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ANOTHER GENERALIZATION OF EDELSTEIN'S FIXED POINT THEOREM IN GENERALIZED METRIC SPACES

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Abstract. We prove another generalization of Edelstein's fixed point theorem in compact ν-generalized metric spaces.

1. Introduction

We define the meaning of "$\{x_n\}_{n=1}^{\mu}{}^{\neq}$" by that $\{x_n\}_{n=1}^{\mu}$ is a finite sequence and $x_1, x_2, \ldots, x_\mu$ are all different. Similarly we define the meaning of "$\{x_n\}_{n\in\mathbb{N}}{}^{\neq}$" by that $\{x_n\}$ is a sequence and $x_1, x_2, \ldots$ are all different.

In 2000, Branciari introduced the following very interesting concept.

Definition 1.1 (Branciari [2])**.** Let X be a set, let d be a function from $X \times X$ into $[0, \infty)$ and let $\nu \in \mathbb{N}$. Then (X, d) is said to be a *ν-generalized metric space* if the following hold:

- (N1) $d(x, y) = 0$ iff $x = y$ for any $x, y \in X$.
- (N2) $d(x, y) = d(y, x)$ for any $x, y \in X$.
- (N3) $d(x, y) \leq D(x, u_1, u_2, \ldots, u_\nu, y)$ for any $x, u_1, u_2, \ldots, u_\nu, y \in X$ such that $x, u_1, u_2, \ldots, u_\nu, y$ are all different, where $D(x, u_1, u_2, \ldots, u_\nu, y) = d(x, u_1) + d(u_1, u_2) + \cdots + d(u_\nu, y)$.

It is obvious that (X, d) is a metric space if and only if (X, d) is a 1-generalized metric space. We found that not every generalized metric space has the compatible topology. See Example 7 in [4] and Example 4.2 in [6]. In [1] and [7], we discussed the completeness and compactness of ν-generalized metric spaces, respectively. See also [5].

In 1962, Edelstein proved the following famous fixed point theorem.

Theorem 1.2 (Edelstein [3])**.** *Let (X, d) be a compact metric space and let T be a mapping on X such that $d(Tx, Ty) < d(x, y)$ for any $x, y \in X$ with $x \neq y$. Then T has a unique fixed point.*

Very recently, we prove the following generalization of Theorem 1.2.

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Theorem 1.3 ([7]). *Let (X,d) be a ν-generalized metric space such that X is compact in the strong sense. Let T be a mapping on X such that $d(Tx,Ty) < d(x,y)$ for any $x,y \in X$ with $x \neq y$. Then T has a unique fixed point z. Moreover for any $x \in X$, $\{T^n x\}$ converges to z in the strong sense.*

In this paper, we prove another generalization of Theorem 1.2, which difffers from Theorem 1.3.

2. Preliminaries

In this section, we give some preliminaries.

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers. For an arbitrary set A, we also denote by $\#A$ the number of the elements of A.

Lemma 2.1. *Let ν be an odd positive integer and let $\mu \in \mathbb{N}$ with $\mu \geq \nu + 3$. Then there exists a positive integer M satisfying the following*:

- *If (X,d) is a ν-generalized metric space, ε is a positive real number and $\{x_j\}_{j=1}^{\mu}{}^{\neq}$ is a finite sequence in X such that $d(x_j, x_{j+1}) \leq \varepsilon$ for any $j = 1,2,\dots,\mu-1$, then*
$$d(x_i, x_j) \leq M\,\varepsilon$$
holds for any $i,j \in \{1,2,\dots,\mu\}$.

Proof. In the case where $\nu = 1$, the conclusion is obvious. So we assume $\nu \geq 3$. We have

$$\begin{aligned}
d(x_1, x_{\nu+2}) &\leq D(x_1, x_2, \dots, x_{\nu+2}) \leq (\nu+1)\,\varepsilon \\
d(x_2, x_{\nu+3}) &\leq D(x_2, x_3, \dots, x_{\nu+3}) \leq (\nu+1)\,\varepsilon \\
d(x_1, x_4) &\leq D(x_1, x_2, x_{\nu+3}, x_{\nu+2}, \dots, x_4) \leq (2\,\nu+1)\,\varepsilon \\
d(x_1, x_6) &\leq D(x_1, x_4, x_3, x_2, x_{\nu+3}, \dots, x_6) \leq (4\,\nu+1)\,\varepsilon \\
d(x_1, x_8) &\leq D(x_1, x_6, \dots, x_2, x_{\nu+3}, \dots, x_8) \leq (6\,\nu+1)\,\varepsilon \\
&\vdots \\
d(x_1, x_{\nu+1}) &\leq D(x_1, x_{\nu-1}, \dots, x_2, x_{\nu+3}, \dots, x_{\nu+1}) \leq (\nu^2 - \nu + 1)\,\varepsilon \\
d(x_1, x_{\nu}) &\leq D(x_1, x_{\nu+2}, x_{\nu+3}, x_2, \dots, x_{\nu}) \leq (3\,\nu+1)\,\varepsilon \\
d(x_1, x_{\nu-2}) &\leq D(x_1, x_{\nu}, \dots, x_{\nu+3}, x_2, \dots, x_{\nu-2}) \leq (5\,\nu+1)\,\varepsilon \\
&\vdots \\
d(x_1, x_3) &\leq D(x_1, x_5, \dots, x_{\nu+3}, x_2, x_3) \leq (\nu^2+1)\,\varepsilon
\end{aligned}$$

by (N3). Putting $N_1 = \nu^2 + 1$, we have
$$d(x_1, x_j) \leq N_1\,\varepsilon$$
for any $j \in \{1,2,\dots,\nu+2\}$. We have
$$d(x_1, x_j) \leq D(x_1, x_{j-\nu}, \dots, x_j) \leq (N_1 + \nu)\,\varepsilon$$
for any $j \in \{\nu+3, \dots, 2\,\nu+2\}$. Continuing this calculations, we have
$$d(x_1, x_j) \leq D(x_1, x_{j-\nu}, \dots, x_j) \leq (N_1 + k\,\nu)\,\varepsilon$$

for any $j, k \in \mathbb{N}$ with $k\,\nu + 2 < j \leq (k+1)\,\nu + 2$ and $j \leq \mu$. Therefore there exists $N_2 \in \mathbb{N}$ satisfying

$$d(x_1, x_j) \leq N_2\,\varepsilon$$

for any $j \in \{1, 2, \ldots, \mu\}$. Similarly we can prove

$$d(x_\mu, x_j) \leq N_2\,\varepsilon$$

for any $j \in \{1, 2, \ldots, \mu\}$. We put $M = 3\,N_2 + \nu - 2$ and fix $i, j \in \mathbb{N}$ with $1 < i < i + 1 < j < \mu$. We consider the following three cases:

- $i \geq \nu + 1$
- $i < \nu + 1$ and $j \geq \nu + 2$
- $i < \nu + 1$ and $j < \nu + 2$

In the first case, we have

$$d(x_i, x_j) \leq D(x_i, \ldots, x_{i-\nu+1}, x_1, x_j) \leq (2\,N_2 + \nu - 1)\,\varepsilon \leq M\,\varepsilon.$$

In the second case, we have

$$d(x_i, x_j) \leq D(x_i, \ldots, x_1, x_{i+j-\nu-1}, \ldots, x_j) \leq (N_2 + \nu)\,\varepsilon \leq M\,\varepsilon.$$

In the third case, we have

$$d(x_i, x_j) \leq D(x_i, \ldots, x_1, x_{i+1}, \ldots, x_{j-1}, x_\mu, x_{\nu+1}, \ldots, x_j) \leq M\,\varepsilon.$$

Noting M does not depend on (X, d) and $\{x_j\}_{j=1}^{\mu}$, we completes the proof. □

Lemma 2.2. *Let ν be an even positive integer and let $\mu \in \mathbb{N}$ with $\mu \geq \nu + 3$. Then there exists a positive integer M satisfying the following*:

- *If (X, d) is a ν-generalized metric space, ε is a positive real number and $\{x_j\}_{j=1}^{\mu\,\neq}$ is a finite sequence in X such that $d(x_j, x_{j+1}) \leq \varepsilon$ for any $j = 1, 2, \ldots, \mu - 1$, then*
$$d(x_i, x_j) \leq M\,\varepsilon$$
holds for any $i, j \in \{1, 2, \ldots, \mu\}$ such that $i - j$ is odd.

Proof. We have

$$d(x_1, x_{\nu+2}) \leq D(x_1, \ldots, x_{\nu+2}) \leq (\nu + 1)\,\varepsilon$$

by (N3). For the case where $\nu \geq 4$, we further have

$$\begin{aligned} d(x_2, x_{\nu+3}) &\leq D(x_2, \ldots, x_{\nu+3}) \leq (\nu + 1)\,\varepsilon \\ d(x_1, x_4) &\leq D(x_1, x_2, x_{\nu+3}, \ldots, x_4) \leq (2\,\nu + 1)\,\varepsilon \\ &\vdots \\ d(x_1, x_\nu) &\leq D(x_1, x_{\nu-2}, \ldots, x_2, x_{\nu+3}, \ldots, x_\nu) \leq (\nu^2 - 2\,\nu + 1)\,\varepsilon. \end{aligned}$$

Putting $N_1 = \max\{\nu^2 - 2\,\nu + 1, \nu + 1\}$, we have

$$d(x_1, x_j) \leq N_1\,\varepsilon$$

for any $j \in \{2, \dots, \nu+2\}$ such that j is even. As in the proof of Lemma 2.1, there exists $N_2 \in \mathbb{N}$ satisfying

$$d(x_1, x_j) \leq N_2 \, \varepsilon$$

for any $j \in \{2, \dots, \mu\}$ such that j is even. We consider the following two cases:

- μ is odd.
- μ is even.

In the case where μ is odd, we can prove

$$d(x_\mu, x_j) \leq N_2 \, \varepsilon$$

for any $j \in \{2, \dots, \mu - 1\}$ such that j is even. We put $M = 3\, N_2 + \nu - 2$ and fix $i, j \in \mathbb{N}$ such that $1 < i < i+1 < j < \mu$ and $i - j$ is odd. Without loss of generality, we may assume that i is odd and j is even. We consider the following three cases:

- $i \geq \nu + 1$
- $i < \nu + 1$ and $j \geq \nu + 2$
- $i < \nu + 1$ and $j < \nu + 2$

In the first and second cases, we have

$$d(x_i, x_j) \leq M \, \varepsilon.$$

as in the proof of Lemma 2.1. In the third case, noting $i + 3 \leq j$, we have

$$d(x_i, x_j) \leq D(x_i, \dots, x_1, x_{i+1}, \dots, x_{j-2}, x_\mu, x_{\nu+2}, \dots, x_j) \leq M \, \varepsilon.$$

So we obtain the desired result in the case where μ is odd. In the other case, where μ is even, noting $\mu - 1 \geq \nu + 3$, we have

$$d(x_i, x_j) \leq M \, \varepsilon$$

for any $i, j \in \mathbb{N}$ satisfying either of the following:

- $i, j \in \{1, \dots, \mu - 1\}$ and $i - j$ is odd.
- $i, j \in \{2, \dots, \mu\}$ and $i - j$ is odd.

So we only have to consider the case where $i = 1$ and $j = \mu$. In this case we have

$$d(x_1, x_\mu) \leq N_2 \, \mu \leq M \, \varepsilon.$$

Noting M does not depend on (X, d) and $\{x_j\}_{j=1}^{\mu}$, we completes the proof. □

As mentioned in Section 1, in general, ν-generalized metric spaces do not necessarily have the compatible topology; see [4, 6]. So we have to define the concept of the convergence and so on.

Definition 2.3 ([2, 7])**.** Let (X, d) be a ν-generalized metric space.

- A sequence $\{x_n\}$ in X is said to be *Cauchy* iff $\lim_n \sup_{m>n} d(x_m, x_n) = 0$ holds.
- A sequence $\{x_n\}$ in X is said to *converge* to x iff $\lim_n d(x, x_n) = 0$ holds.
- A sequence $\{x_n\}$ in X is said to *converge exclusively* to x iff $\lim_n d(x, x_n) = 0$ holds and $\lim_n d(y, x_{f(n)}) = 0$ does not hold for any $y \in X \setminus \{x\}$ and for any subsequence $\{x_{f(n)}\}$ of $\{x_n\}$.
- A sequence $\{x_n\}$ in X is said to *converge to x in the strong sense* iff $\{x_n\}$ is Cauchy and $\{x_n\}$ converges to x.

Definition 2.4 ([7]). Let (X, d) be a ν-generalized metric space.

- X is *compact* iff for any sequence $\{x_n\}$ in X, there exists a subsequence $\{x_{f(n)}\}$ of $\{x_n\}$ converging to some $z \in X$.
- X is *compact in the strong sense* iff for any sequence $\{x_n\}$ in X, there exists a subsequence $\{x_{f(n)}\}$ of $\{x_n\}$ converging to some $z \in X$ in the strong sense.

Lemma 2.5. *Let (X, d) be a 2-generalized metric space and let T be a mapping on X such that*

$$d(Tx, Ty) \leq d(x, y) \tag{2.1}$$

for any $x, y \in X$. Let $\{x_n\}$ be a sequence in X converging to some $v \in X$. Then $\{Tx_n\}$ converges to Tv. Moreover, if $Tx_n \neq x_n$ and $Tx_n \neq Tv$ for sufficiently large $n \in \mathbb{N}$ and $Tv \neq v$, then

$$d(v, Tv) = \lim_{n\to\infty} d(x_n, Tx_n) \tag{2.2}$$

holds.

3. Main results

In this section, we prove our main results. We begin with the case where $\nu = 2$.

Theorem 3.1. *Let (X, d) be a compact 2-generalized metric space. Let T be a mapping on X such that $d(Tx, Ty) < d(x, y)$ for any $x, y \in X$ with $x \neq y$. Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for any $x \in X$.*

Remark. This theorem is a correction of Theorem 3.1 in [7]. We wrote "$\{T^n x\}$ converges *exclusively* to z" in [7], which is not correct.

Proof. We have proved that T has a unique fixed point z in [7]. We note (2.1) holds for any $x, y \in X$. Fix $x \in X$. Since $\{d(T^n x, z)\}$ is nonincreasing, it converges to some nonnegative real number β. Arguing by contradiction, we assume $\beta > 0$. Then $T^n x \neq z$ holds for any $n \in \mathbb{N}$. So $\{d(T^n x, T^{n+1} x)\}$ is strictly decreasing and hence it converges to some nonegative real number γ and $T^n x$ are all different. Since X is compact, there exists a subsequence $\{T^{f(n)} x\}$ of $\{T^n x\}$ converging to some $v \in X$. By Lemma 2.5 (2.1), $\{T^{f(n)+1} x\}$ and $\{T^{f(n)+2} x\}$ converge to Tv and $T^2 v$, respectively. Since

$$\lim_{n\to\infty} d(T^{f(n)} x, z) = \lim_{n\to\infty} d(T^{f(n)+1} x, z) = \beta > 0,$$

we note $v \neq z$ and $Tv \neq z$, thus, $v \neq Tv$ and $Tv \neq T^2 v$ holds. By Lemma 2.5 (2.2), we have

$$d(Tv, T^2 v) = d(v, Tv) = \gamma,$$

which implies a contradiction. Therefore we obtain $\beta = 0$. □

We give a proof in the case where $\nu \geq 3$.

Theorem 3.2. *Let (X, d) be a compact ν-generalized metric space. Let T be a mapping on X such that*

$$d(Tx, Ty) < d(x, y)$$

for any $x, y \in X$ with $x \neq y$. Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for any $x \in X$.

Proof. In the case where $\nu = 1, 2$, we have already proved this theorem (Theorems 1.2 and 3.1). So we assume $\nu \geq 3$. We first show that there exists at most one fixed point of T. If $z, w \in X$ are distinct fixed points of T, then we have

$$d(z, w) = d(Tz, Tw) < d(z, w),$$

which implies a contradiction. So we have shown that there exists at most one fixed point of T. Fix $u \in X$ and define a sequence $\{u_n\}$ in X by $u_n = T^n u$ for $n \in \mathbb{N}$. We next show that $\{u_n\}$ converges to a fixed point of T, dividing the following three cases:

(i) There exists $k \in \mathbb{N}$ such that $u_{k+1} = u_k$.
(ii) $u_{n+1} \neq u_n$ for all $n \in \mathbb{N}$ and there exist $k, \ell \in \mathbb{N}$ such that $k + 2 \leq \ell$ and $u_k = u_\ell$.
(iii) $\{u_n\}_{n\in\mathbb{N}}{}^{\neq}$.

In the case of (i), u_k is a fixed point of T and $\{u_n\}$ converges to u_k. In the case of (ii), $\{d(u_n, u_{n+1})\}$ is strictly decreasing. So, since $u_{k+1} = u_{\ell+1}$, we have

$$d(u_k, u_{k+1}) = d(u_\ell, u_{\ell+1}) < d(u_k, u_{k+1}),$$

which implies a contradiction. Thus, the case of (ii) cannot be possible. In the case of (iii), since X is compact, there exists a subsequence of $\{T^n x\}$ converging to $z \in X$. Put $\mu = \nu + 3$ and let M be as in Lemmas 2.1 and 2.2. Define a sequence $\{z_n\}$ in X by $z_n = T^n z$ for $n \in \mathbb{N} \cup \{0\}$. Arguing by contradiction, we assume the following:

- z_n is not a fixed point of T for any $n \in \mathbb{N} \cup \{0\}$.

Thus, $\{z_n\}_{n\in\mathbb{N}\cup\{0\}}{}^{\neq}$. We further consider the following three cases:

(iii-a) $\{u_n : n \in \mathbb{N}\} \cap \big\{z_n : n \in \mathbb{N} \cup \{0\}\big\} = \varnothing$ and ν is odd.
(iii-b) $\{u_n : n \in \mathbb{N}\} \cap \big\{z_n : n \in \mathbb{N} \cup \{0\}\big\} = \varnothing$ and ν is even.
(iii-c) $\{u_n : n \in \mathbb{N}\} \cap \big\{z_n : n \in \mathbb{N} \cup \{0\}\big\} \neq \varnothing$.

In the case of (iii-a), we let $\varepsilon > 0$ satisfy

$$d(Tz, T^2 z) + (M + \nu - 1)\,\varepsilon < d(z, Tz). \tag{3.1}$$

Then we can choose $k, \ell \in \mathbb{N}$ satisfying

$$k + 2 \leq \ell, \qquad d(u_k, z) < \varepsilon \quad \text{and} \quad d(u_\ell, z) < \varepsilon.$$

Then we have

$$d(x_{j\,\ell-(j-1)\,k}, z_{j\,\ell-j\,k}) < d(x_k, z) < \varepsilon$$

and

$$d(x_{(j+1)\,\ell-j\,k}, z_{j\,\ell-j\,k}) < d(x_\ell, z) < \varepsilon$$

for any $j \in \mathbb{N}$. Define a finite sequence $\{y_n\}_{n=1}^{\mu}{}^{\neq}$ by

$$\begin{array}{llll} y_1 = z, & y_2 = x_\ell, & y_3 = z_{\ell-k}, & y_4 = x_{2\,\ell-k}, \\ y_5 = z_{2\,\ell-2\,k}, & y_6 = x_{3\,\ell-2\,k}, & y_7 = z_{3\,\ell-3\,k}, & y_8 = x_{4\,\ell-3\,k}, \\ \ldots, & y_\mu = x_{(\nu+3)\,\ell/2-(\nu+1)\,k/2}. & & \end{array}$$

Since $d(y_j, y_{j+1}) < \varepsilon$ for any j, we have $d(y_i, y_j) < M\,\varepsilon$ for any i, j by Lemma 2.1. In the case where $\nu = 3$, we have

$$\begin{aligned} d(z, Tz) &\le D(y_1 = z, y_3 = z_{\ell-k}, Tz_{\ell-k}, Tx_\ell, Tz) \\ &< M\,\varepsilon + d(z_{\ell-k}, Tz_{\ell-k}) + d(Tz_{\ell-k}, Tx_\ell) + d(Tx_\ell, Tz) \\ &< M\,\varepsilon + d(Tz, T^2 z) + d(z_{\ell-k}, x_\ell) + d(x_\ell, z) \\ &< d(Tz, T^2 z) + (M+2)\,\varepsilon \\ &< d(z, Tz), \end{aligned}$$

which implies a contradiction. In the other case, where $\nu \ge 5$, we have

$$\begin{aligned} d(z, Tz) &\le D(y_1, y_\nu, \dots, y_3, Tz_{\ell-k}, Tx_\ell, Tz) \\ &< (M+\nu-3)\,\varepsilon + d(z_{\ell-k}, Tz_{\ell-k}) + d(Tz_{\ell-k}, Tx_\ell) + d(Tx_\ell, Tz) \\ &< d(Tz, T^2 z) + (M+\nu-1)\,\varepsilon < d(z, Tz), \end{aligned}$$

which implies a contradiction. In the case of (iii-b), we let ε, k, ℓ and $\{y_n\}_{n=1}^{\mu}{}^{\neq}$ be as in the case of (iii-a). By Lemma 2.2, we have $d(y_i, y_j) < M\,\varepsilon$ for any i, j such that $i - j$ is odd. Noting that $1 - \nu$ is odd, we have

$$\begin{aligned} d(z, Tz) &\le D(y_1, y_\nu, \dots, y_3, Tz_{\ell-k}, Tx_\ell, Tz) \\ &< (M+\nu-3)\,\varepsilon + d(z_{\ell-k}, Tz_{\ell-k}) + d(Tz_{\ell-k}, Tx_\ell) + d(Tx_\ell, Tz) \\ &< d(Tz, T^2 z) + (M+\nu-1)\,\varepsilon < d(z, Tz), \end{aligned}$$

which implies a contradiction. In the case of (iii-c), we let $\varepsilon > 0$ satisfy (3.1). Then we can choose $\ell \in \mathbb{N}$ satisfying $2 \le \ell$ and $d(z_\ell, z) < \varepsilon$. Then we have

$$d(z_{j\,\ell}, z_{(j-1)\ell}) \le d(z_\ell, z) < \varepsilon$$

for any $j \in \mathbb{N}$. Define a finite sequence $\{y_n\}_{n=1}^{\mu}{}^{\neq}$ by $y_n = z_{(n-1)\,\ell}$. By Lemmas 2.1 and 2.2, we have $d(y_i, y_j) < M\,\varepsilon$ for any i, j (such that $i - j$ is odd, in the case where ν is even). We have

$$\begin{aligned} d(z, Tz) &\le D(z = y_1, y_\nu, \dots, y_3 = z_{2\,\ell}, Tz_{2\,\ell}, Tz_\ell, Tz) \\ &< (M+\nu-3)\,\varepsilon + d(z_{2\,\ell}, Tz_{2\,\ell}) + d(Tz_{2\,\ell}, Tz_\ell) + d(Tz_\ell, Tz) \\ &< d(Tz, T^2 z) + (M+\nu-1)\,\varepsilon < d(z, Tz), \end{aligned}$$

which implies a contradiction. Therefore we obtain that z_κ is a fixed point for some $\kappa \in \mathbb{N} \cup \{0\}$. Since

$$\liminf_{n\to\infty} d(z_\kappa, x_n) \le \liminf_{n\to\infty} d(z, x_{n-\kappa}) = 0$$

and $\{d(z_\kappa, x_n)\}$ is nonincreasing, $\{x_n\}$ itself converges to z_κ. □

Remark. We can prove Theorem 3.1 by a similar method of proof of Theorem 3.2. Indeed, in the case of (iii-b), we have

$$\begin{aligned} d(z, Tz) &\le D(y_1, y_4, y_3, Tz) \\ &\le D(y_1, y_4, y_3, Tz_{\ell-k}, Tx_\ell, Tz) \\ &< d(Tz, T^2 z) + (M+\nu+1)\,\varepsilon. \end{aligned}$$

In the case of (iii-c), we have

$$\begin{aligned} d(z, Tz) &\leq D(y_1, y_4, y_3, Tz) \\ &\leq D(y_1, y_4, y_3, Tz_{2\ell}, Tz_\ell, Tz) \\ &< d(Tz, T^2 z) + (M + \nu + 1)\,\varepsilon. \end{aligned}$$

Using these inequalities, we can prove Theorem 3.1.

4. Example

In order to show that Theorem 3.2 is new, we give an example of ν-generalized metric space which is compact, however, which is not compact in the strong sense.

Proposition 4.1. *Let (X, d) be a ν-generalized metric space and let $\lambda \in \mathbb{N}$ such that λ is divisible by ν. Then (X, d) is a λ-generalized metric space.*

Proof. We only have to show (N3). Let $\{x_n\}_{n=1}^{\lambda+2\neq}$ be a finite sequence in X. Then we have

$$\begin{aligned} d(x_1, x_{\nu+2}) &\leq D(x_1, \dots, x_{\nu+2}) \\ d(x_1, x_{2\nu+2}) &\leq D(x_1, x_{\nu+2}, \dots, x_{2\nu+2}) \leq D(x_1, \dots, x_{2\nu+2}) \\ &\vdots \\ d(x_1, x_{\lambda+2}) &\leq D(x_1, x_{\lambda-\nu+2}, \dots, x_{\lambda+2}) \leq D(x_1, \dots, x_{\lambda+2}). \end{aligned}$$

Therefore we obtain (N3). □

Lemma 4.2. *Let $\nu \in \mathbb{N}$ be odd. Let (X, ρ) be a bounded metric space and let M be a real number satisfying*

$$\sup\left\{\rho(x, y) : \ x, y \in X\right\} \leq M.$$

Let A and B be two subsets of X with $A \cap B = \varnothing$. Assume $\#A \leq (\nu - 1)/2$. Define a function d from $X \times X$ into $[0, \infty)$ by

$$\begin{aligned} &d(x, x) = 0 \\ &d(x, y) = d(y, x) = \rho(x, y) && \text{if } x \in A \text{ and } y \in B \\ &d(x, y) = M && \text{otherwise.} \end{aligned}$$

Then (X, d) is a ν-generalized metric space.

Proof. We only have to show (N3). Let $\{x_n\}_{n=1}^{\nu+2\neq}$ be a finite sequence in X. Then we have

$$d(x_1, x_{\nu+2}) \leq M \leq D(x_1, \dots, x_{\nu+2}).$$

Therefore we obtain (N3). □

Lemma 4.3. *Let $\nu \in \mathbb{N}$ be even. Let (X, ρ) be a bounded metric space and let A and B be two subsets of X with $A \cap B = \varnothing$. Let M and d be as in Lemma 4.2. Then (X, d) is a ν-generalized metric space.*

Proof. We have already proved that (X, d) is a 2-generalized metric space; see Lemma 4 in [4] and Remark 17 in [1]. By Proposition 4.1, we obtain the desired result. □

Example 4.4. Put $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ and define a function ρ from $X \times X$ into $[0, \infty)$ by $\rho(x, y) = |x - y|$. Put $A = \{0\}$, $B = \{1/n : n \in \mathbb{N}\}$ and $M = 1$. Define a function d as in Lemmas 4.2 and 4.3. Then the following holds:

(i) (X, d) is a ν-generalized metric space for any $\nu \in \mathbb{N}$ with $\nu \geq 2$.
(ii) X is compact
(iii) X is not compact in the strong sense.

Proof. By Lemmas 4.2 and 4.3, (X, d) is a ν-generalized metric space for any $\nu \in \mathbb{N}$ with $\nu \geq 2$. We next show (ii). Let $\{x_n\}$ be a sequence in X. We consider the following two cases:

- There exists $y \in X$ such that $\#\{n : x_n = y\} = \infty$.
- For any $y \in X$, $\#\{n : x_n = y\} < \infty$.

In the first case, there exists a subsequence of $\{x_n\}$ converging to y. In the second case, $\{x_n\}$ itself converges to 0. Therefore X is compact. Let us prove (iii). Define a sequence $\{x_n\}$ in X by $x_n = 1/n$. Then $\{x_n\}$ converges exclusively to 0. Since $d(x_m, x_n) = M$ for any $m, n \in \mathbb{N}$ with $m \neq n$, there does not exist a subsequence of $\{x_n\}$ which is Cauchy. Therefore X is not compact in the strong sense. □

References

[1] B. Alamri, T. Suzuki and L. A. Khan, *Caristi's fixed point theorem and Subrahmanyam's fixed point theorem in ν-generalized metric spaces*, J. Funct. Spaces **2015**, 2015:709391, 6 pp.
[2] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publ. Math. Debrecen **57** (2000), 31–37.
[3] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. **37** (1962), 74–79.
[4] T. Suzuki, *Generalized metric spaces do not have the compatible topology*, Abstr. Appl. Anal. **2014**, 2014:458098, 5 pp.
[5] T. Suzuki, B. Alamri and L. A. Khan, *Some notes on fixed point theorems in ν-generalized metric spaces*, Bull. Kyushu Inst. Technol. **62** (2015), 15–23.
[6] T. Suzuki, B. Alamri and M. Kikkawa, *Only 3-generalized metric spaces have a compatible symmetric topology*, Open Math. **13** (2015), 510–517.
[7] T. Suzuki, B. Alamri and M. Kikkawa, *Edelstein's fixed point theorem in generalized metric spaces*, J. Nonlinear Convex Anal. **16** (2015), 2301–2309.

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