrm{Su}(\mu,f)<\infty$ *if* μ *is finite or* f *is* μ*-essentially bounded.*

Proof. (1) Since $f\prec_\mu\|f\|_\mu\cdot\chi_{\{f>0\}}$, the strong monotonicity and the generative property of I imply

$$I(\mu,f)\le I(\mu,\|f\|_\mu\cdot\chi_{\{f>0\}})=\theta(\|f\|_\mu,\mu(\{f>0\})).$$

Thus the most right-hand of the above formula is $\|f\|_\mu\cdot\mu(\{f>0\})$ for $I=\mathrm{Ch},\mathrm{Si},\mathrm{Su}$ and $\|f\|_\mu\wedge\mu(\{f>0\})$ for $I=\mathrm{Su}$. The former is finite if μ is finite and f is μ-essentially bounded and so is the latter if μ is finite or f is μ-essentially bounded.

(2) It can be prove in the same way as (1) since every perturbative functional has a weak type of strong monotonicity by Proposition 4.5. □

5. The monotone convergence theorems

In this section we investigate the monotone convergence theorems for nonlinear integrals on a topological space. In the rest of the paper, X is a Hausdorff space, $\mathcal{A}$ is a field containing all open subsets of X, $\mathcal{F}(X)$ is the set of all $\mathcal{A}$-measurable functions $f\colon X\to\overline{\mathbb{R}}$, and $\mathcal{M}(X)$ is the set of all nonadditive measures $\mu\colon\mathcal{A}\to[0,\infty]$. Recall that a function $f\colon X\to\overline{\mathbb{R}}$ is lower semicontinuous if $\{f>r\}$ is open for any $r\in\overline{\mathbb{R}}$ and upper semicontinuous if $\{f\ge r\}$ is closed for any $r\in\overline{\mathbb{R}}$. If $\{f_\alpha\}_{\alpha\in\Gamma}$ is an increasing net of lower semicontinuous functions on X with pointwise limit f, then f is lower semicontinuous. Similarly, if $\{f_\alpha\}_{\alpha\in\Gamma}$ is a decreasing net of upper semicontinuous functions on X with pointwise limit f, then f is upper semicontinuous. Every lower or upper semicontinuous function on X is $\mathcal{A}$-measurable if $\mathcal{A}$ is a σ-field, but this is not the case when $\mathcal{A}$ is a field.

Definition 5.1. Let $\mu\in\mathcal{M}(X)$.

(1) μ is called *totally o-continuous* if $\mu(U)=\sup_{\alpha\in\Gamma}\mu(U_\alpha)$ whenever $\{U_\alpha\}_{\alpha\in\Gamma}$ is an increasing net of open sets with $U=\bigcup_{\alpha\in\Gamma}U_\alpha$.

(2) μ is called *totally c-continuous* if $\mu(C)=\inf_{\alpha\in\Gamma}\mu(C_\alpha)$ whenever $\{C_\alpha\}_{\alpha\in\Gamma}$ is a decreasing net of closed sets with $C=\bigcap_{\alpha\in\Gamma}C_\alpha$.

(3) μ is called *totally conditional c-continuous* if $\mu(C)=\inf_{\alpha\in\Gamma}\mu(C_\alpha)$ whenever $\{C_\alpha\}_{\alpha\in\Gamma}$ is a decreasing net of closed sets with $C=\bigcap_{\alpha\in\Gamma}C_\alpha$ and $\mu(C_{\alpha_0})<\infty$ for some $\alpha_0\in\Gamma$.

Remark 5.2. (1) The total c-continuity implies the total conditional c-continuity and they coincide for finite nonadditive measures.

(2) The total c-continuity and the total o-continuity were already discussed in [6] with applications to convergence theorems for Choquet integrals.

(3) The sequential versions of the total c-continuity and the total o-continuity were introduced and discussed in [13] for nonadditive measures on a locally compact space.

A finite nonadditive measure μ is totally c-continuous (totally o-continuous) if and only if its dual $\bar{\mu}$ is totally o-continuous (totally c-continuous). In general, the total c-continuity and the total o-continuity are independent of each other [7, Examples 2 and 3]. Some examples of nonadditive measures with total continuities will be given in Section 7.

Lemma 5.3. *Let $\mu \in \mathcal{M}(X)$.*

(1) *If μ is totally o-continuous and $\{f_\alpha\}_{\alpha\in\Gamma}$ is a uniformly μ-essentially bounded, increasing net of lower semicontinuous functions in $\mathcal{F}(X)$ with pointwise limit $f \in \mathcal{F}(X)$, then f is lower semicontinuous and μ-essentially bounded.*
(2) *If μ is totally c-continuous and $\{f_\alpha\}_{\alpha\in\Gamma}$ is a uniformly μ-essentially bounded, decreasing net of upper semicontinuous functions in $\mathcal{F}(X)$ with pointwise limit $f \in \mathcal{F}(X)$, then f is upper semicontinuous and μ-essentially bounded.*

Proof. (1) We only prove that f is μ-essentially bounded. Since $\{f_\alpha\}_{\alpha\in\Gamma}$ is uniformly μ-essentially bounded, there is $M > 0$ such that for any $\alpha \in \Gamma$,

$$\mu(\{f_\alpha > M\}) = 0 \text{ and } \mu(\{f_\alpha > -M\}) = \mu(X). \tag{5.1}$$

Since $\{\{f_\alpha > M\}\}_{\alpha\in\Gamma}$ is an increasing net of open sets with $\{f > M\} = \bigcup_{\alpha\in\Gamma}\{f_\alpha > M\}$, by (5.1) and the total o-continuity of μ we have $\mu(\{f > M\}) = 0$ and thus $\mu(\{f \geq 2M\}) = 0$. Similarly, since $\{\{f_\alpha > -M\}\}_{\alpha\in\Gamma}$ is an increasing net of open sets with $\{f > -M\} = \bigcup_{\alpha\in\Gamma}\{f_\alpha > -M\}$, we have $\mu(\{f > -M\}) = \mu(X)$ and thus $\mu(\{f \geq -2M\}) = \mu(X)$. Therefore f is μ-essentially bounded.

(2) It is similar to the proof of (1). □

Now we give a net version of the monotone increasing convergence theorem for an integral functional, which is applicable to the Lebesgue, the Choquet, the Šipoš, the Sugeno, and the Shilkret integrals.

Theorem 5.4. *Let $I\colon \mathcal{M}(X) \times \mathcal{F}^+(X) \to [0,\infty]$ be an integral functional. Let $\mu \in \mathcal{M}(X)$. Consider the following two assertions*:

(i) *μ is totally o-continuous.*
(ii) *For every uniformly μ-essentially bounded, increasing net $\{f_\alpha\}_{\alpha\in\Gamma}$ of lower semicontinuous functions in $\mathcal{F}^+(X)$ with pointwise limit $f \in \mathcal{F}^+(X)$, $I(\mu, f) = \lim_{\alpha\in\Gamma} I(\mu, f_\alpha) = \sup_{\alpha\in\Gamma} I(\mu, f_\alpha)$.*

If μ is finite and I is perturbative and elementary with generator of finite and continuous type, then (i) *implies* (ii). *If I is generative with limit preserving generator, then* (ii) *implies* (i).

Proof. (i)⇒(ii): Let μ be totally o-continuous and finite. Assume that I is perturbative and elementary with generator θ of finite and continuous type. Let $\{f_\alpha\}_{\alpha\in\Gamma}$ be a uniformly μ-essentially bounded, increasing net of lower semicontinuous functions in $\mathcal{F}^+(X)$ with pointwise limit $f \in \mathcal{F}^+(X)$. By Lemma 5.3, f is lower semicontinuous and μ-essentially bounded. Thus $I(\mu, f_\alpha) \leq I(\mu, f) < \infty$ for all $\alpha \in \Gamma$ by (2) of Proposition 4.6.

To begin with, we consider the case that $\{f_\alpha\}_{\alpha\in\varGamma}$ is uniformly bounded. Then there is $M > 0$ such that $0 \le f_\alpha \le f \le M$ for any $\alpha \in \varGamma$. For each $\alpha \in \varGamma$ and $n \in \mathbb{N}$, let $G_{\alpha,i} := \{f_\alpha > (i-1)M/n\}$, $G_i := \{f > (i-1)M/n\}$ $(i = 1, 2, \dots, n)$, and

$$f_{\alpha,n} := \bigoplus_{i=1}^{n} (r_i \ominus r_{i-1})\, \chi_{G_{\alpha,i}}, \tag{5.2}$$

$$f_n := \bigoplus_{i=1}^{n} (r_i \ominus r_{i-1})\, \chi_{G_i}, \tag{5.3}$$

where $r_i := iM/n$ $(i = 0, 1, \dots, n)$. Then $0 \le f_{\alpha,n}(x) \le M$, $0 \le f_n(x) \le M$, $|f_{\alpha,n}(x) - f_\alpha(x)| < M/n$, and $|f_n(x) - f(x)| < M/n$ for any $x \in X$.

First we prove

$$I(\mu, f_{\alpha,n}) \to I(\mu, f_n) \tag{5.4}$$

for every $n \in \mathbb{N}$. For each $i = 1, 2, \dots, n$, $\{G_{\alpha,i}\}_{\alpha\in\varGamma}$ is an increasing net of open sets with $G_i = \bigcup_{\alpha\in\varGamma} G_{\alpha,i}$. Thus $\mu(G_{\alpha,i}) \to \mu(G_i)$ by the total o-continuity of μ, so that $\theta(r_i \ominus r_{i-1}, \mu(G_{\alpha,i})) \to \theta(r_i \ominus r_{i-1}, \mu(G_i))$ since θ is of continuous type. Since I is elementary and θ is its generator, by (5.2) and (5.3),

$$I(\mu, f_{\alpha,n}) = \bigoplus_{i=1}^{n} \theta\big(r_i \ominus r_{i-1}, \mu(G_{\alpha,i})\big),$$

$$I(\mu, f_n) = \bigoplus_{i=1}^{n} \theta\big(r_i \ominus r_{i-1}, \mu(G_i)\big).$$

So (5.4) follows from the continuity of $\oplus$.

Next we prove

$$I(\mu, f_\alpha) \to I(\mu, f). \tag{5.5}$$

Observe that for each $\alpha \in \varGamma$ and $n \in \mathbb{N}$, $f_{\alpha,n} \prec_\mu f_\alpha + M/n$, $f_\alpha \prec_\mu f_{\alpha,n} + M/n$, $f_n \prec_\mu f + M/n$, and $f \prec_\mu f_n + M/n$. Let $p := M + 1 > 0$ and $q := \mu(X) + 1 > 0$. Since I is perturbative, there are control functions $\varphi := \varphi_{p,q}, \psi := \psi_{p,q} \in \varPhi$ such that

$$|I(\mu, f_{\alpha,n}) - I(\mu, f_\alpha)| \le \varphi(0) + \psi(M/n), \tag{5.6}$$

$$|I(\mu, f_n) - I(\mu, f)| \le \varphi(0) + \psi(M/n) \tag{5.7}$$

for every $\alpha \in \varGamma$ and $n \in \mathbb{N}$. Consequently, (5.4), (5.6), and (5.7) imply

$$0 \le \limsup_{\alpha\in\varGamma} |I(\mu, f_\alpha) - I(\mu, f)| \le 2\psi(M/n).$$

So (5.5) follows from $\lim_{n\to\infty} \psi(M/n) = 0$. Thus the theorem has been proved in the case that $\{f_\alpha\}_{\alpha\in\varGamma}$ is uniformly bounded.

Now we consider the general case. Since $\{f_\alpha\}_{\alpha\in\varGamma}$ is uniformly μ-essentially bounded and f is μ-essentially bounded, there is $M > 0$ such that $\mu(\{f \ge M\}) = \mu(\{f_\alpha \ge M\}) = 0$ for all $\alpha \in \varGamma$. Then $\mu(\{f \ge t\}) = \mu(\{f \wedge M \ge t\})$ for any $t \in \mathbb{R}$. Thus $I(\mu, f) = I(\mu, f \wedge M)$ by Proposition 4.5. Similarly, $I(\mu, f_\alpha) = I(\mu, f_\alpha \wedge M)$

for all $\alpha \in \Gamma$. Since $\{f_\alpha \wedge M\}_{\alpha\in\Gamma}$ is a uniformly bounded net of lower semicontinuous functions in $\mathcal{F}^+(X)$ converging pointwise to $f \wedge M \in \mathcal{F}^+(X)$, we have $I(\mu, f) = I(\mu, f \wedge M) = \lim_{\alpha\in\Gamma} I(\mu, f_\alpha \wedge M) = \lim_{\alpha\in\Gamma} I(\mu, f_\alpha)$. Since $\{I(\mu, f_\alpha)\}_{\alpha\in\Gamma}$ is increasing, we also have $I(\mu, f) = \sup_{\alpha\in\Gamma} I(\mu, f_\alpha)$.

(ii)⇒(i): Let $\{G_\alpha\}_{\alpha\in\Gamma}$ be an increasing net of open sets with $G = \bigcup_{\alpha\in\Gamma} G_\alpha$. For each $r > 0$, let $f := r\chi_G$ and $f_\alpha := r\chi_{G_\alpha}$ ($\alpha \in \Gamma$). Then $\{f_\alpha\}_{\alpha\in\Gamma}$ is a uniformly μ-essentially bounded, increasing net of lower semicontinuous functions in $\mathcal{F}^+(X)$ with pointwise limit $f \in \mathcal{F}^+(X)$. Since I is generative with generator θ, assertion (ii) implies $\theta(r, \mu(G_\alpha)) = I(\mu, f_\alpha) \to I(\mu, f) = \theta(r, \mu(G))$ and thus $\mu(G_\alpha) \to \mu(G)$ since θ is limit preserving. So μ is totally o-continuous. □

The following net version of the monotone decreasing convergence theorem can be proved in the same manner as Theorem 5.4.

Theorem 5.5. *Let $I\colon \mathcal{M}(X) \times \mathcal{F}^+(X) \to [0,\infty]$ be an integral functional. Let $\mu \in \mathcal{M}(X)$. Consider the following two assertions*:

(i) *μ is totally c-continuous.*

(ii) *For every uniformly μ-essentially bounded, decreasing net $\{f_\alpha\}_{\alpha\in\Gamma}$ of upper semicontinuous functions in $\mathcal{F}^+(X)$ with pointwise limit $f \in \mathcal{F}^+(X)$, $I(\mu, f) = \lim_{\alpha\in\Gamma} I(\mu, f_\alpha) = \inf_{\alpha\in\Gamma} I(\mu, f_\alpha)$.*

If μ is finite and I is perturbative and elementary with generator of finite and continuous type, then (i) *implies* (ii). *If I is generative with limit preserving generator, then* (ii) *implies* (i).

To prove implication (i) ⇒ (ii) in Theorems 5.4 and 5.5 we assume that μ is finite. In what follows, we consider the case that μ is not necessarily finite. As to the monotone increasing convergence theorem, we obtain the same conclusion as Theorem 5.4 only appending the measure-truncated property of the integral functional.

Theorem 5.6. *Let $I\colon \mathcal{M}(X) \times \mathcal{F}^+(X) \to [0,\infty]$ be an integral functional. Assume that I is perturbative, measure-truncated, and elementary with generator of finite and continuous type. If $\mu \in \mathcal{M}(X)$ is totally o-continuous and $\{f_\alpha\}_{\alpha\in\Gamma}$ is a uniformly μ-essentially bounded, increasing net of lower semicontinuous functions in $\mathcal{F}^+(X)$ with pointwise limit $f \in \mathcal{F}^+(X)$, then $I(\mu, f) = \lim_{\alpha\in\Gamma} I(\mu, f_\alpha) = \sup_{\alpha\in\Gamma} I(\mu, f_\alpha)$.*

Proof. For each $s > 0$, define the finite nonadditive measure $\mu \wedge s$ by $(\mu \wedge s)(A) := \mu(A) \wedge s$ for every $A \in \mathcal{A}$. Then $\mu \wedge s$ is totally o-continuous and $\{f_\alpha\}_{\alpha\in\Gamma}$ is uniformly $\mu \wedge s$-essentially bounded. Thus $I(\mu \wedge s, f) = \sup_{\alpha\in\Gamma} I(\mu \wedge s, f_\alpha)$ by Theorem 5.4. Since I is measure-truncated, we have

$$I(\mu, f) = \sup_{s>0} I(\mu \wedge s, f) = \sup_{\alpha\in\Gamma} \sup_{s>0} I(\mu \wedge s, f_\alpha) = \sup_{\alpha\in\Gamma} I(\mu, f_\alpha).$$

Since $\{I(\mu, f_\alpha)\}_{\alpha\in\Gamma}$ is increasing, we also have $I(\mu, f) = \lim_{\alpha\in\Gamma} I(\mu, f_\alpha)$. □

By contrast, as to the monotone decreasing convergence theorem we need to assume some additional conditions on the finiteness of the functional I and the measure μ. For each $\beta \in \Gamma$, let $\Gamma(\beta) := \{\alpha \in \Gamma \colon \alpha \geq \beta\}$, which is also a directed set.

Theorem 5.7. *Let $I = \mathrm{Ch}, \mathrm{Si}$. Let $\mu \in \mathcal{M}(X)$ and let $\{f_\alpha\}_{\alpha\in\Gamma}$ be a uniformly μ-essentially bounded, decreasing net of upper semicontinuous functions in $\mathcal{F}^+(X)$ with pointwise limit $f \in \mathcal{F}^+(X)$. Assume $I(\mu, f_{\alpha_0}) < \infty$ for some $\alpha_0 \in \Gamma$. If μ is totally conditional c-continuous, then $I(\mu, f) = \lim_{\alpha\in\Gamma} I(\mu, f_\alpha) = \inf_{\alpha\in\Gamma} I(\mu, f_\alpha)$.*

Proof. By assumption, there is $\alpha_0 \in \Gamma$ such that $I(\mu, f) \le I(\mu, f_\alpha) \le I(\mu, f_{\alpha_0}) < \infty$ for all $\alpha \in \Gamma(\alpha_0)$. In what follows we prove

$$I(\mu, f) = \lim_{\alpha\in\Gamma(\alpha_0)} I(\mu, f_\alpha) = \inf_{\alpha\in\Gamma(\alpha_0)} I(\mu, f_\alpha), \tag{5.8}$$

which yields the conclusion since $\{I(\mu, f_\alpha)\}_{\alpha\in\Gamma}$ is decreasing and

$$I(\mu, f) = \inf_{\alpha\in\Gamma(\alpha_0)} I(\mu, f_\alpha) \ge \inf_{\alpha\in\Gamma} I(\mu, f_\alpha) \ge I(\mu, f).$$

Let $\varepsilon > 0$. By Proposition 3.2 of [10] $I = \mathrm{Ch}, \mathrm{Si}$ are lower marginal continuous and horizontally additive and thus we have

$$I(\mu, f_{\alpha_0}) = \sup_{r>0} I(\mu, (f_{\alpha_0} - r)^+)$$

and

$$I(\mu, f_{\alpha_0}) = I(\mu, f_{\alpha_0} \wedge r) + I(\mu, (f_{\alpha_0} - r)^+)$$

for every $r > 0$, so that $\inf_{r>0} I(\mu, f_{\alpha_0} \wedge r) = 0$. Thus there is $r_0 > 0$ such that

$$I(\mu, f_{\alpha_0} \wedge r_0) < \varepsilon. \tag{5.9}$$

Let $g := (f - r_0)^+$ and $g_\alpha := (f_\alpha - r_0)^+$ for every $\alpha \in \Gamma(\alpha_0)$. Then $\{g_\alpha\}_{\alpha\in\Gamma(\alpha_0)}$ is a decreasing net of upper semicontinuous functions in $\mathcal{F}^+(X)$ with pointwise limit $g \in \mathcal{F}^+(X)$.

First we prove

$$I(\mu, g) = \lim_{\alpha\in\Gamma(\alpha_0)} I(\mu, g_\alpha) \tag{5.10}$$

Let $\nu(A) := \mu(A \cap \{f_{\alpha_0} \ge r_0\})$ for every $A \in \mathcal{A}$. Since $r_0 \cdot \mu(\{f_{\alpha_0} \ge r_0\}) \le I(\mu, f_{\alpha_0}) < \infty$ for $I = \mathrm{Ch}, \mathrm{Si}$, the nonadditive measure ν is finite and totally c-continuous. Since $\{f_\alpha\}_{\alpha\in\Gamma}$ is uniformly μ-essentially bounded, there is $M > 0$ such that $\mu(\{f_\alpha \ge M\}) = 0$ for all $\alpha \in \Gamma$. Therefore $0 \le \nu(\{g_\alpha \ge M\}) \le \mu(\{g_\alpha \ge M\}) = \mu(\{f_\alpha \ge M + r_0\}) \le \mu(\{f_\alpha \ge M\}) = 0$ and hence $\nu(\{g_\alpha \ge M\}) = 0$ for any $\alpha \in \Gamma(\alpha_0)$. Thus $\{g_\alpha\}_{\alpha\in\Gamma(\alpha_0)}$ is uniformly ν-essentially bounded. Moreover, since $\mu(\{g \ge t\}) = \nu(\{g \ge t\})$ and $\mu(\{g_\alpha \ge t\}) = \nu(\{g_\alpha \ge t\})$ for any $t > 0$, by the definition of Ch and Si we have $I(\mu, g) = I(\nu, g)$ and $I(\mu, g_\alpha) = I(\nu, g_\alpha)$ for every $\alpha \in \Gamma(\alpha_0)$. Consequently, by Theorem 5.5

$$I(\mu, g) = I(\nu, g) = \lim_{\alpha\in\Gamma(\alpha_0)} I(\nu, g_\alpha) = \lim_{\alpha\in\Gamma(\alpha_0)} I(\mu, g_\alpha)$$

and (5.10) follows.

By (5.9) and the horizontal additivity of I, for any $\alpha \in \Gamma(\alpha_0)$, we have

$$\begin{aligned}|I(\mu, f_\alpha) - I(\mu, f)| &\le |I(\mu, g_\alpha) - I(\mu, g)| + I(\mu, f_\alpha \wedge r_0) + I(\mu, f \wedge r_0)\\ &\le |I(\mu, g_\alpha) - I(\mu, g)| + 2\varepsilon,\end{aligned}$$

so that

$$0 \le \limsup_{\alpha\in\Gamma(\alpha_0)} |I(\mu, f_\alpha) - I(\mu, f)| \le 2\varepsilon.$$

Letting $\varepsilon \to 0$ gives $I(\mu, f) = \lim_{\alpha \in \Gamma(\alpha_0)} I(\mu, f_\alpha)$. Since $\{I(\mu, f_\alpha)\}_{\alpha \in \Gamma(\alpha_0)}$ is decreasing, we also have $I(\mu, f) = \inf_{\alpha \in \Gamma(\alpha_0)} I(\mu, f_\alpha)$. □

For a subset A of X, let A^- be the closure of A, that is, the smallest closed set containing A.

Theorem 5.8. *Let $\mu \in \mathcal{M}(X)$ and let $\{f_\alpha\}_{\alpha \in \Gamma}$ be a uniformly μ-essentially bounded, decreasing net of upper semicontinuous functions in $\mathcal{F}^+(X)$ with pointwise limit $f \in \mathcal{F}^+(X)$. Assume $\mu(\{f_{\alpha_0} > \mathrm{Su}(\mu, f)\}^-) < \infty$ for some $\alpha_0 \in \Gamma$. If μ is totally conditional c-continuous, then $\mathrm{Su}(\mu, f) = \lim_{\alpha \in \Gamma} \mathrm{Su}(\mu, f_\alpha) = \inf_{\alpha \in \Gamma} \mathrm{Su}(\mu, f_\alpha)$.*

Proof. The conclusion is obvious if $\mathrm{Su}(\mu, f) = \infty$ or if $\mathrm{Su}(\mu, f) < \infty$ and $\mathrm{Su}(\mu, f) = \mathrm{Su}(\mu, f_{\beta_0})$ for some $\beta_0 \in \Gamma$. Therefore we consider the case that $\mathrm{Su}(\mu, f) < \infty$ and $\mathrm{Su}(\mu, f_\alpha) > \mathrm{Su}(\mu, f)$ for all $\alpha \in \Gamma$.

By assumption, there is $\alpha_0 \in \Gamma$ such that $\mu(\{f_{\alpha_0} > \mathrm{Su}(\mu, f)\}^-) < \infty$. Let $r := \mathrm{Su}(\mu, f)$ and $\nu(A) := \mu(A \cap \{f_{\alpha_0} > r\}^-)$ for every $A \in \mathcal{A}$. Then the finite non-additive measure ν is totally c-continuous and $\{f_\alpha\}_{\alpha \in \Gamma(\alpha_0)}$ is uniformly ν-essentially bounded. First we prove

$$\mathrm{Su}(\mu, f_\alpha) = \mathrm{Su}(\nu, f_\alpha) \tag{5.11}$$

for all $\alpha \in \Gamma(\alpha_0)$. Fix $\alpha \in \Gamma(\alpha_0)$. Then

$$r = \mathrm{Su}(\mu, f) < \mathrm{Su}(\mu, f_\alpha) \leq r \vee \mu(\{f_\alpha > r\})$$

and hence $\mu(\{f_\alpha > r\}) > r$. Therefore

$$\sup_{t \in [r, \infty]} \bigl[t \wedge \mu(\{f_\alpha > t\})\bigr] \geq r \wedge \mu(\{f_\alpha > r\}) = r. \tag{5.12}$$

On the other hand,

$$r \geq \sup_{t \in [0, r]} \bigl[t \wedge \mu(\{f_\alpha > t\})\bigr] \geq r \wedge \mu(\{f_\alpha > r\}) = r$$

and hence

$$\sup_{t \in [0, r]} \bigl[t \wedge \mu(\{f_\alpha > t\})\bigr] = r. \tag{5.13}$$

Therefore by (5.12) and (5.13) we have

$$\mathrm{Su}(\mu, f_\alpha) = \sup_{t \in [r, \infty]} \bigl[t \wedge \mu(\{f_\alpha > t\})\bigr]. \tag{5.14}$$

Since $\mu(\{f_\alpha > t\}) = \nu(\{f_\alpha > t\})$ for any $t \in [r, \infty]$, by (5.14)

$$\mathrm{Su}(\nu, f_\alpha) \leq \mathrm{Su}(\mu, f_\alpha) = \sup_{t \in [r, \infty]} \bigl[t \wedge \nu(\{f_\alpha > t\})\bigr] \leq \mathrm{Su}(\nu, f_\alpha)$$

and (5.11) follows. Consequently, by Theorem 5.5

$$\mathrm{Su}(\mu, f) \leq \inf_{\alpha \in \Gamma} \mathrm{Su}(\mu, f_\alpha) \leq \inf_{\alpha \in \Gamma(\alpha_0)} \mathrm{Su}(\nu, f_\alpha) = \mathrm{Su}(\nu, f) \leq \mathrm{Su}(\mu, f)$$

and thus $\mathrm{Su}(\mu, f) = \inf_{\alpha \in \Gamma} \mathrm{Su}(\mu, f_\alpha)$. Since $\{\mathrm{Su}(\mu, f_\alpha)\}_{\alpha \in \Gamma}$ is decreasing, we also have $\mathrm{Su}(\mu, f) = \lim_{\alpha \in \Gamma} \mathrm{Su}(\mu, f_\alpha)$. □

Theorem 5.9. *Let $\mu \in \mathcal{M}(X)$ and let $\{f_\alpha\}_{\alpha\in\Gamma}$ be a uniformly μ-essentially bounded, decreasing net of upper semicontinuous functions in $\mathcal{F}^+(X)$ with pointwise limit $f \in \mathcal{F}^+(X)$. Assume $\mu(\{f_{\alpha_0} > 0\}^-) < \infty$ for some $\alpha_0 \in \Gamma$. If μ is totally conditional c-continuous, then $\mathrm{Sh}(\mu, f) = \lim_{\alpha\in\Gamma} \mathrm{Sh}(\mu, f_\alpha) = \inf_{\alpha\in\Gamma} \mathrm{Sh}(\mu, f_\alpha)$.*

Proof. By assumption, there is $\alpha_0 \in \Gamma$ such that $\mu(\{f_{\alpha_0} > 0\}^-) < \infty$. Let $\nu(A) := \mu(A\cap\{f_{\alpha_0} > 0\}^-)$ for every $A \in \mathcal{A}$. Then the finite nonadditive measure ν is totally c-continuous and $\{f_\alpha\}_{\alpha\in\Gamma(\alpha_0)}$ is uniformly ν-essentially bounded. In addition, for any $\alpha \in \Gamma(\alpha_0)$, $\mathrm{Sh}(\mu, f_\alpha) = \mathrm{Sh}(\nu, f_\alpha)$ since $\mu(\{f_\alpha > t\}) = \nu(\{f_\alpha > t\})$ for every $t \in [0,\infty)$. Consequently, by Theorem 5.5

$$\mathrm{Sh}(\mu, f) \leq \inf_{\alpha\in\Gamma} \mathrm{Sh}(\mu, f_\alpha) \leq \inf_{\alpha\in\Gamma(\alpha_0)} \mathrm{Sh}(\nu, f_\alpha) = \mathrm{Sh}(\nu, f) \leq \mathrm{Sh}(\mu, f)$$

and thus $\mathrm{Sh}(\mu, f) = \inf_{\alpha\in\Gamma} \mathrm{Sh}(\mu, f_\alpha)$. Since $\{\mathrm{Sh}(\mu, f_\alpha)\}_{\alpha\in\Gamma}$ is decreasing, we also have $\mathrm{Sh}(\mu, f) = \lim_{\alpha\in\Gamma} \mathrm{Sh}(\mu, f_\alpha)$. □

Example 5.10. Let $X := (0,\infty)$, $\mathcal{A}$ the σ-field of all Lebesgue measurable subsets of X, and λ the Lebesgue measure on $(X,\mathcal{A})$. Define the decreasing sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+(X)$ converging to 0 as below. Then the following show that the finiteness assumptions on the functional I and the measure μ in Theorems 5.7, 5.8 and 5.9 cannot be dropped.

(1) Let $f_n(x) := \chi_{[n,\infty)}$ for every $x \in X$ and $n \in \mathbb{N}$. Then $\mathrm{Ch}(\lambda, f_n) = \mathrm{Si}(\lambda, f_n) = \infty$ for all $n \in \mathbb{N}$, so that $\mathrm{Ch}(\lambda, f_n) \not\to 0$ and $\mathrm{Si}(\lambda, f_n) \not\to 0$.

(2) Let $f_n(x) := 1 \wedge (x/n)$ for every $x \in X$ and $n \in \mathbb{N}$. Then $\lambda(\{f_n > \mathrm{Su}(\lambda, 0)\}) = \lambda(\{f_n > 0\}) = \infty$ and $\mathrm{Su}(\lambda, f_n) = 1$ for all $n \in \mathbb{N}$, so that $\mathrm{Su}(\lambda, f_n) \not\to 0$.

(3) Let $f_n(x) := 1/(x + n)$ for every $x \in X$ and $n \in \mathbb{N}$. Then $\lambda(\{f_n > 0\}) = \infty$ and $\mathrm{Sh}(\lambda, f_n) = 1$ for all $n \in \mathbb{N}$, so that $\mathrm{Sh}(\lambda, f_n) \not\to 0$.

6. Extensions to symmetric and asymmetric integrals

In this section we extend our net versions of the monotone convergence theorem to symmetric and asymmetric nonlinear integrals. Let $I\colon \mathcal{M}(X) \times \mathcal{F}^+(X) \to [0,\infty]$ be an integral functional. The *symmetric extension* of I is the functional $I^s\colon \mathcal{M}(X) \times \mathcal{F}(X) \to [-\infty,\infty]$ defined by $I^s(\mu, f) := I(\mu, f^+) - I(\mu, f^-)$ for every $(\mu, f) \in \mathcal{M}(X) \times \mathcal{F}(X)$. By contrast, the *asymmetric extension* of I is the functional $I^a\colon \mathcal{M}_b(X) \times \mathcal{F}(X) \to [-\infty,\infty]$ defined by $I^a(\mu, f) := I(\mu, f^+) - I(\bar{\mu}, f^-)$ for every $(\mu, f) \in \mathcal{M}_b(X) \times \mathcal{F}(X)$. Both extensions are not defined if the right hand side of the equation is of the form $\infty - \infty$. Note that the asymmetric extension is defined only for finite nonadditive measures. The symmetric extension I^s is *symmetric*: $I^s(\mu, -f) = -I^s(\mu, f)$ and the asymmetric extension I^a is *asymmetric*: $I^a(\mu, -f) = -I^a(\bar{\mu}, f)$. If $I = \mathrm{Ch}$, I^s and I^a are called the *symmetric Choquet integral* and the *asymmetric Choquet integral*, respectively. The same is the case with $I = \mathrm{Si}, \mathrm{Su}, \mathrm{Sh}$. As to the asymmetric extension we have the following limit theorem as a corollary to Theorems 5.4 and 5.5.

Corollary 6.1. *Let $I\colon \mathcal{M}(X) \times \mathcal{F}^+(X) \to [0,\infty]$ be an integral functional. Let $\mu \in \mathcal{M}_b(X)$. Assume that I is perturbative and elementary with generator of finite and continuous type.*

(1) *If μ is totally o-continuous and $\{f_\alpha\}_{\alpha\in\Gamma}$ is a uniformly μ-essentially bounded, increasing net of lower semicontinuous functions in $\mathcal{F}(X)$ with pointwise limit $f \in \mathcal{F}(X)$, then $I^a(\mu, f) = \lim_{\alpha\in\Gamma} I^a(\mu, f_\alpha) = \sup_{\alpha\in\Gamma} I^a(\mu, f_\alpha)$.*

(2) *If μ is totally c-continuous and $\{f_\alpha\}_{\alpha\in\Gamma}$ is a uniformly μ-essentially bounded, decreasing net of upper semicontinuous functions in $\mathcal{F}(X)$ with pointwise limit $f \in \mathcal{F}(X)$, then $I^a(\mu, f) = \lim_{\alpha\in\Gamma} I^a(\mu, f_\alpha) = \inf_{\alpha\in\Gamma} I^a(\mu, f_\alpha)$.*

Proof. (1) By Lemma 5.3 and (2) of Proposition 2.6, $\{f_\alpha^+\}_{\alpha\in\Gamma}$ is a uniformly μ-essentially bounded, increasing net of lower semicontinuous functions in $\mathcal{F}^+(X)$ converging to the μ-essential bounded $f^+ \in \mathcal{F}^+(X)$. Similarly, $\{f_\alpha^-\}_{\alpha\in\Gamma}$ is a uniformly $\bar{\mu}$-essentially bounded, decreasing net of upper semicontinuous functions in $\mathcal{F}^+(X)$ converging to the $\bar{\mu}$-essentially bounded $f^- \in \mathcal{F}^+(X)$. Therefore by Theorem 5.4

$$I(\mu, f^+) = \lim_{\alpha\in\Gamma} I(\mu, f_\alpha^+) = \sup_{\alpha\in\Gamma} I(\mu, f_\alpha^+) \tag{6.1}$$

and by Theorem 5.5

$$I(\bar{\mu}, f^-) = \lim_{\alpha\in\Gamma} I(\bar{\mu}, f_\alpha^-) = \inf_{\alpha\in\Gamma} I(\bar{\mu}, f_\alpha^-) \tag{6.2}$$

Since the generator of I is of finite type, by (2) of Proposition 4.6, $I(\mu, f^+)$, $I(\bar{\mu}, f^-)$, $I(\mu, f_\alpha^+)$, and $I(\bar{\mu}, f_\alpha^-)$ are all finite. Therefore it follows from (6.1) and (6.2) that $I^a(\mu, f) = \lim_{\alpha\in\Gamma} I^a(\mu, f_\alpha) = \sup_{\alpha\in\Gamma} I^a(\mu, f_\alpha)$.

(2) It can be proved in the same way as (1). □

As to the symmetric extension, by Theorems 5.6, 5.7, 5.8, and 5.9 we obtain the following forms that can be proved in a similar way as Corollary 6.1.

Corollary 6.2. *Let $I = \mathrm{Ch}, \mathrm{Si}$. Let $\mu \in \mathcal{M}(X)$ be totally o-continuous and totally conditional c-continuous.*

(1) *Let $\{f_\alpha\}_{\alpha\in\Gamma}$ be a uniformly μ-essentially symmetric bounded, increasing net of lower semicontinuous functions in $\mathcal{F}(X)$ with pointwise limit $f \in \mathcal{F}(X)$. Assume $I(\mu, f_{\alpha_0}^-) < \infty$ for some $\alpha_0 \in \Gamma$. Then the symmetric extensions $I^s(\mu, f)$, $I^s(\mu, f_\alpha)$ are defined for all $\alpha \in \Gamma(\alpha_0)$ and $I^s(\mu, f) = \lim_{\alpha\in\Gamma(\alpha_0)} I^s(\mu, f_\alpha) = \sup_{\alpha\in\Gamma(\alpha_0)} I^s(\mu, f_\alpha)$.*

(2) *Let $\{f_\alpha\}_{\alpha\in\Gamma}$ be a uniformly μ-essentially symmetric bounded, decreasing net of upper semicontinuous functions in $\mathcal{F}(X)$ with pointwise limit $f \in \mathcal{F}(X)$. Assume $I(\mu, f_{\alpha_0}^+) < \infty$ for some $\alpha_0 \in \Gamma$. Then the symmetric extensions $I^s(\mu, f)$, $I^s(\mu, f_\alpha)$ are defined for all $\alpha \in \Gamma(\alpha_0)$ and $I^s(\mu, f) = \lim_{\alpha\in\Gamma(\alpha_0)} I^s(\mu, f_\alpha) = \inf_{\alpha\in\Gamma(\alpha_0)} I^s(\mu, f_\alpha)$.*

Corollary 6.3. *Let $\mu \in \mathcal{M}(X)$ be totally o-continuous and totally conditional c-continuous.*

(1) *Let $\{f_\alpha\}_{\alpha\in\Gamma}$ be a uniformly μ-essentially symmetric bounded, increasing net of lower semicontinuous functions in $\mathcal{F}(X)$ with pointwise limit $f \in \mathcal{F}(X)$. Assume $\mu(\{f_{\alpha_0}^- > \mathrm{Su}(\mu, f^-)\}^-) < \infty$ for some $\alpha_0 \in \Gamma$. Then the symmetric extensions $\mathrm{Su}^s(\mu, f)$, $\mathrm{Su}^s(\mu, f_\alpha)$ are defined for all $\alpha \in \Gamma(\alpha_0)$ and $\mathrm{Su}^s(\mu, f) = \lim_{\alpha\in\Gamma(\alpha_0)} \mathrm{Su}^s(\mu, f_\alpha) = \sup_{\alpha\in\Gamma(\alpha_0)} \mathrm{Su}^s(\mu, f_\alpha)$.*

(2) *Let* $\{f_\alpha\}_{\alpha\in\Gamma}$ *be a uniformly* μ*-essentially symmetric bounded, decreasing net of upper semicontinuous functions in* $\mathcal{F}(X)$ *with pointwise limit* $f \in \mathcal{F}(X)$*. Assume* $\mu(\{f_{\alpha_0}^+ > \mathrm{Su}(\mu, f^+)\}^-) < \infty$ *for some* $\alpha_0 \in \Gamma$*. Then the symmetric extensions* $\mathrm{Su}^s(\mu, f)$*,* $\mathrm{Su}^s(\mu, f_\alpha)$ *are defined for all* $\alpha \in \Gamma(\alpha_0)$ *and* $\mathrm{Su}^s(\mu, f) = \lim_{\alpha\in\Gamma(\alpha_0)} \mathrm{Su}^s(\mu, f_\alpha) = \inf_{\alpha\in\Gamma(\alpha_0)} \mathrm{Su}^s(\mu, f_\alpha)$.

Corollary 6.4. *Let* $\mu \in \mathcal{M}(X)$ *be totally o-continuous and totally conditional c-continuous.*

(1) *Let* $\{f_\alpha\}_{\alpha\in\Gamma}$ *be a uniformly* μ*-essentially symmetric bounded, increasing net of lower semicontinuous functions in* $\mathcal{F}(X)$ *with pointwise limit* $f \in \mathcal{F}(X)$*. Assume* $\mu(\{f_{\alpha_0}^- > 0\}^-) < \infty$ *for some* $\alpha_0 \in \Gamma$*. Then the symmetric extensions* $\mathrm{Sh}^s(\mu, f)$*,* $\mathrm{Sh}^s(\mu, f_\alpha)$ *are defined for all* $\alpha \in \Gamma(\alpha_0)$ *and* $\mathrm{Sh}^s(\mu, f) = \lim_{\alpha\in\Gamma(\alpha_0)} \mathrm{Sh}^s(\mu, f_\alpha) = \sup_{\alpha\in\Gamma(\alpha_0)} \mathrm{Sh}^s(\mu, f_\alpha)$.

(2) *Let* $\{f_\alpha\}_{\alpha\in\Gamma}$ *be a uniformly* μ*-essentially symmetric bounded, decreasing net of upper semicontinuous functions in* $\mathcal{F}(X)$ *with pointwise limit* $f \in \mathcal{F}(X)$*. Assume* $\mu(\{f_{\alpha_0}^+ > 0\}^-) < \infty$ *for some* $\alpha_0 \in \Gamma$*. Then the symmetric extensions* $\mathrm{Sh}^s(\mu, f)$*,* $\mathrm{Sh}^s(\mu, f_\alpha)$ *are defined for all* $\alpha \in \Gamma(\alpha_0)$ *and* $\mathrm{Sh}^s(\mu, f) = \lim_{\alpha\in\Gamma(\alpha_0)} \mathrm{Sh}^s(\mu, f_\alpha) = \inf_{\alpha\in\Gamma(\alpha_0)} \mathrm{Sh}^s(\mu, f_\alpha)$.

7. Examples of nonadditive measures with total continuities

This last section gives some examples of nonadditive measures with total continuities. To this end let us recall some basic definitions concerning the continuity and the quasi-additivity of nonadditive measures.

Definition 7.1. Let $\mu\colon \mathcal{A} \to [0, \infty]$ be a nonadditive measure.

(1) μ is called *continuous from above* if $\mu(A_n) \to \mu(A)$ whenever $A \in \mathcal{A}$ and $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ is a decreasing sequence with $A = \bigcap_{n=1}^\infty A_n$.
(2) μ is called *continuous from below* if $\mu(A_n) \to \mu(A)$ whenever $A \in \mathcal{A}$ and $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ is an increasing sequence with $A = \bigcup_{n=1}^\infty A_n$.
(3) μ is called *continuous* if it is continuous from above and below.
(4) μ is called *conditional continuous from above* if $\mu(A_n) \to \mu(A)$ whenever $A \in \mathcal{A}$ and $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ is a decreasing sequence with $A = \bigcap_{n=1}^\infty A_n$ and $\mu(A_1) < \infty$.
(5) μ is called *totally continuous from above* if $\mu(A_\alpha) \to \mu(A)$ whenever $A \in \mathcal{A}$ and $\{A_\alpha\}_{\alpha\in\Gamma} \subset \mathcal{A}$ is a decreasing net with $A = \bigcap_{\alpha\in\Gamma} A_\alpha$.
(6) μ is called *totally continuous from below* if $\mu(A_\alpha) \to \mu(A)$ whenever $A \in \mathcal{A}$ and $\{A_\alpha\}_{\alpha\in\Gamma} \subset \mathcal{A}$ is an increasing net with $A = \bigcup_{\alpha\in\Gamma} A_\alpha$.
(7) μ is *subadditive* if $\mu(A \cup B) \leq \mu(A) + \mu(B)$ whenever $A, B \in \mathcal{A}$.
(8) μ is *weakly null-additive* if $\mu(A \cup B) = 0$ whenever $A, B \in \mathcal{A}$ and $\mu(A) = \mu(B) = 0$.
(9) μ is *null-additive* if $\mu(A \cup B) = \mu(A)$ whenever $A, B \in \mathcal{A}$ and $\mu(B) = 0$.
(10) μ is *asymptotic null-additive* if $\mu(A \cup B_n) \to \mu(A)$ whenever $A \in \mathcal{A}$ and $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ is a decreasing sequence with $\mu(B_n) \to 0$.
(11) μ is *asymptotic null-subtractive* if $\mu(A \setminus B_n) \to \mu(A)$ whenever $A \in \mathcal{A}$ and $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ is a decreasing sequence with $\mu(B_n) \to 0$.
(12) μ is *autocontinuous from above* if $\mu(A \cup B_n) \to \mu(A)$ whenever $A \in \mathcal{A}$ and $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ is a sequence with $\mu(B_n) \to 0$.

(13) μ is *autocontinuous from below* if $\mu(A \setminus B_n) \to \mu(A)$ whenever $A \in \mathcal{A}$ and $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ is a sequence with $\mu(B_n) \to 0$.

Obviously, every asymptotic null-additive or asymptotic null-subtractive nonadditive measure is null-additive and thus weakly null-additive. In addition, every autocontinuous from above (below) nonadditive measure is asymptotic null-additive (null-subtractive) and every continuous from above (below) and null-additive nonadditive measure is asymptotic null-additive (null-subtractive); see [6, Proposition 1].

The following special nonadditive measures are essential in Dempster-Shafer theory of evidence and dealing with uncertainty in economics. See [25, Chapter 4] for their basic properties and historical notes.

Definition 7.2. Let X be a nonempty set and 2^X the family of all subsets of X.

(1) A set function $m\colon 2^X \to [0,1]$ is called a *basic probability assignment* if $m(\emptyset) = 0$ and

$$\sum_{A\in 2^X} m(A) := \sup\left\{\sum_{A\in\mathcal{D}} m(A)\colon \mathcal{D} \in \Omega\right\} = 1,$$

where $\mathcal{D}$ is a family of a finite number of subsets of X and Ω is the set of all such families. Then the set function $\mathrm{Bel}\colon 2^X \to [0,1]$ defined by

$$\mathrm{Bel}(A) := \sum_{B\subset A} m(B), \quad A \in 2^X,$$

is called a *belief measure* on X and the set function $\mathrm{Pla}\colon 2^X \to [0,1]$ defined by

$$\mathrm{Pla}(A) := \sum_{B\cap A\neq\emptyset} m(B), \quad A \in 2^X,$$

is called a *plausibility measure* on X.

(2) A nonadditive measure $\mathrm{Pos}\colon 2^X \to [0,1]$ is called a *possibility measure* if

$$\mathrm{Pos}\left(\bigcup_{i\in I} A_i\right) = \sup_{i\in I} \mathrm{Pos}(A_i)$$

for any family $\{A_i\}_{i\in I} \subset 2^X$ and a nonadditive measure $\mathrm{Nec}\colon \mathcal{A} \to [0,1]$ is called a *necessity measure* if

$$\mathrm{Nec}\left(\bigcap_{i\in I} A_i\right) = \inf_{i\in I} \mathrm{Nec}(A_i)$$

for any family $\{A_i\}_{i\in I} \subset 2^X$.

A nonadditive measure $\mu\colon \mathcal{A} \to [0,\infty]$ is called *Radon* if for every $A \in \mathcal{A}$ and every $\varepsilon > 0$, there are a compact set K and an open set U such that $K \subset A \subset U$ and $\mu(U \setminus K) < \varepsilon$. For instance, every weakly null-additive, continuous nonadditive measure $\mu\colon \mathcal{A} \to [0,\infty]$ is Radon if X is a complete or locally compact, separable metric space and $\mathcal{A}$ is the Borel σ-field of X, that is, the smallest σ-field containing all open subsets of X. This and other related results were already proved in [5] for

Riesz space-valued nonadditive measures and they can be proved for not necessarily finite, extended real-valued nonadditive measures by exactly the same idea.

Proposition 7.3. *Let $\mu\colon \mathcal{A} \to [0,\infty]$ be a nonadditive measure.*

(1) *If μ is Radon and asymptotic null-subtractive, then it is totally o-continuous.*
(2) *If μ is Radon and asymptotic null-additive, then it is totally c-continuous.*

Proof. The proof is the same as [6, Theorem 6]. □

Recall that X is strongly Lindelöf if every family of open subsets of X has a countable subfamily with the same union [19, Definition 7, p. 103]. For instance, any Suslin space and any space with a countable base for open sets are strongly Lindelöf [19, p. 104].

Proposition 7.4. *Let X be strongly Lindelöf. Let $\mu\colon \mathcal{A} \to [0,\infty]$ be a nonadditive measure.*

(1) *If μ is continuous from below, then it is totally o-continuous.*
(2) *If μ is continuous from above, then it is totally c-continuous.*
(3) *If μ is conditional continuous from above, then it is totally conditional c-continuous.*

Proof. (1) Let $\{G_\alpha\}_{\alpha\in\Gamma}$ be an increasing net of open subsets of X with $U = \bigcup_{\alpha\in\Gamma} U_\alpha$. Since X is strongly Lindelöf, there is an increasing sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subset \Gamma$ such that $U = \bigcup_{n=1}^{\infty} U_{\alpha_n}$. Then the continuity of μ from below implies $\mu(U_{\alpha_n}) \to \mu(U)$, so that $\mu(U) = \sup_{\alpha\in\Gamma} \mu(U_\alpha)$. Thus μ is totally o-continuous.

The proofs of (2) and (3) are similar. □

Proposition 7.5. *Let $\mu\colon \mathcal{A} \to [0,\infty)$ be a totally o-continuous, finite nonadditive measure. Let $\varphi\colon [0,\infty) \to [0,\infty]$ be an increasing function with $\varphi(0) = 0$. Define the nonadditive measure $\varphi(\mu)\colon \mathcal{A} \to [0,\infty]$ by $\varphi(\mu)(A) := \varphi(\mu(A))$ for every $A \in \mathcal{A}$.*

(1) *If φ is lower semicontinuous, then $\varphi(\mu)$ is totally o-continuous.*
(2) *If φ is upper semicontinuous, then $\varphi(\mu)$ is totally c-continuous.*

Proof. (1) Let $\{U_\alpha\}_{\alpha\in\Gamma}$ be an increasing net of open subsets of X with $U = \bigcup_{\alpha\in\Gamma} U_\alpha$. By the total o-continuity of μ we have $\mu(U_\alpha) \to \mu(U)$ and thus the lower semicontinuity of φ implies

$$\varphi(\mu)(U) = \varphi(\mu(U)) \le \liminf_{\alpha\in\Gamma} \varphi(\nu(U_\alpha)) = \liminf_{\alpha\in\Gamma} \varphi(\mu)(U_\alpha). \tag{7.1}$$

Since $\varphi(\mu)(U_\alpha) = \inf\{\varphi(\mu)(U_\beta)\colon \beta\in\Gamma, \beta\ge\alpha\}$ for every $\alpha\in\Gamma$, by (7.1) we have $\varphi(\mu)(U) \le \sup_{\alpha\in\Gamma}\varphi(\mu)(U_\alpha) \le \varphi(\mu)(U)$. Thus $\varphi(\mu)$ is totally o-continuous.

The proof of (2) is similar. □

¿From the above propositions we can verify that not a few nonadditive measures are totally o-continuous and totally c-continuous.

Proposition 7.6. *The following are typical examples of nonadditive measures with total continuities.*

(1) *Every Radon nonadditive measure $\mu\colon \mathcal{A} \to [0,\infty]$ that is subadditive or satisfies* $\inf\{\mu(A)\colon A \in \mathcal{A}, A \ne \emptyset\} > 0$ *is totally o-continuous and totally c-continuous.*

(2) *Let X be strongly Lindelöf. Every σ-additive measure $\mu\colon \mathcal{A} \to [0,\infty]$ is totally o-continuous and totally conditional c-continuous.*

(3) *Let $\mu\colon \mathcal{A} \to [0,\infty)$ be a totally o-continuous, finitely additive measure with $M := 2\mu(X)/\pi < \infty$. Then the nonadditive measures $\mu^2, \sqrt{\mu}, \mu^2 + \sqrt{\mu}$, $\tan(\mu/M)$ are totally o-continuous and totally c-continuous. Note that $\mu^2 + \sqrt{\mu}$ is neither subadditive nor superadditive and $\tan(\mu/M)(X) = \tan(\pi/2) = \infty$. Moreover, $\lfloor\mu\rfloor$ is totally c-continuous and $\lceil\mu\rceil$ is totally o-continuous, where $\lfloor t\rfloor$ is the floor function, that is, the largest integer not greater than a real number t and $\lceil t\rceil$ is the ceiling function, that is, the smallest integer not less than t.*

(4) *Every belief measure* Bel *on X is totally continuous from above and thus totally c-continuous, while every plausibility measure* Pla *on X is totally continuous from below and thus totally o-continuous.*

(5) *Every possibility measure* Pos *on X is totally continuous from below and thus totally o-continuous, while every necessity measure* Nec *on X is totally continuous from above and thus totally c-continuous.*

Proof. (1) Every nonadditive measure that is subadditive or satisfies

$$\inf\{\mu(A)\colon A \in \mathcal{A}, A \neq \emptyset\} > 0$$

is autocontinuous from above and below [25, Theorem 6.5], so that it is asymptotic null-additive and asymptotic null-subtractive. Thus by Proposition 7.3 it is totally c-continuous and totally o-continuous.

(2) Every σ-additive measure is continuous from below and conditional continuous from above. Thus by Proposition 7.4 it is totally o-continuous and totally conditional c-continuous.

(3) The increasing functions $\varphi(t) := t^2, \sqrt{t}, t^2 + \sqrt{t}, \tan(t/M)$ on $[0,\infty)$ satisfy $\varphi(0) = 0$. Since they are continuous, by Proposition 7.5 the corresponding nonadditive measures $\mu^2, \sqrt{\mu}, \mu^2 + \sqrt{\mu}, \tan(\mu/M)$ are totally o-continuous and totally c-continuous. In addition, since the floor function $\lfloor t\rfloor$ is upper semicontinuous and the ceiling function $\lceil t\rceil$ is lower semicontinuous, $\lfloor\mu\rfloor$ is totally c-continuous and $\lceil\mu\rceil$ is totally o-continuous.

(4) The proof (BM4) of [25, Theorem 4.13] shows that the belief measure Bel is totally continuous from above. Thus the plausibility measure Pla is totally continuous from below since Pla is the dual of Bel [25, Theorem 4.16].

(5) It is obvious that the possibility measure Pos is totally continuous from below. Thus the necessity measure Nec is totally continuous from above since Nec is the dual of Pos [25, Theorem 4.25]. □

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