Linear and Nonlinear Analysis
Volume 2, Number 2, 2016, 253–261

Yokohama Publishers
ISSN 2188-8167

ITERATIVE METHODS FOR NONEXPANSIVE MAPPINGS ON HADAMARD SPACES AND THEIR COEFFICIENT CONDITIONS

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Abstract. We consider the Halpern type iterative method for mappings on a Hadamard space and attempt to extend the range of the coefficients used in the scheme. It seems that the characterization of the limit point of the iterative scheme will change according to the limit value of coefficients. However, we obtain its characterization by a single unified expression.

1. Introduction

Halpern type method is one of the most popular approximation techniques to fixed points and it was orginally introduced by Halpern [4] in 1967. Since Wittmann [12] proved strong convergence of the sequence generated by this scheme in Hilbert spaces, the method has been extended by many researchers. Shimizu and Takahashi [11] proved that a Halpern type iteration with two nonexpansive mappings converges strongly to a common fixed point in a Hilbert space. Kimura, Takahashi, and Toyoda [6] showed an approximation theorem with a finite family of nonexpansive mappings in a Banach space with certain assumptions.

On the other hand, Halpern type iterations with a finite family of nonexpansive mappings are also considered in Hadamard spaces. The authors introduce one of their iterations which is generated by a convex combination of Halpern type construction.

Theorem 1.1 (Kimura-Wada [7]). *Let X be a Hadamard space and $T_1, T_2, \dots, T_r : X \to X$ nonexpansive mappings with $F = \bigcap_{i=1}^r F(T_i) \neq \emptyset$. Let $u_1, u_2, \dots, u_r, x_1$ be arbitrary points in X and let $\{x_n\}$ be iteratively generated by*

$$\begin{cases} t_n^i = \alpha_n u_i \oplus (1-\alpha_n) T_i x_n, \quad i = 1, 2, \dots, r, \\ y_n^1 = t_n^1, \\ y_n^j = \beta_n^{j-1} t_n^j \oplus (1-\beta_n^{j-1}) y_n^{j-1}, \quad j = 2, 3, \dots, r, \\ x_{n+1} = y_n^r \end{cases}$$

2010 *Mathematics Subject Classification.* 47H09.

Key words and phrases. Common fixed point, approximation theorem, Halpern type iteration, characterization of fixed points, geodesic space.

*The first author is supported by JSPS KAKENHI Grant Number 15K05007.

for all $n \in \mathbb{N}$, *where* $\{\alpha_n\}$ *is a sequence in* $]0,1[$ *such that*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

and $\{\beta_n^k\}$ *are sequences in* $[a,b] \subset]0,1[$ *such that*

$$\sum_{n=1}^{\infty} |\beta_{n+1}^k - \beta_n^k| < \infty$$

for $k = 1,2,\dots,r-1$. *Then* $\{x_n\}$ *converges to a unique minimizer of* $g(x) = \sum_{i=1}^{r} \gamma_i d(u_i,x)^2$ *on* F, *where* $\gamma_k = \beta^{k-1} \prod_{j=k}^{r-1}(1-\beta^j)$ *for* $k = 1,2,\dots,r-1$ *and* $\gamma_r = \beta^{r-1}$ *for* $\beta^0 = 1$ *and* $\beta^i = \lim_{n\to\infty} \beta_n^i$ *for* $i = 1,2,\dots,r-1$.

Let us consider the range of the coefficients $\{\beta_n^k\}$. In Theorem 1.1, $\{\beta_n^k\}$ belong to $[a,b]$ in $]0,1[$ for all k. We also know that the limit point of the iterative scheme does not guarantee to be a common fixed point of the mappings if $\{\beta_n^k\}$ converges to 0 or 1 for some k. This fact suggests that we need to consider another approach to characterize the limit point of the scheme if we only assume the coefficients to be in $]0,1[$.

In this paper, we attempt to extend the range of the coefficients used in the Halpern type iterative scheme for two mappings. In our first observation, it seems that the characterization of the limit point of the iterative scheme will change according to the limit value of coefficients. However, we finally obtain its characterization by a single unified expression.

2. Preliminaries

Let X be a metric space with a metric d. For $x,y \in X$, a mapping $c : [0,l] \to X$, where $l \geq 0$, is called a geodesic with endpoints x and y if $c(0) = x, c(l) = y$ and $d(c(u),c(v)) = |u-v|$ for $u,v \in [0,l]$. Then, the image of a geodesic c with endpoints x,y is called a geodesic segment joining x and y, and is denoted by $[x,y]$. If a geodesic segment exists for any $x,y \in X$, we call X a geodesic metric space. Furthermore, if a geodesic segment is unique for each $x,y \in X$, we call X a uniquely geodesic space. Then, for $t \in [0,1]$ and $x,y \in X$, there exists a unique point $z \in [x,y]$ such that $d(x,z) = (1-t)d(x,y)$ and $d(z,y) = td(x,y)$. We denote such z by $tx \oplus (1-t)y$.

Let X be a uniquely geodesic space. A geodesic triangle $\triangle(x_1,x_2,x_3)$ with vertices x_1,x_2,x_3 in X is the union of geodesic segments joining each pair of vertices. A comparison triangle $\overline{\triangle}(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in $\mathbb{R}^2$ for $\triangle(x_1,x_2,x_3)$ is a triangle such that $d(x_i,x_j) = \|\bar{x}_i - \bar{x}_j\|$ for all $i,j = 1,2,3$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^2$. A point $\bar{p} \in [\bar{x}_1,\bar{x}_2]$ is a comparison point of $p \in [x_1,x_2]$ if $d(x_1,p) = \|\bar{x}_1 - \bar{p}\|$. For any $p,q \in \triangle(x_1,x_2,x_3)$ and their comparison points $\bar{p},\bar{q} \in \overline{\triangle}(\bar{x}_1,\bar{x}_2,\bar{x}_3)$, the inequality

$$d(p,q) \leq \|\bar{p} - \bar{q}\|$$

is called the CAT(0) inequality. If the CAT(0) inequality holds for any triangles in X, then we call X a CAT(0) space, and we define a Hadamard space as a complete CAT(0) space.

By the CAT(0) inequality, we obtain the inequality

$$d(x, ty \oplus (1-t)z)^2 \leq td(x,y)^2 + (1-t)d(x,z)^2 - t(1-t)d(y,z)^2$$

for any $x, y, z \in X$ and $t \in [0,1]$. This inequality is very important for our results.

Let T be a mapping on a metric space X. We call T a nonexpansive mapping if T satisfies $d(Tx, Ty) \leq d(x,y)$ for any $x, y \in X$. A point $z \in X$ is called a fixed point of T if $Tz = z$ holds. We denote the set of all fixed points of T by $F(T)$.

A subset C in a geodesic space is said to be convex if, for any $x, y \in C$, $[x, y]$ is included in C. If S is a nonexpansive mapping in a Hadamard space, we know $F(S)$ is a closed convex subset.

Let C be a nonempty closed convex subset in a Hadamard space X. Then, for any $x \in X$, there exists a unique nearest point y in C to x, that is, y satisfies that $d(x,y) = \inf_{z \in C} d(x,z)$.

For other properties of Hadamard spaces, see [2].

Next, we define Δ-convergence of a sequence. The notion of Δ-convergence was proposed by Lim [9] in a general metric space setting. Kirk and Panyanak [8] applied it to Hadamard spaces. Let $\{x_n\}$ be a bounded sequence in metric space X. For any $x \in X$, we put

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n), \quad r(\{x_n\}) = \inf_{x \in X} r(x, \{x_n\}).$$

Then, if there exists $x \in X$ such that $r(x, \{x_n\}) = r(\{x_n\})$, we call x an asymptotic center of $\{x_n\}$. We say that $\{x_n\}$ is Δ-convergent to x if the asymptotic center of any subsequence of $\{x_n\}$ is a unique point x. We know that any bounded sequences of a Hadamard space has a Δ-converging subsequence; see [3, 8].

Now, we show several lemmas for our results.

Lemma 2.1 (Aoyama-Kimura-Takahashi-Toyoda [1], Xu [13]). *Let $\{s_n\}$ be a nonnegative real sequence, $\{\alpha_n\}$ a sequence in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{u_n\}$ a nonnegative real sequence with $\sum_{n=1}^{\infty} u_n < \infty$, and $\{t_n\}$ a real sequence with $\limsup_{n\to\infty} t_n \leq 0$. Suppose*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n t_n + u_n$$

for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.2 (He-Fang-López-Li [5]). *Let X be a Hadamard space and $\{x_n\}$ a bounded sequence of X. If $\{x_n\}$ is Δ-convergent to $x \in X$, then*

$$d(u,x)^2 \leq \liminf_{n\to\infty} d(u, x_n)^2$$

for all $u \in X$.

Lemma 2.3 (Kirk-Panyanak [8]). *Let T be a nonexpansive mapping in a Hadamard space X and let $\{x_n\} \subset X$ be Δ-convergent to $x \in X$. If $d(x_n, Tx_n) \to 0$, then x is a fixed point of T.*

3. Main results

First, we provide some conditions for main result. In what follows, $\{\alpha_n\}, \{\beta_n\}$ are sequences in $]0,1[$ and $\{\alpha_n\}$ satisfies the following conditions:

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

We suppose that X is a Hadamard space and consider the iteration $\{x_n\}$ generated by

$$(*) \qquad \begin{cases} x_1 \in X, \\ r_n = \alpha_n u \oplus (1-\alpha_n) R x_n, \\ s_n = \alpha_n v \oplus (1-\alpha_n) S x_n, \\ x_{n+1} = \beta_n r_n \oplus (1-\beta_n) s_n \end{cases}$$

for all $n \in \mathbb{N}$, where u, v are arbitrary points in X and R, S are nonexpansive mappings on X such that $F(R) \cap F(S) \neq \emptyset$.

By Theorem 1.1, we obtain the following result.

Theorem 3.1. *Let $\{\beta_n\}$ be a sequence in $[a,b] \subset]0,1[$. If $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, then $\{x_n\}$ generated by $(*)$ converges to x_0, which is a unique minimizer of $g(x) = \beta d(u,x)^2 + (1-\beta) d(v,x)^2$ on $F(R) \cap F(T)$, where $\beta = \lim_{n\to\infty} \beta_n$.*

We consider the case where $\{\beta_n\}$ does not belong to $[a,b]$ for any $a, b \in]0,1[$ and observe the behavior of the sequence $\{x_n\}$ generated by $(*)$. It is easy to see that $\{x_n\}$ is bounded. Furthermore, we also obtain the following result; see [7].

Lemma 3.2. *If $\{\beta_n\}$ satisfies $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, then $\{d(x_{n+1}, x_n)\}$ converges to 0.*

We show the following approximation theorem. Notice that $\{\beta_n\}$ converges to 1 when $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$.

Theorem 3.3. *Suppose that $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$. Then $\{x_n\}$ converges to x_0 which is the nearest point of $F(R)$ to u.*

Proof. Since $\{x_n\}$ is bounded, $\{Rx_n\}, \{Sx_n\}$ are also bounded. Moreover, since

$$\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| \le \sum_{n=1}^{\infty}(1-\beta_n) + \sum_{n=1}^{\infty}(1-\beta_{n+1}) < \infty,$$

we have $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$ by Lemma 3.2. Let

$$\begin{aligned} f_n &= \beta_n d(u, x_0)^2 + (1-\beta_n) d(v, x_0)^2, \\ b_n &= \beta_n d(u, Rx_n)^2 + (1-\beta_n) d(v, Sx_n)^2, \\ c_n &= f_n - (1-\alpha_n) b_n, \end{aligned}$$

and we show that

$$\limsup_{n\to\infty} c_n \le 0.$$

Since $\beta_n \to 1$, we get

$$|f_n - d(u, x_0)^2| = (1-\beta_n) \left| d(v, x_0)^2 - d(u, x_0)^2 \right| \to 0.$$

Further, since $\{\alpha_n\}$ converges to 0, we obtain that

$$\begin{aligned} d(Rx_n, x_n) &\leq d(Rx_n, x_{n+1}) + d(x_{n+1}, x_n) \\ &\leq \beta_n d(Rx_n, r_n) + (1-\beta_n) d(Rx_n, s_n) + d(x_{n+1}, x_n) \\ &= \beta_n \alpha_n d(u, Rx_n) + (1-\beta_n) d(Rx_n, s_n) + d(x_{n+1}, x_n) \\ &\to 0. \end{aligned}$$

Thus, we get

$$\begin{aligned} &|b_n - d(u, x_n)^2| \\ &\leq \beta_n |d(u, Rx_n)^2 - d(u, x_n)^2| + (1-\beta_n)|d(v, Sx_n)^2 - d(u, x_n)^2| \\ &\leq \beta_n (d(u, Rx_n) + d(u, x_n)) d(Rx_n, x_n) + (1-\beta_n)|d(v, Sx_n)^2 - d(u, x_n)^2| \\ &\to 0. \end{aligned}$$

Summarizing the results above, we have

$$|c_n - (d(u, x_0)^2 - d(u, x_n)^2)| \leq \left| f_n - d(u, x_0)^2 \right| + \left| b_n - d(u, x_n)^2 \right| + \alpha_n \left| b_n \right| \to 0,$$

and thus

$$\limsup_{n\to\infty} c_n = \limsup_{n\to\infty} (d(u, x_0)^2 - d(u, x_n)^2).$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} (d(u, x_0)^2 - d(u, x_n)^2) = \lim_{i\to\infty} (d(u, x_0)^2 - d(u, x_{n_i})^2),$$

and $\{x_{n_i}\}$ has a Δ-convergent subsequence. Without loss of generality, we may assume that $\{x_{n_i}\}$ is Δ-convergent to $x \in X$. By Theorem 2.2, we have that

$$\begin{aligned} \lim_{i\to\infty} (d(u, x_0)^2 - d(u, x_{n_i})^2) &= d(u, x_0)^2 - \liminf_{i\to\infty} d(u, x_{n_i})^2 \\ &\leq d(u, x_0)^2 - d(u, x)^2. \end{aligned}$$

Since $d(x_n, Rx_n) \to 0$, we have that x belongs to $F(R)$ by Theorem 2.3. Moreover, since x_0 is a nearest point of $F(R)$ to u, we have that

$$\limsup_{n\to\infty} c_n \leq d(u, x_0)^2 - d(u, x)^2 \leq 0.$$

Then, since x_0 is a point of $F(R)$, it follows that

$$\begin{aligned} &d(x_{n+1}, x_0)^2 \\ &= d(\beta_n r_n \oplus (1-\beta_n) s_n, x_0)^2 \\ &\leq \beta_n d(r_n, x_0)^2 + (1-\beta_n) d(s_n, x_0)^2 \\ &\leq \beta_n (\alpha_n d(u, x_0)^2 + (1-\alpha_n) d(Rx_n, x_0)^2 - \alpha_n (1-\alpha_n) d(u, Rx_n)^2) \\ &\quad + (1-\beta_n)(\alpha_n d(v, x_0)^2 + (1-\alpha_n) d(Sx_n, x_0)^2 - \alpha_n (1-\alpha_n) d(v, Sx_n)^2) \\ &\leq (1-\alpha_n) d(x_n, x_0)^2 + \alpha_n c_n + (1-\beta_n) d(Sx_n, x_0)^2. \end{aligned}$$

Since $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} (1-\beta_n) < \infty$, we obtain that $\{x_n\}$ converges to x_0 by Lemma 2.1. □

Similarly, it follows that the result in the case where $\{\beta_n\}$ converges to 0. The method of proof is almost the same as the previous one.

Theorem 3.4. *Suppose that $\sum_{n=1}^{\infty} \beta_n < \infty$. Then $\{x_n\}$ converges to x_0 which is the nearest point of $F(S)$ to v.*

Let $\beta = \lim_{n\to\infty} \beta_n$. By those results, we can characterize x_0 as follows according to the value of β;

$$x_0 = \begin{cases} \arg\min_{x\in F(R)} d(u,x)^2 & (\beta = 1), \\ \arg\min_{x\in F(R)\cap F(S)} (\beta d(u,x)^2 + (1-\beta)d(v,x)^2) & (\beta \in\,]0,1[), \\ \arg\min_{x\in F(S)} d(v,x)^2 & (\beta = 0). \end{cases}$$

Now, we consider a mapping $\beta R \oplus (1-\beta)S$. It is obvious that $F(\beta R \oplus (1-\beta)S) = F(R)$ if $\beta = 1$, and $F(\beta R \oplus (1-\beta)S) = F(S)$ if $\beta = 0$. If β belongs to $]0,1[$, the following result holds.

Lemma 3.5 (Seajung [10])**.** *Let X be a Hadamard space and R, S nonexpansive mappings on X such that $F(R) \cap F(S) \neq \emptyset$. If β belongs to $]0,1[$, then $F(\beta R \oplus (1-\beta)S) = F(R) \cap F(S)$.*

Therefore, we can characterize x_0 as a single unified expression in the following way:

$$x_0 = \operatorname*{arg\,min}_{x\in F(\beta R\oplus(1-\beta)S)} (\beta d(u,x)^2 + (1-\beta)d(v,x)^2).$$

On the other hand, to obtain the convergence of $\{x_n\}$, we need to suppose the following condition for the coefficients $\{\beta_n\}$ for each case.

$$\begin{cases} \displaystyle\sum_{n=1}^{\infty} (1-\beta_n) < \infty & (\beta = 1), \\ \displaystyle\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty & (\beta \in\,]0,1[), \\ \displaystyle\sum_{n=1}^{\infty} \beta_n < \infty & (\beta = 0). \end{cases}$$

To obtain a condition regardless of the value of β, we use the following lemma.

Lemma 3.6. *Let $\{\beta_n\}$ be a real sequence with $\sum_{n=1}^{\infty} |\beta_n - \beta| < \infty$ for some $\beta \in \mathbb{R}$. Then, $\{\beta_n\}$ is convergent to β, and satisfies $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.*

Suppose that $\{\beta_n\}$ satisfies $\sum_{n=1}^{\infty} |\beta_n - \beta| < \infty$ for some $\beta \in [0,1]$. Then, we have

$$\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$$

Moreover, if $\lim_{n\to\infty} \beta_n = 1$, then since $\beta = \lim_{n\to\infty} \beta_n$, we have

$$\sum_{n=1}^{\infty} (1-\beta_n) = \sum_{n=1}^{\infty} (\beta - \beta_n) = \sum_{n=1}^{\infty} |\beta_n - \beta| < \infty,$$

and if $\lim_{n\to\infty} \beta_n = 0$, we also get

$$\sum_{n=1}^{\infty} \beta_n = \sum_{n=1}^{\infty} (\beta_n - \beta) = \sum_{n=1}^{\infty} |\beta_n - \beta| < \infty.$$

By these facts, we get the convergence theorem with a unified condition of $\{\beta_n\}$.

Theorem 3.7. *Let X be a Hadamard space and R, S nonexpansive mappings in X with common fixed points. Let u, v be arbitrary points in X and $\{x_n\}$ iteratively generated by $(*)$, where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $]0,1[$ such that*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

and, for some $\beta \in [0,1]$,

$$\sum_{n=1}^{\infty} |\beta_n - \beta| < \infty.$$

Then $\{x_n\}$ converges to a unique minimizer of $g(x) = \beta d(u,x)^2 + (1-\beta)d(v,x)^2$ on $F(\beta R \oplus (1-\beta)S)$.

4. Characterization of the limit points in other spaces

In Theorem 3.7, we characterize the limit point x_0 of the iterative scheme as a unique minimizer of the function g. This g only appears in the result in Hadamard spaces and not in Hilbert spaces. In fact, since Hadamard spaces are generalization of Hilbert spaces, we can show our results even in Hilbert spaces. However, in Hilbert spaces, we can characterize the limit point of the iteration without using the function g.

Corollary 4.1. *Let C be a nonempty closed convex subset of a Hilbert space and R, S nonexpansive mappings on C such that $F(R) \cap F(S) \neq \emptyset$. Suppose u, v, x_1 are arbitrary points in C and define an iteration $\{x_n\}$ by*

$$x_{n+1} = \beta_n(\alpha_n u + (1-\alpha_n)Rx_n) + (1-\beta_n)(\alpha_n v + (1-\alpha_n)Sx_n)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $]0,1[$ such that

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

and, for some $\beta \in [0,1]$,

$$\sum_{n=1}^{\infty} |\beta_n - \beta| < \infty.$$

Then, $\{x_n\}$ converges to a unique nearest point of $F(\beta R + (1-\beta)S)$ to $\beta u + (1-\beta)v$.

Proof. By Theorem 3.7, $\{x_n\}$ is convergent to x_0 which is a minimizer of $g(x) = \beta \|u-x\|^2 + (1-\beta)\|v-x\|^2$ on $F(\beta R + (1-\beta)S)$, where $\beta = \lim_{n\to\infty} \beta_n$. Then, for any $y \in F(\beta R + (1-\beta)S)$, we have

$$\begin{aligned}
\|(\beta u + (1-\beta)v) - x_0\|^2 &= \beta \|u - x_0\|^2 + (1-\beta)\|v - x_0\|^2 - \beta(1-\beta)\|u-v\|^2 \\
&\leq \beta \|u-y\|^2 + (1-\beta)\|v-y\|^2 - \beta(1-\beta)\|u-v\|^2 \\
&= \|(\beta u + (1-\beta)v) - y\|^2 .
\end{aligned}$$

Thus, x_0 is a unique nearest point of $F(\beta R + (1-\beta)S)$ to $\beta u + (1-\beta)v$. $\square$

Since the CAT(0) inequality always holds with equality in Hilbert spaces, the limit point of the iteration coincides with a nearest point of $F(\beta R + (1-\beta)S)$ to $\beta u + (1-\beta)v$. However, since it is not always true in Hadamard spaces, the limit point must be expressed by a minimizer of the function g on $F(\beta R \oplus (1-\beta)S)$. Thus, it is one of the characteristic of general Hadamard spaces that the limit point is expressed by g.

Moreover, we can apply our results to a Banach space. The techniques for the proof are similar to our main results. For more details, see [6].

Theorem 4.2. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and C a closed convex subset of E. Let $T_1, T_2, \dots, T_r : C \to C$ be nonexpansive mappings such that the common fixed point set is nonempty. Let u, x_1 be arbitrary points in C and let $\{x_n\}$ be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\sum_{k=1}^{r} \beta_n^k T_k x_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $]0,1[$ such that

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

and, for each $k = 1, 2, \dots, r$, $\{\beta_n^k\}$ are sequences in $]0,1[$ such that

$$\sum_{k=1}^{r} \beta_n^k = 1$$

for $n \in \mathbb{N}$ and, for some $\beta^k \in [0,1]$,

$$\sum_{n=1}^{\infty}\sum_{k=1}^{r} |\beta_n^k - \beta^k| < \infty.$$

Then, $\{x_n\}$ converges to the point Pu, where P is a sunny nonexpansive retraction of C onto $F(\beta^1 T_1 + \beta^2 T_2 + \cdots + \beta^{r-1} T_{r-1} + \beta^r T_r)$.

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Manuscript received 29 February 2016

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