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A NOTE ON THE COMPLEX INTERPOLATION OF MORREY SPACES

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Dedicated to Professor Masami Okada for his retirement

Abstract. In this paper, a complex interpolation functor is introduced. We show that the complex interpolation of Morrey spaces and closed subspaces using our functor are equal to the spaces produced by Calderón's second complex method.

1. Introduction

The aim of this paper is to propose a complex interpolation functor $\mathfrak{H}_\theta$ for $\theta \in (0,1)$ and to show some interpolation results on Morrey spaces.

Let $1 \le q \le p < \infty$. For an $L^q_{\rm loc}$-function f its Morrey norm is defined by:

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup_{x\in\mathbb{R}^n,\, r>0} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^q\, dy\right)^{\frac{1}{q}}. \tag{1.1}$$

The Morrey space $\mathcal{M}^p_q$ is the set of all L^q-locally integrable functions f for which the norm $\|f\|_{\mathcal{M}^p_q}$ is finite.

In the present paper, we suppose that we are given 7 parameters $1 \le q_0 \le p_0 < \infty$, $1 \le q_1 \le p_1 < \infty$, $1 \le q \le p < \infty$, and $0 < \theta < 1$ satisfying

$$p_0 \neq p_1, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{q}{p} = \frac{q_0}{p_0} = \frac{q_1}{p_1}. \tag{1.2}$$

We recall some notions of interpolation theory before introducing the new interpolation functor. Let X be a Hausdorff linear space, X_0, X_1 be subspaces of X which are continuously embedded in X, and let $\|\cdot\|_{X_0} : X_0 \to [0,\infty)$, $\|\cdot\|_{X_1} : X_1 \to [0,\infty)$ be the norms defined on X_0 and X_1, respectively. Sometimes we write $\overline{X} = (X_0, X_1)$. Let

$$X_0 + X_1 \equiv \{x \in X \,:\, x = x_0 + x_1,\ x_0 \in X_0,\ x_1 \in X_1\}$$

be the vector sum of X_0 and X_1. In this case, the couple $\overline{X} \equiv (X_0, X_1)$ is called a compatible couple of Banach spaces.

(1) Set $S \equiv \{z \in \mathbb{C} \,:\, 0 < {\rm Re}\,(z) < 1\}$ and $\overline{S} \equiv \{z \in \mathbb{C} \,:\, 0 \le {\rm Re}\,(z) \le 1\}$.

(2) Define $\mathcal{H}(X_0, X_1)$ as the set of all functions $H : \bar{S} \to X_0 + X_1$ such that

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(a) H is continuous on $\bar{S}$ and $\sup_{z\in\bar{S}} \left\| \frac{H(z)}{1+|z|^2} \right\|_{X_0+X_1} < \infty$,

(b) H is holomorphic on S,

(c) for all $j = 0, 1$, $H(j + i(t + s)) - 2H(j + it) + H(j + i(t - s)) \in X_j$ and

$$\sup_{t,s\in\mathbb{R},s\neq 0} \frac{\|H(j + i(t + s)) - 2H(j + it) + H(j + i(t - s))\|_{X_j}}{s^2} < \infty. \tag{1.3}$$

The space $\mathcal{H}(X_0, X_1)$ is equipped with the norm

$$\|H\|_{\mathcal{H}(X_0,X_1)} \equiv \max_{j=0,1} \sup_{t,s\in\mathbb{R},s\neq 0} \frac{\|H(j + i(t + s)) - 2H(j + it) + H(j + i(t - s))\|_{X_j}}{s^2}.$$

(3) Let $\theta \in (0, 1)$. Define the complex interpolation space $\mathfrak{H}_\theta[X_0, X_1]$ with respect to $\overline{X} = (X_0, X_1)$ to be the set of all vectors $f \in X_0 + X_1$ such that $f = H''(\theta)$ for some $H \in \mathcal{H}(X_0, X_1)$. The norm on $\mathfrak{H}_\theta[X_0, X_1]$ is defined by

$$\|f\|_{\mathfrak{H}_\theta[X_0,X_1]} \equiv \inf\{\|H\|_{\mathcal{H}(X_0,X_1)} : f = H''(\theta) \text{ for some } H \in \mathcal{H}(X_0, X_1)\}.$$

The first main result of this paper is as follows:

Theorem 1.1. *Under* (1.2), *we have*

$$\mathfrak{H}_\theta[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}] = \mathcal{M}^p_q.$$

Based on Theorem 1.1, we consider the role of this functor. Let U be a linear subspace of L^0 having the lattice property; $|g| \leq |f|$ with $f \in U$ implies $g \in U$. Denote by $U\mathcal{M}^p_q$ the closure of $U \cap \mathcal{M}^p_q$ in $\mathcal{M}^p_q$. Our second theorem is as follows:

Theorem 1.2. *Under* (1.2), *we have*

$$\mathfrak{H}_\theta[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}] = \bigcap_{0<a<b<\infty} \{f \in \mathcal{M}^p_q : \chi_{[a,b]}(|f|)f \in U\mathcal{M}^p_q\}.$$

Theorems 1.1 and 1.2 will be proved in Section 3. From Theorems 1.1 and 1.2, the output of this functor is the same as that of the Calderón second complex interpolation functor, which we recall in Section 2.

The interpolation of Morrey spaces has a long history due to some bad aspects of Morrey spaces we describe below. Due to this fact, we have the following difficulties when $1 < q < p < \infty$:

(1) The Morrey space $\mathcal{M}^p_q$ is not reflexive; see [16, Example 5.2] and [19, Theorem 1.3].
(2) The Morrey space $\mathcal{M}^p_q$ does not have $\mathcal{D}(\mathbb{R}^n)$ as a dense closed subspace; see [18, Proposition 2.16].
(3) The Morrey space $\mathcal{M}^p_q$ is not separable; see [18, Proposition 2.16].
(4) The Morrey space is not a Banach function space; see [16, Example 3.3].

Among difficulties we encounter when we handle Morrey spaces, the control of the boundary is the most serious one. In fact, when we deal with the Calderón first complex interpolation functor, we have to consider seriously the function $F : \overline{S} \to \mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}$ defined in (2.5) such that F is continous in the boundary $\overline{S}\setminus S$. When we deal with the second complex interpolation functor this problem can be overcome.

However, it is not the case that Lipschitz continuity of the functions having the value in Banach spaces guarantees the almost everywhere differentiability; see [9, Proposition 5]. This misleading fact comes from the Rademacher theorem which asserts that Lipschitz continuous functions are almost everywhere differentiable. This theorem is true for $\mathbb{R}^n$, whose proof requires the special case in $\mathbb{R}$; see [6, Section 3.1, p. 81]. In the case of $\mathbb{R}$, we relied upon the fact that any Lipschitz functions can be decomposed into a difference of monotone functions. So, we can not consider the derivative on the boundary when the function assumes its value in a Banach space; see the proof of [8, Theorem 2] for how we coped with the problem.

Despite a counterexample by Blasco, Ruiz, and Vega [2, 15], the interpolation theory of Morrey spaces progressed so much. As for the real interpolation results, Burenkov and Nursultanov obtained an interpolation result in local Morrey spaces [3]. Nakai and Sobukawa generalized their results to B^u_w setting [13]. We made a significant progress in the complex interpolation theory of Morrey spaces. In [5, p. 35] Cobos, Peetre, and Persson pointed out that

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \subset \mathcal{M}^p_q$$

as long as $1 \le q_0 \le p_0 < \infty$, $1 \le q_1 \le p_1 < \infty$, and $1 \le q \le p < \infty$ satisfy

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{1.4}$$

As it is shown in [10, Theorem 3(ii)], when an interpolation functor F satisfies

$$F[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}] = \mathcal{M}^p_q$$

under the condition (1.4), then

$$\frac{q_0}{p_0} = \frac{q_1}{p_1} \tag{1.5}$$

holds. Lemarié-Rieusset showed this assertion by using the counterexample by Ruiz and Vega [15]. Lemarié-Rieusset also proved that we can choose the second complex interpolation functor introduced by Calderón [4] in 1964. Meanwhile, as for the interpolation result under (1.4) and (1.5) by using the first complex interpolation functor by Calderón [4], Lu, Yang, and Yuan obtained the following description:

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta = \overline{\mathcal{M}^{p_0}_{q_0} \cap \mathcal{M}^{p_1}_{q_1}}^{\mathcal{M}^p_q}$$

in [12, Theorem 1.2]. They also extended this result by placing themselves in the setting of a metric measure space. Their technique is to calculate the Calderón product.

More and more attention has been paid for the closed subspaces of the Morrey space $\mathcal{M}^p_q$ with $1 \le q < p < \infty$. Here, we list some of them as examples of U.

Definition 1.3. Let $1 \le q \le p < \infty$.

(1) [16, Definition 4.5] A function f in $\mathcal{M}^p_q$ is said to have "absolutely continuous norm" in $\mathcal{M}^p_q$ if $\|f\chi_{E_k}\|_{\mathcal{M}^p_q} \to 0$ for every sequence $\{E_k\}_{k=1}^\infty$ satisfying $E_k \to \emptyset$ a.e. The set of all functions in $\mathcal{M}^p_q$ of absolutely continuous norm is denoted by $\widehat{\mathcal{M}}^p_q$.

(2) [21, Definition 2.23] $\overset{\circ}{\mathcal{M}}{}^p_q$ denotes the closure with respect to $\mathcal{M}^p_q$ of C^∞_{c}.

(3) [21, Section 2] $*\mathcal{M}^p_q$ denotes the closure with respect to $\mathcal{M}^p_q$ the set formed by all compactly supported functions in $\mathcal{M}^p_q$.
(4) Let E be a measurable set and denote by $L^0(E)$ the set of all measurable functions that vanish outside E.

Another important space is $\overset{\diamond}{\mathcal{M}}{}^p_q$. Recall that $\overset{\diamond}{\mathcal{M}}{}^p_q$ denotes the closure with respect to $\mathcal{M}^p_q$ of the set of all smooth functions f such that $\partial^\alpha f \in \mathcal{M}^p_q$ for all multi-indexes α [21, Definition 2.23]. Note that this space does not have the lattice property. The closed subspaces $\widehat{\mathcal{M}}{}^p_q$ and $\overset{\diamond}{\mathcal{M}}{}^p_q$ arise naturally. We refer to [21, Theorem 2.29] for $\overset{\diamond}{\mathcal{M}}{}^p_q$ and to [16, Theorems 4.3 and 4.6] for $\widetilde{\mathcal{M}}{}^p_q = \widehat{\mathcal{M}}{}^p_q$.

Proposition 1.4 ([9, 21])**.** *Let $1 < q \le p < \infty$, $1 < q_0 \le p_0 < \infty$, and $1 < q_1 \le p_1 < \infty$ satisfy $p_0 < p < p_1$ and* (1.2).

We have

$$\overset{\circ}{\mathcal{M}}{}^p_q \subset \widetilde{\mathcal{M}}{}^p_q = \widehat{\mathcal{M}}{}^p_q \subset *\mathcal{M}^p_q \tag{1.6}$$

$$\overset{\circ}{\mathcal{M}}{}^p_q \subset \overset{\diamond}{\mathcal{M}}{}^p_q \tag{1.7}$$

$$\widetilde{\mathcal{M}}{}^p_q \subset \overline{\mathcal{M}^{p_0}_{q_0} \cap \mathcal{M}^{p_1}_{q_1}}^{\mathcal{M}^p_q} \tag{1.8}$$

$$\widetilde{\mathcal{M}}{}^p_q \subset [\widetilde{\mathcal{M}}{}^{p_0}_{q_0}, \widetilde{\mathcal{M}}{}^{p_1}_{q_1}]^\theta. \tag{1.9}$$

$$\overline{\mathcal{M}^{p_0}_{q_0} \cap \mathcal{M}^{p_1}_{q_1}}^{\mathcal{M}^p_q} \subsetneq \overset{\diamond}{\mathcal{M}}{}^p_q. \tag{1.10}$$

We have no inclusionship other than (1.6)–(1.10) *and all the inclusions above are strict.*

Finally we compare our interpolation functors with the ones defined in the previous papers. In [7] they defined an interpolation functors as follows:

Definition 1.5 ([7, Definition 2.1])**.** Let X_0 and X_1 be as above. Let $n = 1, 2, \ldots$ and $1 \le p < \infty$.

(1) Let $\mathbb{D} \equiv \{z \in \mathbb{C} : |z| < 1\}$.
(2) Define $P(\mathbb{D}, X_0 \cap X_1)$ to be the set of all polynomials in $\mathbb{D}$ of the form
$$p(z) = \sum_{j=1}^N z^j a_j,$$
where $N \in \mathbb{N}$ is arbitrary and $a_j \in X_0 \cap X_1$ for $j = 1, 2, \ldots, N$.
(3) Denote by $A(\mathbb{D}, \overline{X})$ the closure of $P(\mathbb{D}, X_0 \cap X_1)$ with respect to the norm
$$\|p\| \equiv \sup_{z \in \mathbb{D}} \|p(z)\|_{X_0 \cap X_1}.$$
(4) The space $[\overline{X}]_{D(n,\infty)}$ as a set is the set of all $x \in X_0 + X_1$ for which it is realized as $x = f^{(n)}(0)$ for some $f \in A(\mathbb{D}, \overline{X})$. As a Banach space, it is realized as the quotient space of $A(\mathbb{D}, \overline{X})$ modulo $K_{(n)}$, where
$$K_{(n)} \equiv \{f \in A : f^{(n)}(0) = 0\}.$$

(5) For a simply connected domain D having C^1-boundary, use a homeomorphism $\Phi : \overline{D} \to \overline{\mathbb{D}}$ such that $\Phi|D$ is analytic on D to define $A(D, \overline{X})$.

The fact that $[\overline{X}]_{D(n,\infty)}$ is the interpolation space of (X_0, X_1) can be found in [7, Theorem 2.8].

Meanwhile, Schechter defined the space $X_{\theta,\rho}^{(n)}$ as follows:

Definition 1.6. Let $\rho : \bar{S} \to \mathbb{C}$ be a continuous function on $\bar{S}$ such that ρ is holomorphic on S.

(1) The space $\tilde{H}(X_0, X_1; \rho)$ denotes the set of all continuous functions F in $\bar{S}$ such that $\rho^{-1}F$ is holomorphic in S and that

$$\|f(j+i\eta)\|_{X_j} \le C|\rho(j+i\eta)| \quad (j = 0, 1, \eta \in \mathbb{R}) \tag{1.11}$$

for some constant C. The minimum constant C satisfying (1.11) is denoted by $\|f\|_{\tilde{H}(X_0,X_1;\rho)}$.

(2) The space $X_{\theta,\rho}^{(n)}$ denotes the set of all $x \in X_0 + X_1$ for which there exists $f \in \tilde{H}(X_0, X_1; \rho)$ such that $x = f^{(n)}(\theta)$. The norm is given by

$$\|x\|_{X_{\theta,\rho}^{(n)}} = \inf\{\|f\|_{\tilde{H}(X_0,X_1;\rho)} : x = f^{(n)}(\theta)\}.$$

Note that the notation $\tilde{H}(X_0, X_1; \rho)$ stands for the space $H(X_0, X_1; \rho)$, defined in [17, p.119]. As it is written in [17, Proposition 2.7], one can consider the functor for the first order difference. In fact, Schechter proposed to consider

$$\|f(j+it_2) - f(j+it_1)\|_{X_j} \le M \int_{t_1}^{t_2} |\rho(j+it)|\, dt$$

for $M > 0$ and $t_1, t_2 \in \mathbb{R}$ satisfying $t_1 < t_2$ together with the smallest M in [17, (2.9)]. The resulting space is called the primed space. In his terminology, our space can be understood as the doubly primed space with $\rho = 1$.

2. Preliminaries

2.1. **Interpolation functors of the first and the second kind.** We recall the definition of the complex interpolation functors as follows:

Definition 2.1 ([1,4], Calderón's first complex interpolation space)**.** Suppose that $\overline{X} = (X_0, X_1)$ is a compatible couple of Banach spaces.

(1) The space $\mathcal{F}(X_0, X_1)$ is defined as the set of all functions $F : \bar{S} \to X_0 + X_1$ such that
  (a) F is continuous on $\bar{S}$ and $\sup_{z \in \bar{S}} \|F(z)\|_{X_0+X_1} < \infty$,
  (b) F is holomorphic on S,
  (c) the functions $t \in \mathbb{R} \mapsto F(j+it) \in X_j$ are bounded and continuous on $\mathbb{R}$ for $j = 0, 1$.

The space $\mathcal{F}(X_0, X_1)$ is equipped with the norm

$$\|F\|_{\mathcal{F}(X_0,X_1)} \equiv \max\left\{\sup_{t\in\mathbb{R}} \|F(it)\|_{X_0},\ \sup_{t\in\mathbb{R}} \|F(1+it)\|_{X_1}\right\}.$$

(2) Let $\theta \in (0,1)$. Define the complex interpolation space $[X_0,X_1]_\theta$ with respect to $\overline{X}=(X_0,X_1)$ to be the set of all $f \in X_0+X_1$ such that $f=F(\theta)$ for some $F \in \mathcal{F}(X_0,X_1)$. The norm on $[X_0,X_1]_\theta$ is defined by

$$\|f\|_{[X_0,X_1]_\theta} \equiv \inf\{\|F\|_{\mathcal{F}(X_0,X_1)} : f=F(\theta) \text{ for some } F \in \mathcal{F}(X_0,X_1)\}.$$

It is known in [4] that $[X_0,X_1]_\theta$ and $\mathcal{F}(X_0,X_1)$ are Banach spaces. See also [1, Theorem 4.1.2].

Let X be a Banach space. The space $\mathrm{Lip}(\mathbb{R},X)$ is defined to be the set of all functions $F:\mathbb{R}\to X$ for which the quantity

$$\|F\|_{\mathrm{Lip}(\mathbb{R},X)} \equiv \sup_{-\infty<s<t<\infty} \frac{\|F(t)-F(s)\|_X}{|t-s|} < \infty.$$

Definition 2.2 ([1, 4], Calderón's second complex interpolation space)**.** Suppose that $\overline{X}=(X_0,X_1)$ is a compatible couple of Banach spaces.

(1) Define $\mathcal{G}(X_0,X_1)$ as the set of all functions $G:\bar{S}\to X_0+X_1$ such that
  (a) G is continuous on $\bar{S}$ and $\sup\limits_{z\in\bar{S}} \left\|\frac{G(z)}{1+|z|}\right\|_{X_0+X_1} < \infty$,
  (b) G is holomorphic on S,
  (c) the functions
$$t \in \mathbb{R} \mapsto G(j+it)-G(j) \in X_j$$
  are Lipschitz continuous on $\mathbb{R}$ for $j=0,1$.

The space $\mathcal{G}(X_0,X_1)$ is equipped with the norm

$$\|G\|_{\mathcal{G}(X_0,X_1)} \equiv \max\left\{\|G(i\cdot)\|_{\mathrm{Lip}(\mathbb{R},X_0)},\ \|G(1+i\cdot)\|_{\mathrm{Lip}(\mathbb{R},X_1)}\right\}. \tag{2.1}$$

(2) Let $\theta \in (0,1)$. Define the complex interpolation space $[X_0,X_1]^\theta$ with respect to $\overline{X}=(X_0,X_1)$ to be the set of all functions $f \in X_0+X_1$ such that $f=G'(\theta)$ for some $G \in \mathcal{G}(X_0,X_1)$. The norm on $[X_0,X_1]^\theta$ is defined by

$$\|f\|_{[X_0,X_1]^\theta} \equiv \inf\{\|G\|_{\mathcal{G}(X_0,X_1)} : f=G'(\theta) \text{ for some } G \in \mathcal{G}(X_0,X_1)\}.$$

The space $[X_0,X_1]^\theta$ is called the Calderón's second complex interpolation space, or the upper complex interpolation space of (X_0,X_1).

The following theorem is known:

Theorem 2.3. *Assume* (1.2)*.*

(1) [12] $[\mathcal{M}^{p_0}_{q_0},\mathcal{M}^{p_1}_{q_1}]_\theta$ *coincides with the closure of* $\mathcal{M}^{p_0}_{q_0}\cap\mathcal{M}^{p_1}_{q_1}$ *in* $\mathcal{M}^p_q$.
(2) [11] $[\mathcal{M}^{p_0}_{q_0},\mathcal{M}^{p_1}_{q_1}]^\theta=\mathcal{M}^p_q$.

2.2. **Some inequalities on complex analysis.** The estimates in the following lemmas are trivial since

$$\frac{e^z+e^{-z}-2}{z^2}-1=\frac{1}{12}z^2+\cdots \quad (z\in\mathbb{C}).$$

We prove Lemmas 2.4 and 2.5 in the appendix.

Lemma 2.4. *Let* $A>0$, $\varepsilon\in(0,\frac{1}{2})$, *and* $w\in\mathbb{C}$. *Assume that* $\varepsilon>2|w|$. *Then the following statements hold:*

(1) *If* $0 < A \le 1$,

$$A^{\varepsilon}\left|\frac{\exp(w\log A)-2+\exp(-w\log A)}{(w\log A)^2}-1\right| \le C_{\varepsilon}|w|. \tag{2.2}$$

(2) *If* $A > 1$,

$$A^{-\varepsilon}\left|\frac{\exp(w\log A)-2+\exp(-w\log A)}{(w\log A)^2}-1\right| \le C_{\varepsilon}|w|. \tag{2.3}$$

Lemma 2.5. *Let* $q_0 > q_1$ *and* $f \in L^0$. *First define* $q : \overline{S} \to \mathbb{C}$, $F : \overline{S} \to L^0$, $G : \overline{S} \to L^0$, *and* $H : \overline{S} \to L^0$ *by:*

$$\frac{1}{q(z)} \equiv \frac{1-z}{q_0}+\frac{z}{q_1} \quad (z \in \overline{S}), \tag{2.4}$$

$$F(z) \equiv \operatorname{sgn}(f)\exp\left(\frac{q}{q(z)}\log|f|\right) \quad (z \in \overline{S}), \tag{2.5}$$

$$G(z) \equiv (z-\theta)\int_0^1 F(\theta+(z-\theta)t)\,dt \quad (z \in \overline{S}), \tag{2.6}$$

$$H(z) \equiv (z-\theta)\int_0^1 G(\theta+(z-\theta)t)\,dt \quad (z \in \overline{S}), \tag{2.7}$$

respectively. Define $F_0, F_1, G_0, G_1, H_0, H_1 : \overline{S} \to L^0$ *by:*

$$F_0(z) \equiv F(z)\chi_{\{|f|\le 1\}}, \quad F_1(z) \equiv F(z)\chi_{\{|f|>1\}}, \tag{2.8}$$

$$G_0(z) \equiv G(z)\chi_{\{|f|\le 1\}}, \quad G_1(z) \equiv G(z)\chi_{\{|f|>1\}}, \tag{2.9}$$

and

$$H_0(z) \equiv H(z)\chi_{\{|f|\le 1\}}, \quad H_1(z) \equiv H(z)\chi_{\{|f|>1\}}. \tag{2.10}$$

Then, for any $z \in \overline{S}$, *we have*

$$|H(z)| \le 5(1+|z|^2)(|f|^{q/q_0}+|f|^{q/q_1}). \tag{2.11}$$

For any $z \in \mathbb{C}$ *with* $\varepsilon < \operatorname{Re}(z) < 1-\varepsilon$, *we have*

$$\left|\frac{H_0(z+w)-2H_0(z)+H_0(z-w)}{w^2}-F_0(z)\right| \le C_{\varepsilon}|w|\cdot|f|^{\frac{q}{q_0}} \tag{2.12}$$

and

$$\left|\frac{H_1(z+w)-2H_1(z)+H_1(z-w)}{w^2}-F_1(z)\right| \le C_{\varepsilon}|w|\cdot|f|^{\frac{q}{q_1}}, \tag{2.13}$$

where the constant C_{ε} *depending only on* $\varepsilon \in (0, 1/2)$.

2.3. **Closed subspaces.** For later consideration, we need the following lemmas:

Lemma 2.6 ([14, Lemma 2.5])**.** *Let $0<\eta$ and $0<q\le p<\infty$. If $f\in\mathcal{M}^p_q$, then $|f|^\eta\in\mathcal{M}^{p/\eta}_{q/\eta}$ with*

$$\||f|^\eta\|_{\mathcal{M}^{p/\eta}_{q/\eta}}=\|f\|^\eta_{\mathcal{M}^p_q}.$$

Lemma 2.7. *Let U be a closed subspace enjoying the lattice property. Then $U\mathcal{M}^p_q$ has the lattice property.*

Proof. Let $0\le|g|\le|f|$ and $f\in U\mathcal{M}^p_q$. Then we can choose a sequence $\{f_j\}_{j=1}^\infty\subseteq U\cap\mathcal{M}^p_q$ converges to f in $\mathcal{M}^p_q$. Set $g_j\equiv\chi_{\{f\ne0\}}g\cdot f^{-1}\cdot f_j$. Then $\{g_j\}_{j=1}^\infty$ is convergent to g in $\mathcal{M}^p_q$. □

Lemma 2.8. *Let U be a subspace of measurable functions enjoying the lattice property. If E is a measurable subset such that $\chi_E\in U\mathcal{M}^p_q$, then $\chi_E\in U\mathcal{M}^{p_0}_{q_0}\cap U\mathcal{M}^{p_1}_{q_1}$.*

Proof. Let $\chi_E\in U\mathcal{M}^p_q$ and $\varepsilon>0$. Choose $g_\varepsilon\in U\cap\mathcal{M}^p_q$ such that

$$\|\chi_E-g_\varepsilon\|_{\mathcal{M}^p_q}<\varepsilon.$$

Define $h_\varepsilon\equiv\chi_{\{g_\varepsilon\ne0\}\cap E}$. Then

$$|\chi_E-h_\varepsilon|=\chi_E-h_\varepsilon\le|\chi_E-g_\varepsilon|.$$

Consequently, for $j=0,1$, we have

$$\|\chi_E-h_\varepsilon\|_{\mathcal{M}^{p_j}_{q_j}}=\|\chi_E-h_\varepsilon\|^{q/q_j}_{\mathcal{M}^p_q}<\varepsilon^{q/q_j}.$$

This shows that $\chi_E\in U\mathcal{M}^{p_0}_{q_0}\cap U\mathcal{M}^{p_1}_{q_1}$. □

Lemma 2.9 ([9])**.** *Let U be a subspace of measurable functions enjoying the lattice property. Then we have*

$$\mathcal{M}^p_q\cap\overline{U\mathcal{M}^p_q}^{\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}}\subset\bigcap_{0<a<b<\infty}\left\{f\in\mathcal{M}^p_q:\chi_{\{a\le|f|\le b\}}f\in U\mathcal{M}^p_q\right\}.$$

3. Proofs

3.1. **Proof of Theorem 1.1.** Note that, by our assumption, we have

$$\frac{p_0}{q_0}=\frac{p_1}{q_1}=\frac{p}{q}.$$

Let $f\in\mathcal{M}^p_q$. Then define H, F, and G as in Lemma 2.5. By using the inequality (2.11), we get

$$\sup_{z\in\overline{S}}\left\|\frac{H(z)}{1+|z|^2}\right\|_{\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}}\le5\left(\|f\|^{q/q_0}_{\mathcal{M}^p_q}+\|f\|^{q/q_1}_{\mathcal{M}^p_q}\right).$$

For $\varepsilon\in(0,\frac12)$, define $S_\varepsilon\equiv\{z\in\mathbb{C}:\varepsilon<\mathrm{Re}(z)<1-\varepsilon\}$. Let $z\in S_\varepsilon$ and $w\in\mathbb{C}$ with $2|w|<\varepsilon$. By combining the inequalities (2.12) and (2.13), we have

$$\left\|\frac{H(z+w)-2H(z)+H(z-w)}{w^2}-F(z)\right\|_{\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}}$$

$$\le C_\varepsilon \left(\|f\|_{\mathcal{M}^p_q}^{\frac{q}{q_0}} + \|f\|_{\mathcal{M}^p_q}^{\frac{q}{q_1}} \right) |w|,$$

This shows that H is holomorphic. Note that

$$H''(z)(x) = G'(z)(x) = F(z)(x).$$

Thus, for $j = 0, 1$ and $s, t \in \mathbb{R}$ with $s \neq 0$, we have

$$\begin{aligned} &\left\| \frac{H(j+i(t+s)) - 2H(j+it) + H(j+i(t-s))}{s^2} \right\|_{\mathcal{M}^{p_j}_{q_j}} \\ &= \frac{1}{s^2} \left\| \int_0^s \int_{-u}^u H''(j+i(t+v))\, dv\, du \right\|_{\mathcal{M}^{p_j}_{q_j}} \\ &\le \frac{1}{s^2} \left\| \int_0^s \int_{-u}^u |H''(j+i(t+v))|\, dv\, du \right\|_{\mathcal{M}^{p_j}_{q_j}} \\ &= \frac{1}{s^2} \left\| \int_0^s \int_{-u}^u |F(j+i(t+v))| dv du \right\|_{\mathcal{M}^{p_j}_{q_j}} \\ &= \left\| |f|^{\frac{q}{q_j}} \right\|_{\mathcal{M}^{p_j}_{q_j}} = \|f\|_{\mathcal{M}^p_q}^{q/q_j} \end{aligned}$$

This implies that H satisfies (1.3). Then $H \in \mathcal{H}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$. We have $f \in \mathfrak{H}_\theta[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]$, since $H''(\theta) = F(\theta) = f$.

Conversely, we assume that $f \in \mathfrak{H}_\theta[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]$. Then, $f = H''(\theta)$ in the topology of $\mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}$ for some $H \in \mathcal{H}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$. Define

$$K_k(z) = \frac{H(z+i2^{-k}) - 2H(z) + H(z-i2^{-k})}{(2^{-k}i)^2} \quad (z \in S).$$

Observe that $f = \lim_{k\to\infty} K_k(\theta)$ in the topology of $\mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}$. Hence, there exists a subsequence $\{K_{k_l}(\theta)\}_{l=1}^\infty \subseteq \{K_k(\theta)\}_{k=1}^\infty$ such that

$$\lim_{l\to\infty} K_{k_l}(\theta)(x) = f(x) \quad \text{a.e. } x \in \mathbb{R}^n. \tag{3.1}$$

Since

$$\begin{aligned} &\|K_k\|_{\mathcal{F}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})} \\ &= \max_{j=0,1} \sup_{t\in\mathbb{R}} \|K_k(j+it)\|_{\mathcal{M}^{p_j}_{q_j}} \\ &= \max_{j=0,1} \sup_{t\in\mathbb{R}} \left\| \frac{H(j+i(t+2^{-k})) - 2H(j+it) + H(j+i(t-2^{-k}))}{(2^{-k})^2} \right\|_{\mathcal{M}^{p_j}_{q_j}} \\ &\le \|H\|_{\mathcal{H}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})} \end{aligned}$$

we have $K_k \in \mathcal{F}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$. Consequently, $K_k(\theta) \in [\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \subseteq \mathcal{M}^p_q$. By Fatou's lemma, we get

$$\begin{aligned} \|f\|_{\mathcal{M}^p_q} &\le \liminf_{l\to\infty} \|K_{k_l}(\theta)\|_{\mathcal{M}^p_q} \\ &\lesssim \liminf_{l\to\infty} \|K_{k_l}(\theta)\|_{[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta} \end{aligned}$$

$$\leq \liminf_{l\to\infty} \|K_{k_l}\|_{\mathcal{F}(\mathcal{M}^{p_0}_{q_0},\mathcal{M}^{p_1}_{q_1})}$$
$$\leq \|H\|_{\mathcal{H}(\mathcal{M}^{p_0}_{q_0},\mathcal{M}^{p_1}_{q_1})} < \infty,$$

as desired.

3.2. **Proof of Theorem 1.2.** Let $f \in \mathfrak{H}_\theta[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]$. By Theorem 1.1, we have $f \in \mathcal{M}^p_q$. Choose $H \in \mathcal{H}(U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1})$ such that $H''(\theta) = f$. Let $z \in \overline{S}$ and $k \in \mathbb{N}$. Define

$$F_k(z) \equiv \frac{H(z+i/k) - 2H(z) + H(z-i/k)}{(i/k)^2}.$$

Observe that $F_k(\theta) \in [U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]_\theta \subseteq U\mathcal{M}^p_q$. By combining Lemma 2.9 and $f = \lim\limits_{k\to\infty} F_k(\theta)$ in $U\mathcal{M}^{p_0}_{q_0} + U\mathcal{M}^{p_1}_{q_1}$, we have

$$f \in \bigcap_{0<a<b<\infty} \left\{ f \in \mathcal{M}^p_q : \chi_{\{a\leq|f|\leq b\}} f \in U\mathcal{M}^p_q \right\}.$$

Conversely, let $f \in \mathcal{M}^p_q$ be such that $\chi_{\{a\leq|f|\leq b\}} f \in U\mathcal{M}^p_q$ for all $0 < a < b < \infty$. Without loss of generality, we may assume that $q_0 > q_1$. Then define $H(z)$ by (2.7). For $a \in (0,1)$, write $H_a(z) \equiv \chi_{[a,a^{-1}]}(|f|)H(z)$ by the lattice property $H_a(z) \in U\mathcal{M}^{p_0}_{q_0} \cap U\mathcal{M}^{p_1}_{q_1}$. Define $H_0(z)$ and $H_1(z)$ by (2.10). Write $H_{a,0}(z) \equiv \chi_{\{a\leq|f|\leq a^{-1}\}} H_0(z)$ and $H_{a,1}(z) \equiv \chi_{\{a\leq|f|\leq a^{-1}\}} H_1(z)$. For $z \in \overline{S}$, we have

$$|H_0(z) - H_{a,0}(z)| \leq \chi_{\{|f|\leq a\}} |H(z)|,$$

$$\chi_{\{|f|\leq a\}} |F(z)| = \chi_{\{|f|\leq a\}} |f|^{\frac{q}{q_0}} |f|^{q\mathrm{Re}(w)\left(\frac{1}{q_1}-\frac{1}{q_0}\right)} \leq \chi_{\{|f|\leq a\}} |f|^{\frac{q}{q_0}}, \tag{3.2}$$

and

$$G(z) = \int_\theta^z F(w)\,dw = \int_\theta^z \mathrm{sgn(f)} |f|^{\frac{q(1-w)}{q_0}+\frac{qw}{q_1}}\,dw = \frac{F(z) - F(\theta)}{q\left(\frac{1}{q_1} - \frac{1}{q_0}\right)\log|f|}. \tag{3.3}$$

By combining (3.2) and (3.3), we have

$$\begin{aligned}
\chi_{\{|f|\leq a\}} |G(z)| &\leq \chi_{\{|f|\leq a\}} \frac{|F(z)| + |F(\theta)|}{q\left(\frac{1}{q_1} - \frac{1}{q_0}\right)|\log|f||} \\
&\leq \chi_{\{|f|\leq a\}} \frac{|F(z)| + |F(\theta)|}{q\left(\frac{1}{q_1} - \frac{1}{q_0}\right)|\log|f||} \\
&\leq 2\chi_{\{|f|\leq a\}} \frac{|f|^{\frac{q}{q_0}}}{q\left(\frac{1}{q_1} - \frac{1}{q_0}\right)\log(a^{-1})}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
|H_0(z) - H_{a,0}(z)| &\leq \chi_{\{|f|\leq a\}} |H(z)| \\
&= \chi_{\{|f|\leq a\}} \left| \int_\theta^z G(w)\,dw \right| \\
&\lesssim |z - \theta| \chi_{\{|f|\leq a\}} \frac{|f|^{q/q_0}}{\log(a^{-1})}
\end{aligned}$$

When $a \to 0^+$, we have

$$\|H_0(z) - H_{a,0}(z)\|_{\mathcal{M}^{p_0}_{q_0}} \lesssim \frac{|z-\theta|}{\log(a^{-1})} \||f|^{q/q_0}\|_{\mathcal{M}^{p_0}_{q_0}} = \frac{|z-\theta|}{\log(a^{-1})} \|f\|^{q/q_0}_{\mathcal{M}^p_q} \to 0.$$

Therefore, $H_0(z) \in U\mathcal{M}^{p_0}_{q_0}$. By a similar argument, we also have

$$\|H_1(z) - H_{a,1}(z)\|_{\mathcal{M}^{p_1}_{q_1}} \lesssim \frac{|z-\theta|}{-q\left(\frac{1}{q_1}-\frac{1}{q_0}\right)\log a} \|f\|^{q/q_1}_{\mathcal{M}^p_q} \to 0,$$

as $a \to 0$. This implies $H_1(z) \in U\mathcal{M}^{p_1}_{q_1}$. Thus, $H(z) \in U\mathcal{M}^{p_0}_{q_0} + U\mathcal{M}^{p_1}_{q_1}$.

Write $H_a^+ \equiv \chi_{(a^{-1},\infty)}(|f|)H$ and $H_a^- \equiv \chi_{(0,a)}(|f|)H$. Let $t, s \in \mathbb{R}$ and $j = 0, 1$. By using the identity

$$\begin{aligned} H(j+i(t+s)) &- 2H(j+it) + H(j+i(t-s)) \\ &= \frac{F(j+i(t+s)) - 2F(j+it) + F(j+i(t-s))}{\left(q\left(\frac{1}{q_1}-\frac{1}{q_0}\right)\log|f|\right)^2}, \end{aligned}$$

we have

$$\begin{aligned} &\|(1-\chi_{(a,a^{-1})}(|f|))(H(j+i(t+s)) - 2H(j+it) + H(j+i(t-s)))\|_{\mathcal{M}^{p_j}_{q_j}} \\ &\quad \le \|H_a^+(j+i(t+s)) - 2H_a^+(j+it) + H_a^+(j+i(t-s))\|_{\mathcal{M}^{p_j}_{q_j}} \\ &\qquad + \|H_a^-(j+i(t+s)) - 2H_a^-(j+it) + H_a^-(j+i(t-s))\|_{\mathcal{M}^{p_j}_{q_j}} \\ &\quad \le \frac{C\|f\|^{q/q_j}_{\mathcal{M}^p_q}}{(\log a)^2}. \end{aligned}$$

As a result,

$$\begin{aligned} &H(j+i(t+s)) - 2H(j+it) + H(j+i(t-s)) \\ &= \lim_{a \downarrow 0} (H_a(j+i(t+s)) - 2H_a(j+it) + H_a(j+i(t-s))) \end{aligned}$$

in $\mathcal{M}^{p_j}_{q_j}$ and hence

$$H(j+i(t+s)) - 2H(j+it) + H(j+i(t-s)) \in U\mathcal{M}^{p_j}_{q_j}.$$

Since $H \in \mathcal{H}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$, we have

$$\begin{aligned} &\max_{j=0,1} \sup_{t,s\in\mathbb{R}, s\neq 0} \frac{\|H(j+i(t+s)) - 2H(j+it) + H(j+i(t-s))\|_{U\mathcal{M}^{p_j}_{q_j}}}{s^2} \\ &= \max_{j=0,1} \sup_{t,s\in\mathbb{R}, s\neq 0} \frac{\|H(j+i(t+s)) - 2H(j+it) + H(j+i(t-s))\|_{\mathcal{M}^{p_j}_{q_j}}}{s^2} \\ &< \infty. \end{aligned}$$

Therefore $H \in \mathcal{H}(U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1})$. Thus, $H(\theta) \in \mathfrak{H}_\theta[U\mathcal{M}^{p_0}_{q_0}, U\mathcal{M}^{p_1}_{q_1}]$.

4. Appendix–some fundamental calculations

4.1. Proof of Lemma 2.4. Since

$$\begin{aligned}&\left|\exp(w\log A)-2+\exp(-w\log A)-(w\log A)^2\right|\\&\le\sum_{n=2}^{\infty}\frac{|w\log A|^{2n}}{(2n)!}\\&\le|w\log A|^3\sum_{n=2}^{\infty}\frac{|w\log A|^{2n-3}}{(2n-3)!}\\&\le|w\log A|^3\exp(|w\log A|)\end{aligned}$$

we have $\lim\limits_{A\to1}\frac{\exp(w\log A)-2+\exp(-w\log A)}{(w\log A)^2}-1=0$ and

$$\left|\frac{\exp(w\log A)-2+\exp(-w\log A)}{(w\log A)^2}-1\right|\le|w\log A|\exp\left(\frac{\varepsilon}{2}|\log A|\right),$$

for all $A\neq1$. Consequently, for every $0<A<1$, we have

$$A^{\varepsilon}\left|\frac{\exp(w\log A)-2+\exp(-w\log A)}{(w\log A)^2}-1\right|\le-A^{\varepsilon/2}\log(A)|w|.$$

We can check that $\sup\limits_{0<A\le1}-A^{\varepsilon/2}\log(A)=\frac{2}{e\varepsilon}$. Thus, we choose $C_\varepsilon\equiv\frac{2}{e\varepsilon}$ to get (2.2). Similarly, for $A>1$, we also have

$$A^{-\varepsilon}\left|\frac{\exp(w\log A)-2+\exp(-w\log A)}{(w\log A)^2}-1\right|\le A^{-\varepsilon/2}\log(A)|w|\le C_\varepsilon|w|,$$

as desired.

4.2. Proof of Lemma 2.5. For $z\in\overline{S}$, we have

$$|F(z)|=|f|^{\frac{q(1-\mathrm{Re}(z))}{q_0}+\frac{q\mathrm{Re}(z)}{q_1}}\le|f|^{\frac{q}{q_0}}+|f|^{\frac{q}{q_1}}.$$

Thus, $|G(z)|\le|z-\theta|\left(|f|^{q/q_0}+|f|^{q/q_1}\right)\le(1+|z|)\left(|f|^{q/q_0}+|f|^{q/q_1}\right)$. Consequently,

$$|H(z)|\le|z-\theta|(1+\theta+|z|)\left(|f|^{q/q_0}+|f|^{q/q_1}\right)\le5(1+|z|^2)\left(|f|^{q/q_0}+|f|^{q/q_1}\right).$$

Let $\varepsilon\in(0,1/2)$, $z\in\mathbb{C}$ be such that $\varepsilon<\mathrm{Re}(z)<1-\varepsilon$, and $w\in\mathbb{C}$ with $|w|<\frac{\varepsilon}{2}$. Write $A=|f|^{q\left(\frac{1}{q_1}-\frac{1}{q_0}\right)}$. By using the identity

$$H_0(z+w)-2H_0(z)+H_0(z-w)=\frac{F_0(z+w)-2F_0(z)+F_0(z-w)}{\left(q\left(\frac{1}{q_1}-\frac{1}{q_0}\right)\log|f|\right)^2},$$

and also the inequality (2.2), we have

$$\begin{aligned}&\left|\frac{H_0(z+w)-2H_0(z)+H_0(z-w)}{w^2}-F_0(z)\right|\\&=|F_0(z)|\left|\frac{\exp(w\log A)-2+\exp(-w\log A)}{(w\log A)^2}-1\right|\end{aligned}$$

$$\begin{aligned}
&= \chi_{\{|f|\le 1\}}|f|^{q/q_0}A^{\mathrm{Re}(z)}\left|\frac{\exp(w\log A)-2+\exp(-w\log A)}{(w\log A)^2}-1\right| \\
&\le \chi_{\{|f|\le 1\}}|f|^{q/q_0}A^{\varepsilon}\left|\frac{\exp(w\log A)-2+\exp(-w\log A)}{(w\log A)^2}-1\right| \\
&\le C_\varepsilon |f|^{q/q_0}|w|.
\end{aligned}$$

By using a similar argument and the inequality (2.2), we also can obtain (2.13).

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References

[1] J. Bergh and J. Löfström, *Interpolation Spaces*, An introduction, Grundlehren der Mathematischen Wissenschaften, 223, Springer-Verlag, Berlin-New York, 1976.

[2] O. Blasco, A. Ruiz and L. Vega, *Non-interpolation in Morrey-Campanato and block spaces*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **28** (1999), 31–40.

[3] V. I. Burenkov and E. D. Nursultanov, *Description of interpolation spaces for local Morrey-type spaces (Russian)*, Tr. Mat. Inst. Steklova **269** (2010), 52–62 Teoriya Funktsii i Differentsialnye Uravneniya; translation in Proc. Steklov Inst. Math. **269** (2010), 46–56.

[4] A. P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. **14** (1964), 113–190.

[5] F. Cobos, J. Peetre and L. E. Persson, *On the connection between real and complex interpolation of quasi-Banach spaces*, Bull. Sci. Math. **122** (1998), 17–37.

[6] C. Evans and F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, 1999.

[7] M. Fan and S. Kaijser, *Complex interpolation with derivatives of analytic functions*, J. Funct. Anal. **120** (1994), 380–402.

[8] D. I. Hakim and Y. Sawano, *Interpolation property of generalized Morrey spaces*, Revista Mathematica **29** (2016), 295–340.

[9] D. I. Hakim and Y. Sawano, *Calderón's first and second complex Interpolations of closed subspaces of Morrey spaces*, J. Fourier Anal. Appl., online.

[10] P. G. Lemarié-Rieusset, *Multipliers and Morrey spaces*, Potential Anal. **38** (2013), 741–752.

[11] P. G. Lemarié-Rieusset, *Erratum to: Multipliers and Morrey spaces*, Potential Anal **41** (2014), 1359–1362.

[12] Y. Lu, D. Yang and W. Yuan, Interpolation of Morrey Spaces on Metric Measure Spaces, Canad. Math. Bull. **57** (2014), 598–608.

[13] E. Nakai and T. Sobukawa, B_u^w*-function spaces and their interpolation*, Tokyo J. Math., online.

[14] S. Nakamura, T. Noi and Y. Sawano, *Generalized Morrey spaces and trace operator*, Sci. China Math. **59** (2016), 281–336.

[15] A. Ruiz and L. Vega, *Corrigenda to Unique continuation for Schrödinger operators with potential in Morrey spaces and a remark on interpolation of Morrey spaces*, Publ. Mat. **39** (1995), 405–411.

[16] Y. Sawano and H. Tanaka, *The Fatou property of block spaces*, J. Math. Sci. Univ. Tokyo **22** (2015), 663–683.

[17] M. Schechter, *Complex Interpolation*, Comp. Math. **18** (1967), 117–147.

[18] H. Triebel, *Hybrid Function Spaces, Heat and Navier-Stokes Equations*, Tracts in Mathematics 24, European Mathematical Society, 2015.

[19] D. Yang and W. Yuan, *Dual properties of Triebel-Lizorkin-type spaces and their applications*, Z. Anal. Anwend. **30** (2011), 29–58.

[20] D. Yang, W. Yuan and C. Zhuo, *Complex interpolation on Besov-type and Triebel-Lizorkin-type spaces*, Anal. Appl. (Singap.) **11** (2013), 1350021.
[21] W. Yuan, W. Sickel and D. Yang, Interpolation of Morrey-Campanato and Related Smoothness Spaces, Sci. China Math. **58**, no. 9, 1835–1908, (2015).

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