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GENERALIZED TRACE INEQUALITIES RELATED TO FIDELITY AND TRACE DISTANCE

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Abstract. Recently in [8] we obtained non-hermitian extensions of Heisenberg type and Schrödinger type uncertainty relations for generalized metric adjusted skew information or generalized metric adjusted correlation measure and gave the results of Dou-Du ([3,4])as corollaries. In this paper we define generalized quasi-metric adjusted skew information for different two generalized states and obtain corresponding uncertainty relation. The result is applied to the inequalities related to fidelity and trace distance for different two generalized states which were given by Audenaert et al; and Powers-Stϕrmer ([1,2,5]).

1. Introduction

Let $M_n(\mathbb{C})$(resp. $M_{n,sa}(\mathbb{C})$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices), endowed with the Hilbert-Schmidt scalar product $\langle X, Y \rangle = Tr[X^*Y]$. Let $M_{n,+}(\mathbb{C})$ be the set of strictly positive elements of $M_n(\mathbb{C})$. A function $f : (0, +\infty) \to \mathbb{R}$ is said operator monotone if, for any $n \in \mathbb{N}$, and $A, B \in M_{n,+}(\mathbb{C})$ such that $0 \leq A \leq B$, the inequality $0 \leq f(A) \leq f(B)$ holds. An operator monotone function is said symmetric if $f(x) = xf(x^{-1})$ and normalized if $f(1) = 1$.

Definition 1.1. Let $\mathfrak{F}_{op}$ be the class of functions $f : (0, +\infty) \to (0, +\infty)$ satisfying
(1) $f(1) = 1$,
(2) $tf(t^{-1}) = f(t)$,
(3) f is operator monotone.

For $f \in \mathfrak{F}_{op}$ define $f(0) = \lim_{x \to 0} f(x)$. We introduce the sets of regular and non-regular functions

$$\mathfrak{F}_{op}^r = \{f \in \mathfrak{F}_{op} | f(0) \neq 0\}, \quad \mathfrak{F}_{op}^n = \{f \in \mathfrak{F}_{op} | f(0) = 0\}$$

and notice that trivially $\mathfrak{F}_{op} = \mathfrak{F}_{op}^r \cup \mathfrak{F}_{op}^n$. In Kubo-Ando theory of matrix means one associates a mean to each operator monotone function $f \in \mathfrak{F}_{op}$ by the formula

$$m_f(A, B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},$$

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where $A, B \in M_{n,+}(\mathbb{C})$. By using the notion of matrix means we define the generalized monotone metrics for $X, Y \in M_n(\mathbb{C})$ by the following formula

$$\langle X, Y\rangle_f = Tr[X^* m_f(L_A, R_B)^{-1} Y],$$

where $L_A(X) = AX, R_B(X) = XB$.

2. Generalized quasi-metric adjusted skew information and correlation measure

Definition 2.1. Let $g, f \in \mathfrak{F}_{op}^r$ satisfy

$$g(x) \geq k\frac{(x-1)^2}{f(x)}$$

for some $k > 0$. We define

$$\Delta_g^f(x) = g(x) - k\frac{(x-1)^2}{f(x)} \in \mathfrak{F}_{op}. \tag{2.1}$$

Definition 2.2. Notation as in Definition 2.1. For $X, Y \in M_n(\mathbb{C})$ and $A, B \in M_{n,+}(\mathbb{C})$, we define the following quantities:

(1) $\Gamma_{A,B}^{(g,f)}(X,Y) = k\langle (L_A - R_B)X, (L_A - R_B)Y\rangle_f$
$= kTr[X^*(L_A - R_B)m_f(L_A, R_B)^{-1}(L_A - R_B)Y]$
$= Tr[X^* m_g(L_A, R_B)Y] - Tr[X^* m_{\Delta_g^f}(L_A, R_B)Y]$,

(2) $I_{A,B}^{(g,f)}(X) = \Gamma_{A,B}^{(g,f)}(X,X)$,

(3) $\Psi_{A,B}^{(g,f)}(X,Y) = Tr[X^* m_g(L_A, R_B)Y] + Tr[X^* m_{\Delta_g^f}(L_A, R_B)Y]$,

(4) $J_{A,B}^{(g,f)}(X) = \Psi_{A,B}^{(g,f)}(X,X)$,

(5) $U_\rho^{(g,f)}(X) = \sqrt{I_{A,B}^{(g,f)}(X) J_{A,B}^{(g,f)}(X)}$.

The quantities $I_{A,B}^{(g,f)}(X)$ and $\Gamma_{A,B}^{(g,f)}(X,Y)$ are said generalized quasi-metric adjusted skew information and generalized quasi-metric adjusted correlation measure, respectively.

Theorem 2.3 (Schrödinger type)**.** *For $f \in \mathfrak{F}_{op}^r$, it holds*

$$I_{A,B}^{(g,f)}(X) \cdot I_{A,B}^{(g,f)}(Y) \geq |\Gamma_{A,B}^{(g,f)}(X,Y)|^2,$$

where $X, Y \in M_n(\mathbb{C})$ and $A, B \in M_{n,+}(\mathbb{C})$.

We use only Schwarz inequality to prove Theorem 2.3 by a similar way as the proof of Theorem 2 in [8]. We note the equation

$$|L_A - R_B| = \sum_{i=1}^n \sum_{j=1}^n |\lambda_i - \mu_j| L_{|\phi_i\rangle\langle\phi_i|} R_{|\psi_j\rangle\langle\psi_j|},$$

where $A = \sum_{i=1}^n \lambda_i |\phi_i\rangle\langle\phi_i|$, $B = \sum_{j=1}^n \mu_j |\psi_j\rangle\langle\psi_j|$ are the spectral decompositions.

Theorem 2.4 (Heisenberg type). *For $f \in \mathfrak{F}_{op}^r$, if*

$$g(x) + \Delta_g^f(x) \geq \ell f(x) \tag{2.2}$$

for some $\ell > 0$, then it holds

$$U_{A,B}^{(g,f)}(X) \cdot U_{A,B}^{(g,f)}(Y) \geq k\ell |Tr[X^*|L_A - R_B|Y]|^2,$$

where $X, Y \in M_n(\mathbb{C})$ and $A, B \in M_{n,+}(\mathbb{C})$. In particular,

$$\begin{aligned} k\ell(Tr[X^*|L_A - R_B|X])^2 \leq & \, Tr[X^*(m_g(L_A, R_B) - m_{\Delta_g^f}(L_A, R_B))X] \\ & \times Tr[X^*(m_g(L_A, R_B) + m_{\Delta_g^f}(L_A, R_B))X], \end{aligned} \tag{2.3}$$

where $X \in M_n(\mathbb{C})$ and $A, B \in M_{n,+}(\mathbb{C})$.

We use refined Schwarz inequality to prove Theorem 2.4 by a similar way as the proof of Theorem 3 in [8].

3. Trace inequalities

We assume that

$$g(x) = \frac{x+1}{2}, \; f(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}, \; k = \frac{f(0)}{2}, \; \ell = 2.$$

Then, since (2.1), (2.2) are satisfied for g, f, k and ℓ, we have the following trace inequality by putting $X = I$ in (2.3).

$$\begin{aligned} & \alpha(1-\alpha)(Tr[|L_A - R_B|I])^2 \\ \leq & \left(\frac{1}{2}Tr[A+B]\right)^2 - \left(\frac{1}{2}Tr[A^\alpha B^{1-\alpha} + A^{1-\alpha}B^\alpha]\right)^2. \end{aligned} \tag{3.1}$$

This is a generalization of trace inequality given in [2]. And also we give the following new inequality by combining the Chernoff type inequality with the above theorem.

Theorem 3.1. *We have the following:*

$$\begin{aligned} & \frac{1}{2}Tr[A + B - |L_A - R_B|I] \leq \inf_{0 \leq \alpha \leq 1} Tr[A^{1-\alpha}B^\alpha] \\ \leq & \; Tr[A^{1/2}B^{1/2}] \leq \frac{1}{2}Tr[A^\alpha B^{1-\alpha} + A^{1-\alpha}B^\alpha] \\ \leq & \sqrt{\left(\frac{1}{2}Tr[A+B]\right)^2 - \alpha(1-\alpha)(Tr[|L_A - R_B|I)^2}. \end{aligned}$$

We need the following lemma in order to prove Theorem 3.1.

Lemma 3.2. *Let $f(s) = Tr[A^{1-s}B^s]$ for $A, B \in M_n(\mathbb{C})$ and $0 \leq s \leq 1$. Then $f(s)$ is convex in s.*

Proof of Lemma 3.2. $f'(s) = Tr[-A^{1-s}\log AB^s + A^{1-s}B^s \log B]$. And then

$$\begin{aligned} f''(s) &= Tr[A^{1-s}(\log A)^2 B^s - A^{1-s}\log AB^s \log B] \\ & \quad - Tr[A^{1^s}\log AB^s \log B - A^{1-s}B^s(\log B)^2] \\ &= Tr[A^{1-s}(\log A)^2 B^s] - Tr[A^{1-s}\log A \log BB^s] \end{aligned}$$

$$\begin{aligned}
&-Tr[\log B\log AA^{1-s}B^s]+Tr[A^{1-s}(\log B)^2B^s]\\
=\ &Tr[A^{1-s}\log A(\log A-\log B)B^s]-Tr[A^{1-s}(\log A-\log B)\log BB^s]\\
=\ &Tr[A^{1-s}(\log A-\log B)B^s\log A]-Tr[A^{1-s}(\log A-\log B)\log BB^s]\\
=\ &Tr[A^{1-s}(\log A-\log B)B^s(\log A-\log B)]\\
=\ &Tr[A^{(1-s)/2}(\log A-\log B)B^s(\log A-\log B)A^{(1-s)/2}]\geq 0.
\end{aligned}$$

Then $f(s)$ is convex in s. □

Proof of Theorem 3.1. The third and fourth inequalities follow from Lemma 3.2 and (3.1), respectively. So we may only prove

$$Tr[A+B-|L_A-R_B|I]\leq 2Tr[A^{1-\alpha}B^\alpha]\quad (0\leq\alpha\leq 1).$$

Let

$$A=\sum_i\lambda_i|\phi_i\rangle\langle\phi_i|=\sum_{i,j}\lambda_i|\phi_i\rangle\langle\phi_i|\psi_j\rangle\langle\psi_j|,$$

$$B=\sum_j\mu_j|\psi_j\rangle\langle\psi_j|=\sum_{i,j}\mu_j|\phi_i\rangle\langle\phi_i|\psi_j\rangle\langle\psi_j|.$$

Then we have

$$Tr[A]=\sum_{i,j}\lambda_i|\langle\phi_i|\psi_j\rangle|^2,Tr[B]=\sum_{i,j}\mu_j|\langle\phi_i|\psi_j\rangle|^2.$$

And since

$$|L_A-R_B|=\sum_{i,j}|\lambda_i-\mu_j|L_{|\phi_i\rangle\langle\phi_i|}R_{|\psi_j\rangle\langle\psi_j|},$$

we have

$$|L_A-R_B|I=\sum_{i,j}|\lambda_i-\mu_j||\phi_i\rangle\langle\phi_i|\psi_j\rangle\langle\psi_j|.$$

Then we have

$$Tr[|L_A-R_B|I]=\sum_{i,j}|\lambda_i-\mu_j||\langle\phi_i|\psi_j\rangle|^2.$$

Therefore

$$Tr[A+B-|L_A-R_B|I]=\sum_{i,j}(\lambda_i+\mu_j-|\lambda_i-\mu_j|)|\langle\phi_i|\psi_j\rangle|^2.$$

On the other hand since

$$A^\alpha=\sum_i\lambda_i^\alpha|\phi_i\rangle\langle\phi_i|=\sum_{i,j}\lambda_i^\alpha|\phi_i\rangle\langle\phi_i|\psi_j\rangle\langle\psi_j|,$$

$$B^{1-\alpha}=\sum_j\mu_j^{1-\alpha}|\psi_j\rangle\langle\psi_j|=\sum_{i,j}\mu_j^{1-\alpha}|\phi_i\rangle\langle\phi_i|\psi_j\rangle\langle\psi_j|,$$

we have

$$A^\alpha B^{1-\alpha}=\sum_{i,j}\lambda_i^\alpha\mu_j^{1-\alpha}|\phi_i\rangle\langle\phi_i|\psi_j\rangle\langle\psi_j|.$$

Then

$$Tr[A^\alpha B^{1-\alpha}]=\sum_{i,j}\lambda_i^\alpha\mu_j^{1-\alpha}|\langle\phi_i|\psi_j\rangle|^2.$$

Thus

$$\begin{aligned} & 2Tr[A^{\alpha}B^{1-\alpha}] - Tr[A+B-|L_A-R_B|I] \\ = & \sum_{i,j}\{2\lambda_i^{\alpha}\mu_j^{1-\alpha} - (\lambda_i+\mu_j-|\lambda_i-\mu_j|)\}|\langle\phi_i|\psi_j\rangle|^2. \end{aligned}$$

Since $2x^{\alpha}y^{1-\alpha} - (x+y-|x-y|) \geq 0$ for $x, y > 0, 0 \leq \alpha \leq 1$ in general, we can get the result. □

Remark 3.3. There are no relationship between $Tr[|A-B|]$ and $Tr[|L_A-R_B|I]$. For example, let

$$A = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $Tr[L_A-R_B|I] = 3$ and $Tr[|A-B|] = \sqrt{10}$.
On the other hand let

$$A = \begin{pmatrix} \frac{13}{2} & \frac{7}{2} \\ \frac{7}{2} & \frac{13}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}.$$

Then $Tr[|L_A-R_B|I] = 8$ and $Tr[|A-B|] = \sqrt{58}$.

Finally we give a refinement of inequality about fidelity.

Theorem 3.4.

$$Tr[|A^{1/2}B^{1/2}|] \geq \frac{1}{1+\sqrt{\lambda_0}}Tr[A] + \frac{\sqrt{\lambda_0}}{1+\sqrt{\lambda_0}}\left(\frac{1}{2}Tr[A+B-|A-B|]\right),$$

where λ_0 is the largest eigenvalue of $B^{-1/2}AB^{-1/2}$.

In order to prove Theorem 3.4, we need the following lemma.

Lemma 3.5. *Let $A, B \in M_{n,+}(\mathbb{C}^n)$ and $\mathfrak{E} = \{E = \{E_j\}; E_j \geq 0, \sum_{j=1}^n E_j = I\}$. We put $a_j = Tr[AE_j], b_j = Tr[BE_j]$. Then*

$$F(A,B) = Tr[|A^{1/2}B^{1/2}|] = \min_{E\in\mathfrak{E}}\sum_{j=1}^n\sqrt{a_jb_j}, \tag{3.2}$$

$$D(A,B) = Tr[|A-B|] = \max_{E\in\mathfrak{E}}\sum_{j=1}^n|a_j-b_j|. \tag{3.3}$$

Proof of Lemma 3.5. First we prove (3.2). For $A, B \in M_{n,+}(\mathbb{C})$, we take the polar decomposition

$$A^{1/2}B^{1/2} = W^*|A^{1/2}B^{1/2}|,$$

where W is a unitary matrix. Then we have

$$\begin{aligned} F(A,B) &= Tr[|A^{1/2}B^{1/2}|] = Tr[WA^{1/2}B^{1/2}] \\ &= \sum_{j=1}^n Tr[WA^{1/2}E_j^{1/2}E_j^{1/2}B^{1/2}] \end{aligned}$$

$$\begin{aligned} &\leq \sum_{j=1}^{n} \sqrt{Tr[AE_j]Tr[BE_j]} \\ &= \sum_{j=1}^{n} \sqrt{a_j b_j}. \end{aligned}$$

The equality is satisfied in the case that for each j

$$E_j^{1/2} B^{1/2} = \alpha_j E_j^{1/2} A^{1/2} W^*$$

for some α_j, or equivalently

$$E_j^{1/2} = \alpha_j E_j^{1/2} B^{-1/2} |A^{1/2} B^{1/2}| B^{-1/2}.$$

Then when $B^{-1/2}|A^{1/2}B^{1/2}|B^{-1/2} = \sum_k \beta_k |k\rangle\langle k|$, $\beta_j = \frac{1}{\alpha_j}$ and $E_j = |j\rangle\langle j|$, the equality is attained. Then $F(A, B) = \min_{E \in \mathfrak{E}} \sum_{j=1}^{n} \sqrt{a_j b_j}$.
Next we prove (3.3). Let $X_{\pm} = \frac{1}{2}(|X| \pm X)$ for selfadjoint X. Since

$$\begin{aligned} |Tr[AE_j - BE_j]| = |Tr[(A-B)E_j]| &= |Tr[((A-B)_+ - (A-B)_-)E_j] \\ &\leq Tr[((A-B)_+ + (A-B)_-)E_j] \\ &= Tr[|A-B|E_j], \end{aligned}$$

we have

$$D(A, B) = Tr[|A - B|] \geq \sum_{j=1}^{n} |a_j - b_j|.$$

The equality is satisfied with the measurement which is made by the projections onto the support of $(A-B)_+$ and the projection onto the support of $(A-B)_-$. Then $D(A, B) = \max_{E \in \mathfrak{E}} \sum_{j=1}^{n} |a_j - b_j|$.

Now we are ready to prove Theorem 3.4. □

Proof of Theorem 3.4. By a similar way as the proof of Lemma 2.5 in [9], we can get the following inequalities; for any λ $(0 \leq \lambda \leq 1)$

$$\begin{aligned} \sum_{j=1}^{n} \sqrt{a_j(\lambda a_j + (1-\lambda)b_j)} + \frac{1-\sqrt{\lambda}}{2} \sum_{j=1}^{n} |a_j - b_j| &\geq \frac{1+\sqrt{\lambda}}{2} \sum_{j=1}^{n} a_j + \frac{1-\sqrt{\lambda}}{2} \sum_{j=1}^{n} b_j \\ &= \frac{1+\sqrt{\lambda}}{2} Tr[A] + \frac{1-\sqrt{\lambda}}{2} Tr[B]. \end{aligned}$$

By taking the minimum over $\mathfrak{E}$ in both sides,

$$F(A, \lambda A + (1-\lambda)B) \geq \frac{1+\sqrt{\lambda}}{2} Tr[A] + \frac{1-\sqrt{\lambda}}{2} Tr[B] - \frac{1-\sqrt{\lambda}}{2} Tr[|A-B|].$$

Let $\lambda_0 = \min\{\lambda > 0; A \leq \lambda B\}$. If $\lambda_0 \leq 1$, then it is clear that $A \leq B$. Since $|A - B| = |B - A| = B - A$, the result of theorem holds. Then we can assume $\lambda_0 > 1$. If we put

$$C = \frac{1}{1-\lambda_0^{-1}} B - \frac{\lambda_0^{-1}}{1-\lambda_0^{-1}} A,$$

then we have

$$\begin{aligned} F(A,B) &= F(A, \lambda_0^{-1}A + (1-\lambda_0^{-1})C) \\ &\geq \frac{1+\sqrt{\lambda_0^{-1}}}{2}Tr[A] + \frac{1-\sqrt{\lambda_0^{-1}}}{2}Tr[C] - \frac{1-\sqrt{\lambda_0^{-1}}}{2}Tr[|A-C|]. \end{aligned}$$

Since

$$Tr[C] = \frac{1}{1-\lambda_0^{-1}}Tr[B] - \frac{\lambda_0^{-1}}{1-\lambda_0^{-1}}Tr[A],$$

we have

$$\begin{aligned} F(A,B) &\geq \frac{1+\sqrt{\lambda_0^{-1}}}{2}Tr[A] + \frac{1-\sqrt{\lambda_0^{-1}}}{2}\left(\frac{1}{1-\lambda_0^{-1}}Tr[B] - \frac{\lambda_0^{-1}}{1-\lambda_0^{-1}}Tr[A]\right) \\ &\quad -\frac{1-\sqrt{\lambda_0^{-1}}}{2}Tr[|A-C|] \\ &= \frac{1+\sqrt{\lambda_0^{-1}}}{2}Tr[A] + \frac{1}{2(1+\sqrt{\lambda_0^{-1}})}Tr[B] - \frac{\lambda_0^{-1}}{2(1+\sqrt{\lambda_0^{-1}})}Tr[A] \\ &\quad -\frac{\sqrt{\lambda_0}}{2(1+\sqrt{\lambda_0})}Tr[|A-B|] \quad \left(\text{by } A-C = \frac{1}{1-\lambda_0^{-1}}(A-B)\right) \\ &= \frac{1+\sqrt{\lambda_0^{-1}}}{2}Tr[A] + \frac{\sqrt{\lambda_0}}{2(1+\sqrt{\lambda_0})}Tr[B] \\ &\quad -\frac{1}{2\sqrt{\lambda_0}(1+\sqrt{\lambda_0})}Tr[A] - \frac{\sqrt{\lambda_0}}{2(1+\sqrt{\lambda_0})}Tr[|A-B|] \\ &= \frac{2+\sqrt{\lambda_0}}{2(1+\sqrt{\lambda_0})}Tr[A] + \frac{\sqrt{\lambda_0}}{2(1+\sqrt{\lambda_0})}Tr[B] - \frac{\sqrt{\lambda_0}}{2(1+\sqrt{\lambda_0})}Tr[|A-B|] \\ &= \frac{1}{1+\sqrt{\lambda_0}}Tr[A] + \frac{\sqrt{\lambda_0}}{2(1+\sqrt{\lambda_0})}(Tr[A+B-|A-B|]). \end{aligned}$$

□

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