Linear and Nonlinear Analysis
Volume 2, Number 2, 2016, 229–235

Yokohama Publishers
ISSN 2188-8167

BOUNDEDNESS OF TOEPLITZ OPERATORS ON PARABOLIC HARDY SPACES

HAYATO NAKAGAWA AND NORIAKI SUZUKI

Abstract. We define a Toeplitz operator on α-parabolic Hardy spaces and give a condition that it is bounded. A relation between Toeplitz operators and Carleson inclusions is important.

1. Introduction

For an integer $n \geq 1$, let $\mathbb{R}^{n+1}_+ := \{(x,t) \in \mathbb{R}^{n+1} \mid x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, t > 0\}$ denote the upper half space. For $0 < \alpha \leq 1$, let $L^{(\alpha)}$ be a parabolic operator

$$L^{(\alpha)} := \partial_t + (-\Delta_x)^\alpha, \quad \Delta_x := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

An $L^{(\alpha)}$-harmonic function is a continuous function u on $\mathbb{R}^{n+1}_+$ satisfying $L^{(\alpha)}u = 0$ in a weak sense; the precise definition will be given in the next section.

For $1 < p < \infty$, we denote by $h^p_\alpha := h^p_\alpha(\mathbb{R}^{n+1}_+)$ the set of all $L^{(\alpha)}$-harmonic functions u with $\|u\|_{h^p_\alpha} < \infty$, where

$$\|u\|_{h^p_\alpha} := \sup_{t>0} \left(\int_{\mathbb{R}^n} |u(x,t)|^p dx \right)^{\frac{1}{p}}. \tag{1.1}$$

We call h^p_α the α-parabolic Hardy space of order p, which is a Banach space under the norm $\|\cdot\|_{h^p_\alpha}$. Remark that

$$\|u\|_{h^p_\alpha} = \lim_{t\to 0} \left(\int_{\mathbb{R}^n} |u(x,t)|^p dx \right)^{\frac{1}{p}} \tag{1.2}$$

(see (3.3) below).

Let μ be a positive Borel measure on $\mathbb{R}^{n+1}_+$. We define the Toeplitz operator T_μ with symbol μ by

$$T_\mu u(x,t) := \iint_{\mathbb{R}^{n+1}_+} W^{(\alpha)}(x-y, t+s) u(y,s)\, d\mu(y,s) \tag{1.3}$$

for $u \in h^p_\alpha$, where $W^{(\alpha)}$ is the fundamental solution of $L^{(\alpha)}$. Let $\tau > 0$. Recall that μ is called a T_τ-Carleson measure if there exists $C \geq 1$ such that

$$\mu(T^{(\alpha)}(x,t)) \leq C t^{\frac{n\tau}{2\alpha}} \tag{1.4}$$

2010 *Mathematics Subject Classification.* Primary 35K05; Secondary 30H10, 26D10.

Key words and phrases. Parabolic Hardy space, Toeplitz operator, Carleson inequality.

holds for all $(x,t) \in \mathbb{R}^{n+1}_+$, where $T^{(\alpha)}(x,t) := \{(y,s) \in \mathbb{R}^{n+1}_+; |y-x|^{2\alpha} + s \le t\}$ (see [2]).

Now we will state our main result.

Theorem 1.1. *Let $1 < p \le q < \infty$, $\tau := 1 + 1/p - 1/q$ and μ be a positive Borel measure on $\mathbb{R}^{n+1}_+$. If μ is a T_τ-Carleson measure, then the Toeplitz operator $T_\mu = T_{\mu,p,q} \colon h^p_\alpha \to h^q_\alpha$ is well defined and bounded.*

We give an explanation for the definition (1.3). The α-parabolic Bergman space b^p_α is the set of all $L^{(\alpha)}$-harmonic functions u with $\|u\|_{b^p_\alpha} < \infty$, where

$$\|u\|_{b^p_\alpha} := \left(\iint_{\mathbb{R}^{n+1}_+} |u(x,t)|^p dxdt \right)^{\frac{1}{p}}.$$

Toeplitz operator T with symbol μ on b^p_α has already discussed in [5] and [6] (see also [7]). Its definition is

$$Tu(x,t) := -2 \iint_{\mathbb{R}^{n+1}_+} \frac{\partial}{\partial t} W^{(\alpha)}(x-y, t+s) u(y,s)\, d\mu(y,s). \tag{1.5}$$

We note that $-2(\partial/\partial t)W^{(\alpha)}(x-y,t+s)$ is the reproducing kernel of b^2_α. Later we will show that $W^{(\alpha)}(x-y,t+s)$ is the reproducing kernel of the Hilbert space h^2_α. Hence (1.3) is a Hardy space version of (1.5).

In Section 2, we recall the definition of $L^{(\alpha)}$-harmonic functions and basic properties of the fundamental solution $W^{(\alpha)}$. In Section 3, we prepare some basic properties of α-parabolic Hardy space h^p_α. In particular, Huygens property and duality are important. The reproducing kernel of the Hilbert space h^2_α is also discussed. Using Carleson inequalities on h^p_α (cf. [2]), we give a proof of Theorem 1.1 in Section 4.

Throughout the paper, we will use the same letter C to denote various positive constants; it may vary even within a line.

2. Preliminalies

When $\alpha = 1$, $L^{(1)}$-harmonic functions are solutions of the heat operator. We go the case $0 < \alpha < 1$. Let $C^\infty_c(\mathbb{R}^{n+1}_+)$ be the set of all C^∞-functions with compact support on $\mathbb{R}^{n+1}_+$. For $\varphi \in C^\infty_c(\mathbb{R}^{n+1}_+)$, we set

$$\widetilde{L}^{(\alpha)}\varphi(x,t) := -\frac{\partial}{\partial t}\varphi(x,t) - c_{n,\alpha} \lim_{\delta \to 0} \int_{|y|>\delta} (\varphi(x+y,t) - \varphi(x,t))|y|^{-n-2\alpha} dy,$$

where

$$c_{n,\alpha} = 4^\alpha \pi^{-\frac{n}{2}} \frac{\Gamma(\frac{2n+\alpha}{2})}{|\Gamma(-\alpha)|}, \quad |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}},$$

and $\Gamma(\cdot)$ is the gamma function. A function h on $\mathbb{R}^{n+1}_+$ is said to be $L^{(\alpha)}$-harmonic if h is continuous,

$$\iint_{\mathbb{R}^n \times [t_1,t_2]} |h(x,t)|(1+|x|)^{-n-2\alpha} dxdt < \infty \tag{2.1}$$

holds for every $0 < t_1 < t_2 < \infty$ and $\iint_{\mathbb{R}^{n+1}_+} h \cdot \widetilde{L}^{(\alpha)}\varphi \, dxdt = 0$ holds for all $\varphi \in C_c^\infty(\mathbb{R}^{n+1}_+)$. Note that the condition (2.1) is equivalent to $\iint_{\mathbb{R}^{n+1}_+} |h \cdot \widetilde{L}^{(\alpha)}\varphi| \, dxdt < \infty$ for all $\varphi \in C_c^\infty(\mathbb{R}^{n+1}_+)$.

We use a fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$, which is defined by

$$W^{(\alpha)}(x,t) = \begin{cases} (2\pi)^{-n} \displaystyle\int_{\mathbb{R}^n} e^{-t|\xi|^{2\alpha}} e^{ix\cdot\xi} \, d\xi & t > 0 \\ 0 & t \le 0 \end{cases}$$

where $x \cdot \xi$ is the inner product of x and ξ. It is known that when $\alpha = 1/2$, $W^{(1/2)}$ coincides with the Poisson kernel on $\mathbb{R}^{n+1}_+$, that is, for $t > 0$,

$$W^{(\frac{1}{2})}(x,t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{t}{(t^2 + |x|^2)^{\frac{n+1}{2}}}. \tag{2.2}$$

Note also that $W^{(1)}(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$ is the Gauss kernel.

It is known that $W^{(\alpha)}(x,t) \ge 0$ and

$$\int_{\mathbb{R}^n} W^{(\alpha)}(x,t) dx = 1$$

for $t > 0$. Note also that

$$W^{(\alpha)}(x,t) = t^{-\frac{n}{2\alpha}} W^{(\alpha)}(t^{-\frac{1}{2\alpha}} x, 1)$$

and

$$W^{(\alpha)}(x,t) = \int_{\mathbb{R}^n} W^{(\alpha)}(x-y, t-s) W^{(\alpha)}(y,s) dy$$

for $0 < s < t$. The following estimate is useful (see [3], [5]): There exists a constant $C > 0$ such that

$$W^{(\alpha)}(x,t) \le C \frac{t}{(t + |x|^{2\alpha})^{\frac{n}{2\alpha}+1}}. \tag{2.3}$$

Using this, we see

$$\begin{aligned} \int_{\mathbb{R}^n} \left(W^{(\alpha)}(x, t+s)\right)^p dx &\le C \int_{\mathbb{R}^n} \left(\frac{t+s}{(t+s+|x|^{2\alpha})^{\frac{n}{2\alpha}+1}}\right)^p dx \\ &= C \frac{\omega_{n-1}}{2\alpha} (t+s)^{\frac{n}{2\alpha}(1-p)} \int_0^\infty \frac{\eta^{\frac{n}{2\alpha}-1}}{(1+\eta)^{(\frac{n}{2\alpha}+1)p}} d\eta, \end{aligned}$$

where ω_{n-1} is the volume of sphere of unit ball in $\mathbb{R}^n$ and $\eta := \dfrac{|x|^{2\alpha}}{t+s}$. In particular, if $1 < p < \infty$, then

$$\|W^{(\alpha)}(\cdot, \cdot + s)\|_{h^p_\alpha} < C s^{\frac{n}{2\alpha}(\frac{1}{p}-1)} < \infty \tag{2.4}$$

for all $s > 0$.

3. α-parabolic Hardy spaces

The following Huygens property is important in our argument.

Definition 3.1. We say that an α-harmonic function u on $\mathbb{R}^{n+1}_+$ satisfies the Huygens property, if $W^{(\alpha)}(x - \cdot, t - s)u(\cdot, s) \in L^1(\mathbb{R}^n)$ and

$$u(x,t) = \int_{\mathbb{R}^n} W^{(\alpha)}(x-y, t-s)u(y,s)dy \tag{3.1}$$

holds for every $x \in \mathbb{R}^n$ and every $0 < s < t$.

We know that every function in b^p_α satisfies the Huygens property (see [3, Theorem 4.1]). If $u \in h^p_\alpha$, then for any $0 < a < b < \infty$,

$$\iint_{\mathbb{R}^n \times [a,b]} |u(x,t)|^p dxdt < \infty.$$

Hence by the same manner as in [3], we have the following proposition.

Proposition 3.2. *Let $1 < p < \infty$. Every function $u \in h^p_\alpha$ satisfies the Huygens property.*

For $f \in L^p(\mathbb{R}^n)$, we set

$$P^{(\alpha)}[f](x,t) := \int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t)f(y)dy. \tag{3.2}$$

The following proposition is shown in [2] (see also [8, p.62]).

Proposition 3.3. *Let $1 < p < \infty$. If $f \in L^p(\mathbb{R}^n)$, then $P^{(\alpha)}[f] \in h^p_\alpha$ and conversely, for $u \in h^p_\alpha$, there exists a unique function $f \in L^p(\mathbb{R}^n)$ such that $u = P^{(\alpha)}[f]$. Moreover, we see $\|P^{(\alpha)}[f]\|_{h^p_\alpha} = \|f\|_{L^p(\mathbb{R}^n)}$ and*

$$\lim_{t\to 0} \|P^{(\alpha)}[f](\cdot, t) - f\|_{L^p(\mathbb{R}^n)} = 0. \tag{3.3}$$

This implies that

$$P^{(\alpha)} : L^p(\mathbb{R}^n) \to h^p_\alpha \tag{3.4}$$

is a linear surjective isometry. When $\alpha = 1/2$, (2.2) shows that $h^p_{1/2}$ is the usual harmonic Hardy spaces on the upper half space, and (3.4) is a generalization of Theorem 7.17 in [1].

Let $1 < p < \infty$ and let $1/p + 1/p' = 1$. If $u := P^{(\alpha)}[f] \in h^p_\alpha$ and $v := P^{(\alpha)}[g] \in h^{p'}_\alpha$, then by (3.3), we have

$$\lim_{t\to 0} \int_{\mathbb{R}^n} u(x,t)v(x,t)dx = \int_{\mathbb{R}^n} f(x)g(x)dx. \tag{3.5}$$

We put

$$\langle u, v\rangle_H := \lim_{t\to 0} \int_{\mathbb{R}^n} u(x,t)v(x,t)\,dx. \tag{3.6}$$

Remark that Proposition 3.3 gives that

$$|\langle u, v\rangle_H| \le \|u\|_{h^p_\alpha}\|v\|_{h^{p'}_\alpha} = \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^{p'}(\mathbb{R}^n)} < \infty. \tag{3.7}$$

In case of $p=2$, h^2_α is a Hilbert space with the inner product (3.6). By (3.1), the reproducing kernel for h^2_α is $W^{(\alpha)}(x-\cdot,t+\cdot)$, that is,

$$\langle u, W^{(\alpha)}(x-\cdot,t+\cdot)\rangle_H = u(x,t) \tag{3.8}$$

for $u\in h^2_\alpha$. Moreover (3.8) holds for all h^p_α with $1<p<\infty$, that is, if $u\in h^p_\alpha$, then by (3.1),

$$\lim_{s\to 0}\int_{\mathbb{R}^n} u(y,s)W^{(\alpha)}(x-y,t+s)\,dy = \lim_{s\to 0} u(x,t+2s) = u(x,t). \tag{3.9}$$

Next we observe the dual space $(h^p_\alpha)^*$ of h^p_α.

Proposition 3.4. *Let $1<p<\infty$ and let $1/p+1/p'=1$. For $v\in h^{p'}_\alpha$, we set $\Lambda_v(u)=\langle u,v\rangle_H$ for $u\in h^p_\alpha$. Then $\Phi: v\to\Lambda_v$ is a linear surjective isometry from $h^{p'}_\alpha$ to $(h^p_\alpha)^*$, that is, $\|\Phi(v)\|_{(h^p_\alpha)^*}=\|v\|_{h^{p'}_\alpha}$ and $(h^p_\alpha)^*\cong h^{p'}_\alpha$ hold.*

Proof. We write $u=P^{(\alpha)}[f]$ and $v=P^{(\alpha)}[g]$ with $f\in L^p(\mathbb{R}^n)$ and $g\in L^{p'}(\mathbb{R}^n)$, respectively. Then by (3.5) and (3.7),

$$\begin{aligned}\|\Phi(v)\|_{(h^p_\alpha)^*} &= \sup_{u\in h^p_\alpha, \|u\|_{h^p_\alpha}=1}\langle u,v\rangle_H\\ &= \sup_{f\in L^p(\mathbb{R}^n),\|f\|_{L^p(\mathbb{R}^n)}=1}\int_{\mathbb{R}^n} f(x)g(x)dx\\ &= \|g\|_{L^{p'}(\mathbb{R}^n)} = \|v\|_{h^{p'}_\alpha}.\end{aligned}$$

This shows that Φ is isometry. To show that Φ is onto, take $\Lambda\in(h^p_\alpha)^*$. Since $f\mapsto\Lambda(P^{(\alpha)}[f])$ is a bounded linear functional on $L^p(\mathbb{R}^n)$, there exists $g\in L^{p'}(\mathbb{R}^n)$ such that

$$\Lambda(P^{(\alpha)}[f]) = \int_{\mathbb{R}^n} f(x)g(x)dx.$$

It is not difficult to show that $\Lambda=\Lambda_{P^{(\alpha)}[g]}$, which shows Φ is surjective. □

Here we recall a main result of [2]. Let $1<p<\infty$ and $1<q<\infty$. We say that a positive Borel measure μ on $\mathbb{R}^{n+1}_+$ satisfies a (p,q)-Carleson inequality on parabolic Hardy spaces if the mapping $\iota_{\mu,p,q}(u)=u$ from h^p_α to $L^q(\mathbb{R}^{n+1}_+,d\mu)$ is bounded, that is,

$$\|\iota_{\mu,p,q}\| := \sup_{u\in h^p_\alpha}\frac{\|u\|_{L^q(\mathbb{R}^{n+1}_+,\,d\mu)}}{\|u\|_{h^p_\alpha}} < \infty. \tag{3.10}$$

We call $\iota_{\mu,p,q}$ the Carleson inclusion, even if it is not necessarily injective.

Proposition 3.5. ([2, Theorem 1]) *Let $1<p\le q<\infty$. Then $\|\iota_{\mu,p,q}\|<\infty$ if and only if μ is a $T_{q/p}$-Carleson measure.*

Let μ be a positive Borel measure on $\mathbb{R}^{n+1}_+$. For functions u and v on $\mathbb{R}^{n+1}_+$, we write

$$\langle u,v\rangle_{L(\mu)} := \iint_{\mathbb{R}^{n+1}_+} u(x,t)v(x,t)d\mu(x,t),$$

if this integral converges.

Proposition 3.6. *Let $1 < p \le q < \infty$, and let $1/p + 1/p' = 1/q + 1/q' = 1$. Put $\tau := 1 + 1/p - 1/q$ and assume that μ is T_τ-Carleson measure. Then there exists a constant $C \ge 1$ such that for $u \in h^p_\alpha$ and $v \in h^{q'}_\alpha$,*

$$\langle |u|, |v| \rangle_{L(\mu)} \le C \|u\|_{h^p_\alpha} \|v\|_{h^{q'}_\alpha}. \tag{3.11}$$

Proof. We note that $1/(\tau p) + 1/(\tau q') = 1$. By Proposition 3.5, $\iota_{\mu,p,\tau p}$ and $\iota_{\mu,q',\tau q'}$ are bounded. Hence the Hölder inequality shows that

$$\begin{aligned}\iint_{\mathbb{R}^{n+1}_+} |u(x,t)v(x,t)| d\mu(x,t) &\le \|u\|_{L^{\tau p}(\mathbb{R}^{n+1}_+,\, d\mu)} \|v\|_{L^{\tau q'}(\mathbb{R}^{n+1}_+,\, d\mu)} \\ &\le C\|u\|_{h^p_\alpha} \|v\|_{h^{q'}_\alpha}.\end{aligned}$$

□

4. Proof of Theorem 1.1

In this section, we will give a proof of Theorem 1.1.

Let $1 < p \le q < \infty$ and $1/p + 1/p' = 1/q + 1/q' = 1$. Let μ be a T_τ-Carleson measure, where $\tau = 1 + 1/p - 1/q$. We note that $\iota_{\mu,p,\tau p}$ and $\iota_{\mu,q',\tau q'}$ are bounded. Let $u \in h^p_\alpha$. Since $W^{(\alpha)}(x - \cdot, t + \cdot) \in h^{q'}_\alpha$, by (3.11)

$$T_\mu u(x,t) := \iint_{\mathbb{R}^{n+1}_+} W^{(\alpha)}(x-y, t+s) u(y,s) d\mu(y,s)$$

converges for every $(x,t) \in \mathbb{R}^{n+1}_+$. We will show that $T_\mu u$ is $L^{(\alpha)}$-harmonic. When $u = P^{(\alpha)}[f]$, we set $\tilde{u} := P^{(\alpha)}[|f|]$. Then for every $0 < t_1 < t_2 < \infty$,

$$\begin{aligned}&\iint_{\mathbb{R}^n \times [t_1,t_2]} |T_\mu u(x,t)|(1+|x|)^{-n-2\alpha} dxdt \\ &\le \iint_{\mathbb{R}^n \times [t_1,t_2]} T_\mu \tilde{u}(x,t)(1+|x|)^{-n-2\alpha} dxdt \\ &\le \iint_{\mathbb{R}^{n+1}_+} \left(\iint_{\mathbb{R}^n \times [t_1,t_2]} W^{(\alpha)}(x-y, t+s)(1+|x|)^{-n-2\alpha} dxdt \right) \tilde{u}(y,s) d\mu(y,s) \\ &< \infty\end{aligned}$$

because $\tilde{u} \in h^p_\alpha$ and

$$\iint_{\mathbb{R}^n \times [t_1,t_2]} W^{(\alpha)}(x - \cdot, t + \cdot)(1+|x|)^{-n-2\alpha} dxdt \in h^{q'}_\alpha.$$

This estimate and the Fubini Theorem show that $\iint_{\mathbb{R}^{n+1}_+} T_\mu u \cdot \widetilde{L}^{(\alpha)} \varphi \, dxdt = 0$ for all $\varphi \in C^\infty_c(\mathbb{R}^{n+1}_+)$.

Next we will show that $T_\mu u \in h^q_\alpha$. Take $v \in h^{q'}_\alpha$ arbitrarily. Then remarking

$$\|v(\cdot, \cdot + 2s) - v(\cdot,\cdot)\|_{L^{\tau q'}(\mu)} \le C \|v(\cdot, \cdot + 2s) - v(\cdot,\cdot)\|_{h^{q'}_\alpha} \to 0$$

as $s \to 0$, we have

$$\langle T_\mu u, v \rangle_H$$

$$
\begin{aligned}
&= \lim_{s\to 0}\int_{\mathbb{R}^n} T_\mu u(y,s)v(y,s)\,dy \\
&= \lim_{s\to 0}\int_{\mathbb{R}^n}\left(\iint_{\mathbb{R}^{n+1}_+} W^{(\alpha)}(y-x,s+t)u(x,t)\,d\mu(x,t)\right)v(y,s)\,dy \\
&= \lim_{s\to 0}\iint_{\mathbb{R}^{n+1}_+}\left(\int_{\mathbb{R}^n} W^{(\alpha)}(x-y,t+s)v(y,s)\,dy\right)u(x,t)\,d\mu(x,t) \\
&= \lim_{s\to 0}\iint_{\mathbb{R}^{n+1}_+} v(x,t+2s)u(x,t)\,d\mu(x,t) \\
&= \iint_{\mathbb{R}^{n+1}_+} v(x,t)u(x,t)\,d\mu(x,t) \\
&= \langle \iota_{\mu,p,\tau p}u, \iota_{\mu,q',\tau q'}v\rangle_{L(\mu)} \\
&= \langle \iota^*_{\mu,q',\tau q'}\iota_{\mu,p,\tau p}u, v\rangle_{H},
\end{aligned}
$$

which implies $T_\mu u \in h^q_\alpha$ and $T_\mu = \iota^*_{\mu,q',\tau q'}\iota_{\mu,p,\tau p}$. Hence $T_\mu = T_{\mu,p,q} : h^p_\alpha \to h^q_\alpha$ is well defined and

$$\|T_{\mu,p,q}\| \le \|\iota^*_{\mu,q',\tau q'}\|\|\iota_{\mu,p,\tau p}\| = \|\iota_{\mu,q',\tau q'}\|\|\iota_{\mu,p,\tau p}\| < \infty.$$

References

[1] S.Axler, P. Bourdon and W. Ramey, *Harmonic Function Theory*, Springer-Verlag, 1992.

[2] H. Nakagawa and N. Suzuki, *Carleson inequalities on parabolic Hardy spaces*, to appear in Hokkaido Math. J.

[3] M. Nishio, K. Shimomura and N. Suzuki, *α-parabolic Bergman spaces*, Osaka J. of Math. **42** (2005), 133–162.

[4] M. Nishio, K. Shimomura and N. Suzuki, *L^p boundedness of Bergman projections for α-parabolic operators*, Potential theory in Matsue, Adv. Stud. Pure Math. **44**, Math. Soc. Japan, Tokyo, 2006, pp. 305–318.

[5] M. Nishio, N. Suzuki and M. Yamada, *Toeplitz operators and Carleson measures on parabolic Bergman spaces*, Hokkaido Math. J. **36** (2007), 563–583.

[6] M. Nishio, N. Suzuki and M. Yamada, *Weighted Berezin transformations with application to Toeplitz operators of Schatten class on parabolic Bergman spaces*, Kodai Math. J. **32** (2009), 501–520.

[7] M. Nishio, N. Suzuki and M. Yamada, *Carleson inequalities on parabolic Bergman spaces*, Tohoku Math. J. **62** (2010), 269–286.

[8] E. M. Stein, *Singular Integrals and Differential Properties of Functions*, Princeton Univ. Press, Princeton, New Jersey, 1970.

Manuscript received 29 February 2016
revised 22 April 2016

H. Nakagawa
Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya, 466-8602, Japan
E-mail address: m04026b@math.nagoya-u.ac.jp

N. Suzuki
Department of Mathematics, Meijo University, Tenpaku-ku, Nagoya, 468-8502, Japan
E-mail address: suzukin@meijo-u.ac.jp