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NEW EXISTENCE RESULTS OF BEST PROXIMITY POINTS AND FIXED POINTS FOR $\mathcal{MT}(\lambda)$-FUNCTIONS WITH APPLICATIONS TO DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we introduce and study the new concept of $\mathcal{MT}(\lambda)$-functions. Some characterizations of $\mathcal{MT}(\lambda)$-functions are established. We obtain some new existence and convergence theorems of iterates of best proximity points for $\mathcal{MT}(\lambda)$-functions on quasiordered metric spaces. Moreover, some new fixed point results on quasiordered metric spaces are given. As applications, we establish the existence and uniqueness results for initial-value problems.

1. Introduction and preliminaries

Fixed point theory and best proximity point theory have attracted much attention as very important and powerful tools in nonlinear sciences and various fields of applied mathematical analysis such as game theory, economics, engineering, physics, and computer sciences. In 1922, Banach proved his famous fixed point theorem (so-called the Banach contraction principle [3]): Every contractive mapping from a complete metric space (X, d) into itself has a unique fixed point in X. An amount of generalizations in various different directions of the Banach contraction principle have been studied by several authors; see [8–14, 20, 23, 25, 27, 28] and references therein. Recent investigations in the fixed point theory and its applications motivate the development of new fixed point techniques for solving practical problems arising in natural sciences. However, as we know, it is very natural that some mappings, especially non-selfmappings T defined on a metric space (X, d), do not necessarily possess a fixed point, that is, the equation $Tx = x$ (or $x \in Tx$ if T is a multivalued mapping) is not necessarily to have a solution. So, in such situations, we often turn to explore the best approximation of the existence of solutions. In other words, one try to puzzled out an approximate solution p (call it a best proximity point of T) that is optimal in the sense that the distance between p and Tp is minimum. In the last decade, several authors have investigated to generalize and extend the best proximity point theory by using new nonlinear conditions or replacing general metric

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spaces with some convenient abstract spaces; see, e.g., [1,2,4–7,15–19,21,22,24,26] and references therein.

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$, the sets of positive integers and real numbers, respectively. Let A and B be nonempty subsets of a nonempty set X. A mapping $S : A \cup B \to A \cup B$ is called *cyclic* if $S(A) \subset B$ and $S(B) \subset A$. Let (X, d) be a metric space and $T : A \cup B \to A \cup B$ be a selfmapping. For any nonempty subsets A and B of X, let

$$dist(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

A point $x \in A \cup B$ is called to be a *best proximity point* for T if $d(x, Tx) = dist(A, B)$.

In 2003, Kirk, Srinavasan and Veeramani [19] extended and generalized the Banach contraction principle that introduced cyclic mappings and best proximity points. In 2006, Eldered and Veeramani [17] established some results about best proximity points of cyclic contraction mappings.

Definition 1.1 (see [17]). Let A and B be nonempty closed subsets of a complete metric space (X, d). A cyclic map $T : A \cup B \to A \cup B$ is called a *cyclic contraction* if for some $\alpha \in (0, 1)$, the condition

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha) dist(A, B)$$

holds for all $x \in A$ and $y \in B$.

Theorem 1.2 (see [17, Proposition 3.2]). *Let A and B be nonempty closed subsets of a complete metric space X. Let $T : A \cup B \to A \cup B$ be a cyclic contraction map, $x_1 \in A$ and define $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. Suppose $\{x_{2n-1}\}$ has a convergent subsequence in A. Then there exists $x \in A$ such that $d(x, Tx) = dist(A, B)$.*

In 2011, Derafshpour, Rezapour and Shahzad [7] proved the following convergence theorem on ordered metric spaces.

Theorem 1.3 (see [7, Theorem 2.2]). *Let $(X, d, \preccurlyeq)$ be an ordered metric space, $A, B \in 2^X$ and T a $\preccurlyeq$-nonincreasing selfmapping on $A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$. Suppose that there exists $x_0 \in A$ such that $x_0 \preccurlyeq T^2 x_0 \preccurlyeq Tx_0$ and*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(dist(A, B))$$

for all $x \in A$ and $y \in B$ with $x \preccurlyeq y$, where $\varphi : [0, \infty) \to [0, \infty)$ is a strictly increasing function. If $x_{n+1} = Tx_n$ and $d_n = d(x_{n+1}, x_n)$ for all $n \geq 0$, then $d_n \to dist(A, B)$ as $n \to \infty$.

Let f be a real-valued function defined on $\mathbb{R}$. For $c \in \mathbb{R}$, we recall

$$\limsup_{x \to c^+} f(x) = \inf_{\varepsilon > 0} \sup_{c < x < c + \varepsilon} f(x).$$

Definition 1.4 (see [8–16]). A function $\varphi : [0, \infty) \to [0, 1)$ is said to be an $\mathcal{MT}$-*function* (or $\mathcal{R}$-*function*) if

$$\limsup_{s \to t^+} \varphi(s) < 1 \quad \text{for all } t \in [0, \infty).$$

It is obvious that if $\varphi : [0, \infty) \to [0, 1)$ is a nondecreasing function or a nonincreasing function, then φ is an $\mathcal{MT}$-function. So the set of $\mathcal{MT}$-functions is a rich class. Some new characterizations of $\mathcal{MT}$-functions were established in [11, Theorem 2.1].

Recently, Pathak, Agarwal and Cho introduced the concept of $\mathcal{P}$-functions [23] as follows.

Definition 1.5 (see [23])**.** A function $\varphi : [0, \infty) \to \left[0, \frac{1}{2}\right)$ is said to be a $\mathcal{P}$-*function* if

$$\limsup_{s \to t^+} \varphi(s) < \frac{1}{2} \quad \text{for all } t \in [0, \infty).$$

By using the same technique in the proof of [11, Theorem 2.1], Pathak, Agarwal and Cho gave some characterizations of $\mathcal{P}$-functions; see [23, Lemma 3.1]. However, we must point out that [23, Lemma 3.1] can actually be proved easily by applying [11, Theorem 2.1] due to the fact that φ is a $\mathcal{P}$-function if and only if 2φ is an $\mathcal{MT}$-function.

Theorem 1.6 (see [23, Lemma 3.1])**.** *Let $\varphi : [0, \infty) \to \left[0, \frac{1}{2}\right)$ be a function. Then the following statements are equivalent.*

(a) *φ is a $\mathcal{P}$-function.*
(b) *For each $t \in [0, \infty)$, there exist $r_t^{(1)} \in \left[0, \frac{1}{2}\right)$ and $\epsilon_t^{(1)} > 0$ such that $\varphi(s) \le r_t^{(1)}$ for all $s \in (t, t + \epsilon_t^{(1)})$.*
(c) *For each $t \in [0, \infty)$, there exist $r_t^{(2)} \in \left[0, \frac{1}{2}\right)$ and $\epsilon_t^{(2)} > 0$ such that $\varphi(s) \le r_t^{(2)}$ for all $s \in [t, t + \epsilon_t^{(2)}]$.*
(d) *For each $t \in [0, \infty)$, there exist $r_t^{(3)} \in \left[0, \frac{1}{2}\right)$ and $\epsilon_t^{(3)} > 0$ such that $\varphi(s) \le r_t^{(3)}$ for all $s \in (t, t + \epsilon_t^{(3)}]$.*
(e) *For each $t \in [0, \infty)$, there exist $r_t^{(4)} \in \left[0, \frac{1}{2}\right)$ and $\epsilon_t^{(4)} > 0$ such that $\varphi(s) \le r_t^{(4)}$ for all $s \in [t, t + \epsilon_t^{(4)})$.*
(f) *For any sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, from and after some fixed term, it is nonincreasing and $0 \le \sup_{n \in \mathbb{N}} \varphi(x_n) < \frac{1}{2}$.*
(g) *φ is a function of semicontractive factor; that is, for any sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, from and after some fixed term, it is strictly decreasing and $0 \le \sup_{n \in \mathbb{N}} \varphi(x_n) < \frac{1}{2}$.*

The article is divided into four sections. In Section 2, motivated by the concept of $\mathcal{P}$-functions, we first introduce and study the new concept of $\mathcal{MT}(\lambda)$-functions which generalizes the concept of $\mathcal{P}$-functions. Some characterizations of $\mathcal{MT}(\lambda)$-functions are presented. In Section 3, we establish some new existence and convergence theorems of iterates of best proximity points for $\mathcal{MT}(\lambda)$-functions on quasiordered metric spaces. Moreover, some new fixed point results on quasiordered metric spaces are given. Finally, as applications, we investigate the existence and uniqueness of solutions of initial-value problems for nonlinear first order ordinary differential equations in Section 4.

2. $\mathcal{MT}$-functions, $\mathcal{MT}(\lambda)$-functions and their characterizations

Recall that a real sequence $\{a_n\}_{n \in \mathbb{N}}$ is called

(i) *eventually strictly decreasing* if there exists $\ell \in \mathbb{N}$ such that $a_{n+1} < a_n$ for all $n \in \mathbb{N}$ with $n \geq \ell$;
(ii) *eventually strictly increasing* if there exists $\ell \in \mathbb{N}$ such that $a_{n+1} > a_n$ for all $n \in \mathbb{N}$ with $n \geq \ell$;
(iii) *eventually nonincreasing* if there exists $\ell \in \mathbb{N}$ such that $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$ with $n \geq \ell$;
(iv) *eventually nondecreasing* if there exists $\ell \in \mathbb{N}$ such that $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$ with $n \geq \ell$.

In this section, we first present some new characterizations of $\mathcal{MT}$-functions linked with eventually nonincreasing and eventually strictly decreasing sequences.

Theorem 2.1. *Let $\varphi : [0, \infty) \to [0, 1)$ be a function. Then the following statements are equivalent.*

(a) *φ is an $\mathcal{MT}$-function.*
(b) *For each $t \in [0, \infty)$, there exist $r_t^{(1)} \in [0, 1)$ and $\varepsilon_t^{(1)} > 0$ such that $\varphi(s) \leq r_t^{(1)}$ for all $s \in (t, t + \varepsilon_t^{(1)})$.*
(c) *For each $t \in [0, \infty)$, there exist $r_t^{(2)} \in [0, 1)$ and $\varepsilon_t^{(2)} > 0$ such that $\varphi(s) \leq r_t^{(2)}$ for all $s \in [t, t + \varepsilon_t^{(2)}]$.*
(d) *For each $t \in [0, \infty)$, there exist $r_t^{(3)} \in [0, 1)$ and $\varepsilon_t^{(3)} > 0$ such that $\varphi(s) \leq r_t^{(3)}$ for all $s \in (t, t + \varepsilon_t^{(3)}]$.*
(e) *For each $t \in [0, \infty)$, there exist $r_t^{(4)} \in [0, 1)$ and $\varepsilon_t^{(4)} > 0$ such that $\varphi(s) \leq r_t^{(4)}$ for all $s \in [t, t + \varepsilon_t^{(4)})$.*
(f) *For any nonincreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.*
(g) *φ is a function of contractive factor; that is, for any strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.*
(h) *For any eventually nonincreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.*
(i) *For any eventually strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.*

Proof. The equivalence of statements (a)-(g) was indeed proved in [11, Theorem 2.1]. The implications "(h) $\Rightarrow$ (f)" and "(i) $\Rightarrow$ (g)" are obvious. Let us prove "(f) $\Rightarrow$ (h)". Suppose that (f) holds. Let $\{x_n\}_{n\in\mathbb{N}}$ be an eventually nonincreasing sequence in $[0, \infty)$. Then there exists $\ell \in \mathbb{N}$ such that $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$ with $n \geq \ell$. Put $y_n = x_{n+\ell-1}$ for $n \in \mathbb{N}$. So $\{y_n\}_{n\in\mathbb{N}}$ is a nonincreasing sequence in $[0, \infty)$. By (f), we obtain

$$0 \leq \sup_{n\in\mathbb{N}} \varphi(y_n) < 1.$$

Let $\gamma := \sup_{n\in\mathbb{N}} \varphi(y_n)$. Then

$$0 \leq \varphi(x_{n+\ell-1}) = \varphi(y_n) \leq \gamma < 1 \quad \text{for all } n \in \mathbb{N}.$$

Hence we get

$$0 \leq \varphi(x_n) \leq \gamma < 1 \quad \text{for all } n \in \mathbb{N} \text{ with } n \geq \ell.$$

Let

$$\eta := \max\{\varphi(x_1), \varphi(x_2), \cdots, \varphi(x_{\ell-1}), \gamma\} < 1.$$

Then $\varphi(x_n) \leq \eta$ for all $n \in \mathbb{N}$. Hence $0 \leq \sup_{n\in\mathbb{N}} \varphi(x_n) \leq \eta < 1$ and (h) holds. Similarly, we can varify "(g) $\Rightarrow$ (i)". Therefore, from above, we prove that the statements (a)-(i) are all logically equivalent. The proof is completed. □

In this paper, motivated by the concept of $\mathcal{P}$-functions, we first introduce the concept of $\mathcal{MT}(\lambda)$-functions as follows.

Definition 2.2. Let $\lambda > 0$. A function $\mu : [0, \infty) \to [0, \lambda)$ is said to be an $\mathcal{MT}(\lambda)$-*function* if $\limsup_{s\to t^+} \mu(s) < \lambda$ for all $t \in [0, \infty)$.

Remark 2.3.
- (i) Obviously, an $\mathcal{MT}$-function is an $\mathcal{MT}(1)$-function and a $\mathcal{P}$-function is an $\mathcal{MT}\left(\frac{1}{2}\right)$-function.
- (ii) It is easy to see that μ is an $\mathcal{MT}(\lambda)$-function if and only if $\lambda^{-1}\mu$ is an $\mathcal{MT}$-function.

The following characterizations of $\mathcal{MT}(\lambda)$-functions is an immediate consequence of Theorem 2.1.

Theorem 2.4. *Let $\lambda > 0$ and let $\mu : [0, \infty) \to [0, \lambda)$ be a function. Then the following statements are equivalent.*

(1) *μ is an $\mathcal{MT}(\lambda)$-function.*
(2) *$\lambda^{-1}\mu$ is an $\mathcal{MT}$-function.*
(3) *For each $t \in [0, \infty)$, there exist $\xi_t^{(1)} \in [0, \lambda)$ and $\epsilon_t^{(1)} > 0$ such that $\mu(s) \leq \xi_t^{(1)}$ for all $s \in (t, t + \epsilon_t^{(1)})$.*
(4) *For each $t \in [0, \infty)$, there exist $\xi_t^{(2)} \in [0, \lambda)$ and $\epsilon_t^{(2)} > 0$ such that $\mu(s) \leq \xi_t^{(2)}$ for all $s \in [t, t + \epsilon_t^{(2)}]$.*
(5) *For each $t \in [0, \infty)$, there exist $\xi_t^{(3)} \in [0, \lambda)$ and $\epsilon_t^{(3)} > 0$ such that $\mu(s) \leq \xi_t^{(3)}$ for all $s \in (t, t + \epsilon_t^{(3)}]$.*
(6) *For each $t \in [0, \infty)$, there exist $\xi_t^{(4)} \in [0, \lambda)$ and $\epsilon_t^{(4)} > 0$ such that $\mu(s) \leq \xi_t^{(4)}$ for all $s \in [t, t + \epsilon_t^{(4)})$.*
(7) *For any nonincreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n\in\mathbb{N}} \mu(x_n) < \lambda$.*
(8) *For any strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n\in\mathbb{N}} \mu(x_n) < \lambda$.*
(9) *For any eventually nonincreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n\in\mathbb{N}} \mu(x_n) < \lambda$.*
(10) *For any eventually strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n\in\mathbb{N}} \mu(x_n) < \lambda$.*

Remark 2.5. [23, Lemma 3.1] is a special case of Theorem 2.4 for $\lambda = \frac{1}{2}$.

3. Best proximity point and fixed point theorems for $\mathcal{MT}(\lambda)$-functions

Let X be a nonempty set and $\preceq$ a quasiorder (preorder or pseudo-order, i.e., a reflexive and transitive relation) on X. Then $(X, \preceq)$ is called a *quasiordered set.* A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called $\preceq$-*nondecreasing* (resp., $\preceq$-*nonincreasing*) if $x_n \preceq x_{n+1}$ (resp., $x_{n+1} \preceq x_n$) for each $n \in \mathbb{N}$. If (X, d) is a metric space with a quasiorder $\preceq$, we call it a *quasiordered metric space* $(X, d, \preceq)$ for short. Let $(X, \preceq)$ be a quasiorder set and $T : X \to X$ be a selfmapping. We say that T is

(i) $\preceq$-*nonincreasing* if $x, y \in X$ with $x \preceq y$ implies $Ty \preceq Tx$ or, equivalently, $Tx \succeq Ty$.

(ii) $\preceq$-*nondecreasing* if $x, y \in X$ with $x \preceq y$ implies $Tx \preceq Ty$.

A point x in X is a *fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $\mathcal{F}(T)$. Let (X, d) be a metric space. Define a function $S : X \times X \to [0, \infty)$ by

$$S(x, y) = d(x, y) + d(x, Tx) + d(y, Ty) + d(x, Ty) + d(y, Tx) \quad \text{for } x, y \in X.$$

In addition, for any $\lambda > 0$, we denote by $\Sigma(\lambda)$ the class of all $\mathcal{MT}(\lambda)$-functions.

First, we establish the following convergence theorem for the best proximity points, which is one of the main results in this paper.

Theorem 3.1. *Let $(X, d, \preceq)$ be a quasiordered metric space, A and B be nonempty subsets of X and T be a cyclic $\preceq$-nondecreasing selfmapping on $A \cup B$. Suppose that there exists $x_0 \in A$ such that $x_0 \preceq Tx_0$ and there exists $\mu \in \Sigma\left(\frac{1}{5}\right)$ such that*

$$d(Tx, Ty) \leq \mu(d(x, y))S(x, y) + (1 - 5\mu(d(x, y)))dist(A, B) \tag{3.1}$$

for all $x \in A$ and $y \in B$ with $x \preceq y$ or $x \succeq y$. Define an iterative sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ by $x_{n+1} = Tx_n$ for $n \in \mathbb{N} \cup \{0\}$. Then the following hold.

(a) *$\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a $\preceq$-nondecreasing sequence in $A \cup B$;*

(b) *$d(x_n, x_{n+1}) - dist(A, B) \leq \frac{3\mu(d(x_{n-1}, x_n))}{1-2\mu(d(x_{n-1}, x_n))}[d(x_{n-1}, x_n) - dist(A, B)]$ for all $n \in \mathbb{N}$;*

(c) *There exists $\gamma \in [0, 1)$, such that $d(x_n, x_{n+1}) \leq \gamma^n d(x_0, x_1) + (1-\gamma^n)dist(A, B)$ for all $n \in \mathbb{N}$;*

(d) *$\lim_{n\to\infty} d(x_{n-1}, x_n) = \inf_{n\in\mathbb{N}} d(x_{n-1}, x_n) = dist(A, B)$.*

Moreover, if $d(Tx, Ty) \leq d(x, y)$ for any $x \in A$ and $y \in B$ and $\{x_{2n}\}_{n\in\mathbb{N}\cup\{0\}}$ has a convergent subsequence in A, then there exists $v \in A$ such that $d(v, Tv) = dist(A, B)$.

Proof. Since T is cyclic and $x_0 \in A$, we know $x_{2n} \in A$ and $x_{2n+1} \in B$ for all $n \in \mathbb{N}\cup\{0\}$. Since T is $\preceq$-nondecreasing and $x_0 \preceq Tx_0 = x_1$, we have

$$x_1 = Tx_0 \preceq Tx_1 = x_2$$

and

$$x_2 = Tx_1 \preceq Tx_2 = x_3.$$

By induction, we know that $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a $\preceq$-nondecreasing sequence in $A \cup B$. Since $x_0 \in A$, $x_1 \in B$ and $x_0 \preceq x_1$, by (3.1), we get

$$\begin{aligned} d(x_1, x_2) &= d(Tx_0, Tx_1) \\ &\leq \mu(d(x_0, x_1))S(x_0, x_1) + (1 - 5\mu(d(x_0, x_1)))dist(A, B) \\ &= \mu(d(x_0, x_1))[d(x_0, x_1) + d(x_0, x_1) + d(x_1, x_2) + d(x_0, x_2) + d(x_1, x_1)] \\ &\quad + (1 - 5\mu(d(x_0, x_1)))dist(A, B) \\ &\leq \mu(d(x_0, x_1))[2d(x_0, x_1) + d(x_1, x_2) + (d(x_0, x_1) + d(x_1, x_2))] \end{aligned}$$

$$
\begin{aligned}
&+ (1 - 5\mu(d(x_0, x_1)))dist(A, B)\\
&= \mu(d(x_0, x_1))[3d(x_0, x_1) + 2d(x_1, x_2)] + (1 - 5\mu(d(x_0, x_1)))dist(A, B).
\end{aligned}
$$

which implies

$$
d(x_1, x_2) \leq \frac{3\mu(d(x_0, x_1))}{1 - 2\mu(d(x_0, x_1))}d(x_0, x_1) + \frac{1 - 5\mu(d(x_0, x_1))}{1 - 2\mu(d(x_0, x_1))}dist(A, B). \tag{3.2}
$$

Since $\mu \in \Sigma\left(\frac{1}{5}\right)$, we have $0 \leq \mu(t) < \frac{1}{5}$ and $1 - 2\mu(t) > \frac{3}{5}$ for $t \in [0, \infty)$. Hence, the last inequality (3.2) deduces

$$
d(x_1, x_2) - dist(A, B) \leq \frac{3\mu(d(x_0, x_1))}{1 - 2\mu(d(x_0, x_1))}\left[d(x_0, x_1) - dist(A, B)\right].
$$

Similarly, since $x_2 \in A$, $x_1 \in B$ and $x_1 \preceq x_2$, by (3.1) again, we have

$$
\begin{aligned}
d(x_3, x_2) &= d(Tx_2, Tx_1)\\
&\leq \mu(d(x_2, x_1))S(x_2, x_1) + (1 - 5\mu(d(x_2, x_1)))dist(A, B)\\
&= \mu(d(x_2, x_1))[d(x_2, x_1) + d(x_2, x_3) + d(x_1, x_2) + d(x_2, x_2) + d(x_1, x_3)]\\
&\quad + (1 - 5\mu(d(x_2, x_1)))dist(A, B)\\
&\leq \mu(d(x_1, x_2))[2d(x_1, x_2) + d(x_2, x_3) + (d(x_1, x_2) + d(x_2, x_3))]\\
&\quad + (1 - 5\mu(d(x_1, x_2)))dist(A, B)\\
&= \mu(d(x_1, x_2))[3d(x_1, x_2) + 2d(x_2, x_3)] + (1 - 5\mu(d(x_1, x_2)))dist(A, B).
\end{aligned}
$$

which implies

$$
d(x_2, x_3) - dist(A, B) \leq \frac{3\mu(d(x_1, x_2))}{1 - 2\mu(d(x_1, x_2))}\left[d(x_1, x_2) - dist(A, B)\right].
$$

Therefore, by induction, we obtain

$$
d(x_n, x_{n+1}) - dist(A, B) \leq \frac{3\mu(d(x_{n-1}, x_n))}{1 - 2\mu(d(x_{n-1}, x_n))}\left[d(x_{n-1}, x_n) - dist(A, B)\right] \tag{3.3}
$$

for all $n \in \mathbb{N}$. Since $0 \leq \mu(t) < \frac{1}{5}$ for all $t \in [0, \infty)$, we have

$$
\frac{3\mu(d(x_{n-1}, x_n))}{1 - 2\mu(d(x_{n-1}, x_n))} < 1 \quad \text{for all } n \in \mathbb{N}. \tag{3.4}
$$

By (3.3) and (3.4), we get

$$
d(x_n, x_{n+1}) - dist(A, B) < d(x_{n-1}, x_n) - dist(A, B)
$$

which implies

$$
d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \quad \text{for all } n \in \mathbb{N}.
$$

So, the sequence $\{d(x_{n-1}, x_n)\}_{n\in\mathbb{N}}$ is a strictly decreasing sequence in $[0, \infty)$. Hence

$$
\lim_{n\to\infty} d(x_{n-1}, x_n) = \inf_{n\in\mathbb{N}} d(x_{n-1}, x_n) \text{ exists.} \tag{3.5}
$$

Since $\mu \in \Sigma\left(\frac{1}{5}\right)$, by Theorem 2.4, we obtain

$$
0 \leq \sup_{n\in\mathbb{N}} \mu(d(x_{n-1}, x_n)) < \frac{1}{5}.
$$

Let $\xi = \sup_{n\in\mathbb{N}} \mu(d(x_{n-1}, x_n))$. Then $0 \le \xi < \frac{1}{5}$. For each $n \in \mathbb{N}$, since $\mu(d(x_{n-1}, x_n)) \le \xi$, we get

$$1 - 2\mu(d(x_{n-1}, x_n)) \ge 1 - 2\xi.$$

and hence

$$\frac{3\mu(d(x_{n-1}, x_n))}{1 - 2\mu(d(x_{n-1}, x_n))} \le \frac{3\xi}{1 - 2\xi} \quad \text{for all } n \in \mathbb{N}.$$

So,

$$0 \le \sup_{n\in\mathbb{N}} \frac{3\mu(d(x_{n-1}, x_n))}{1 - 2\mu(d(x_{n-1}, x_n))} \le \frac{3\xi}{1 - 2\xi} < 1.$$

Let

$$\gamma := \sup_{n\in\mathbb{N}} \frac{3\mu(d(x_{n-1}, x_n))}{1 - 2\mu(d(x_{n-1}, x_n))}.$$

Then $\gamma \in [0, 1)$. For each $n \in \mathbb{N}$, by (3.3) again, we have

$$\begin{aligned} d(x_n, x_{n+1}) - dist(A, B) &\le \frac{3\mu(d(x_{n-1}, x_n))}{1 - 2\mu(d(x_{n-1}, x_n))} [d(x_{n-1}, x_n) - dist(A, B)] \\ &\le \gamma[d(x_{n-1}, x_n) - dist(A, B)] \\ &\le \gamma^2[d(x_{n-2}, x_{n-1}) - dist(A, B)] \\ &\le \cdots \\ &\le \gamma^n[d(x_0, x_1) - dist(A, B)], \end{aligned}$$

and hence

$$d(x_n, x_{n+1}) \le \gamma^n d(x_0, x_1) + (1 - \gamma^n) dist(A, B).$$

Since $\gamma \in [0, 1)$, we have $\lim_{n\to\infty} \gamma^n = 0$. So the last inequality deduces

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = dist(A, B). \tag{3.6}$$

By taking into account (3.5) and (3.6), we get

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = \inf_{n\in\mathbb{N}} d(x_{n-1}, x_n) = dist(A, B). \tag{3.7}$$

Moreover, let us assume that $d(Tx, Ty) \le d(x, y)$ for any $x \in A$ and $y \in B$ and $\{x_{2n}\}_{n\in\mathbb{N}\cup\{0\}}$ has a convergent subsequence $\{x_{2n_k}\}$ in A. Then there exists $v \in A$ such that $x_{2n_k} \to v$ as $k \to \infty$. Since $Tv \in B$ and $d(Tx, Ty) \le d(x, y)$ for any $x \in A$ and $y \in B$, by (3.7), we have

$$dist(A, B) \le d(x_{2n_k}, Tv) \le d(x_{2n_k-1}, v) \le d(v, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}) \quad \text{for all } k \in \mathbb{N}.$$

By taking the limit from both sides of the last inequality as $k \to \infty$, we get $d(v, Tv) = dist(A, B)$. The proof is completed. □

Corollary 3.2. *Let $(X, d, \preceq)$ be a quasiordered metric space, A and B be nonempty subsets of X and T be a cyclic $\preceq$-nondecreasing selfmapping on $A \cup B$. Suppose that there exists $x_0 \in A$ such that $x_0 \preceq Tx_0$ and there exists $\kappa \in \left[0, \frac{1}{5}\right)$ such that*

$$d(Tx, Ty) \le \kappa S(x, y) + (1 - 5\kappa) dist(A, B)$$

for all $x \in A$ and $y \in B$ with $x \preceq y$ or $x \succeq y$. Define an iterative sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ by $x_{n+1} = Tx_n$ for $n \in \mathbb{N} \cup \{0\}$. Then the following hold.

(a) *$\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a $\preceq$-nondecreasing sequence in $A \cup B$;*

(b) $d(x_n, x_{n+1}) - dist(A, B) \leq \frac{3\kappa}{1-2\kappa}\left[d(x_{n-1}, x_n) - dist(A, B)\right]$ *for all* $n \in \mathbb{N}$*;*

(c) $d(x_n, x_{n+1}) \leq \left(\frac{3\kappa}{1-2\kappa}\right)^n d(x_0, x_1) + \left[1 - \left(\frac{3\kappa}{1-2\kappa}\right)^n\right] dist(A, B)$ *for all* $n \in \mathbb{N}$*;*

(d) $\lim_{n\to\infty} d(x_{n-1}, x_n) = \inf_{n\in\mathbb{N}} d(x_{n-1}, x_n) = dist(A, B)$.

Moreover, if $d(Tx, Ty) \leq d(x, y)$ *for any* $x \in A$ *and* $y \in B$ *and* $\{x_{2n}\}_{n\in\mathbb{N}\cup\{0\}}$ *has a convergent subsequence in* A*, then there exists* $v \in A$ *such that* $d(v, Tv) = dist(A, B)$*.*

Next, we present the following existence theorems for best proximity points. It is worth to mention that some conditions assumed in these two results are different.

Theorem 3.3. *Let* $(X, d, \preceq)$ *be a quasiordered metric space,* A *and* B *be nonempty subsets of* X *and* T *be a cyclic* $\preceq$*-nonincreasing selfmapping on* $A \cup B$*. Suppose that there exists* $x_0 \in A$ *such that* $x_0 \preceq T^2x_0 \preceq Tx_0$ *and there exists* $\mu \in \Sigma\left(\frac{1}{5}\right)$ *such that*

$$d(Tx, Ty) \leq \mu(d(x, y))S(x, y) + (1 - 5\mu(d(x, y)))dist(A, B)$$

for all $x \in A$ *and* $y \in B$ *with* $x \preceq y$*. Define an iterative sequence* $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ *by* $x_{n+1} = Tx_n$ *for* $n \in \mathbb{N} \cup \{0\}$*. Then the following hold.*

(i) *For any* $n \in \mathbb{N}$*,* $x_{2n-2} \preceq x_{2n}$*,* $x_{2n-1} \succeq x_{2n+1}$*,* $x_{2n} \preceq x_{2n-1}$ *and* $x_{2n+1} \succeq x_{2n}$*;*

(ii) $d(x_n, x_{n+1}) - dist(A, B) \leq \frac{3\mu(d(x_{n-1},x_n))}{1-2\mu(d(x_{n-1},x_n))}\left[d(x_{n-1}, x_n) - dist(A, B)\right]$ *for all* $n \in \mathbb{N}$*;*

(iii) *There exists* $\gamma \in [0, 1)$*, such that* $d(x_n, x_{n+1}) \leq \gamma^n d(x_0, x_1) + (1-\gamma^n)dist(A, B)$ *for all* $n \in \mathbb{N}$*;*

(iv) $\lim_{n\to\infty} d(x_{n-1}, x_n) = \inf_{n\in\mathbb{N}} d(x_{n-1}, x_n) = dist(A, B)$.

Moreover, if $d(Tx, Ty) \leq d(x, y)$ *for any* $x \in A$ *and* $y \in B$ *and* $\{x_{2n}\}_{n\in\mathbb{N}\cup\{0\}}$ *has a convergent subsequence in* A*, then there exists* $v \in A$ *such that* $d(v, Tv) = dist(A, B)$*.*

Proof. We first prove (i). Since T is cyclic and $x_0 \in A$, we know $x_{2n} \in A$ and $x_{2n+1} \in B$ for all $n \in \mathbb{N}\cup\{0\}$. By our hypothesis, we have

$$x_0 \preceq T^2x_0 = x_2.$$

Since T is $\preceq$-nonincreasing, we get

$$x_1 = Tx_0 \succeq Tx_2 = x_3,$$

$$x_2 = Tx_1 \preceq Tx_3 = x_4$$

and

$$x_3 = Tx_2 \succeq Tx_4 = x_5.$$

By induction, we obtain the following: for any $n \in \mathbb{N}$,

- $x_{2n-2} \preceq x_{2n}$,
- $x_{2n-1} \succeq x_{2n+1}$.

From our hypothesis again, we have

$$x_2 = T^2x_0 \preceq Tx_0 = x_1,$$

$$x_3 = Tx_2 \succeq Tx_1 = x_2,$$

$$x_4 = Tx_3 \preceq Tx_2 = x_3,$$

and

$$x_5 = Tx_4 \succeq Tx_3 = x_4.$$

So, by induction, we have the following: for any $n \in \mathbb{N}$,

- $x_{2n} \preceq x_{2n-1}$,
- $x_{2n+1} \succeq x_{2n}$.

Hence we show that (i) holds. The rest of the proof can be verified by following a similar argument as in the proof of Theorem 3.1 □

Corollary 3.4. *Let $(X, d, \preceq)$ be a quasiordered metric space, A and B be nonempty subsets of X and T be a cyclic $\preceq$-nonincreasing selfmapping on $A \cup B$. Suppose that there exists $x_0 \in A$ such that $x_0 \preceq T^2x_0 \preceq Tx_0$ and there exists $\kappa \in \left[0, \frac{1}{5}\right)$ such that*

$$d(Tx, Ty) \leq \kappa S(x, y) + (1 - 5\kappa)dist(A, B)$$

for all $x \in A$ and $y \in B$ with $x \preceq y$. Define an iterative sequence $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ by $x_{n+1} = Tx_n$ for $n \in \mathbb{N} \cup \{0\}$. Then the following hold.

(i) *For any $n \in \mathbb{N}$, $x_{2n-2} \preceq x_{2n}$, $x_{2n-1} \succeq x_{2n+1}$, $x_{2n} \preceq x_{2n-1}$ and $x_{2n+1} \succeq x_{2n}$;*
(ii) $d(x_n, x_{n+1}) - dist(A, B) \leq \frac{3\kappa}{1-2\kappa}\left[d(x_{n-1}, x_n) - dist(A, B)\right]$ *for all $n \in \mathbb{N}$;*
(iii) $d(x_n, x_{n+1}) \leq \left(\frac{3\kappa}{1-2\kappa}\right)^n d(x_0, x_1) + \left[1 - \left(\frac{3\kappa}{1-2\kappa}\right)^n\right] dist(A, B)$ *for all $n \in \mathbb{N}$;*
(iv) $\lim_{n \to \infty} d(x_{n-1}, x_n) = \inf_{n \in \mathbb{N}} d(x_{n-1}, x_n) = dist(A, B)$.

Moreover, if $d(Tx, Ty) \leq d(x, y)$ for any $x \in A$ and $y \in B$ and $\{x_{2n}\}_{n \in \mathbb{N} \cup \{0\}}$ has a convergent subsequence in A, then there exists $v \in A$ such that $d(v, Tv) = dist(A, B)$.

Let $(X, d, \preceq)$ be a quasiordered metric space. Recall that a nonempty subset M of X is said to be *sequentially $\preceq^{\uparrow}$-complete* [9] if every $\preceq$-nondecreasing Cauchy sequence in M converges. It is obvious that if M is complete then it is sequentially $\preceq^{\uparrow}$-complete. As a consequence of Theorem 3.1, we can prove the following new fixed point theorem for $\mathcal{MT}(\lambda)$-functions on quasiordered metric spaces.

Theorem 3.5. *Let $(X, d, \preceq)$ be a quasiordered metric space and $T : X \to X$ be a $\preceq$–nondecreasing selfmapping on X. Suppose that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and there exists $\mu \in \Sigma\left(\frac{1}{5}\right)$ such that*

$$d(Tx, Ty) \leq \mu(d(x, y))S(x, y) \tag{3.8}$$

for all $x, y \in X$ with $x \preceq y$. Then $\{T^n x_0\}_{n \in \mathbb{N} \cup \{0\}}$ is a $\preceq$-nondecreasing Cauchy sequence in X, where $T^0 = I$ (the identity mapping). Moreover, if $(X, d, \preceq)$ is sequentially $\preceq^{\uparrow}$-complete, then the sequence $\{T^n x_0\}_{n \in \mathbb{N} \cup \{0\}}$ converges to the unique fixed point of T.

Proof. Let $A = B = X$. Then $A \cup B = X$ and $dist(A, B) = 0$. So T is a cyclic $\preceq$-nondecreasing selfmapping on $A \cup B$ and (3.8) deduces

$$d(Tx, Ty) \leq \mu(d(x, y))S(x, y) + (1 - 5\mu(d(x, y)))dist(A, B)$$

for all $x \in A$ and $y \in B$ with $x \preceq y$ or $x \succeq y$. Define an iterative sequence $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ by $x_{n+1} = Tx_n$ for $n \in N \cup \{0\}$. Applying Theorem 3.1, we know that

$\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a $\preceq$-nondecreasing sequence in X and there exists $\gamma \in [0,1)$, such that

$$d(x_n, x_{n+1}) \le \gamma^n d(x_0, x_1) \quad \text{for all } n \in \mathbb{N}. \tag{3.9}$$

We claim that $\{x_n\}_{n=0}^{\infty}$ is Cauchy in X. Let $\alpha_n = \frac{\gamma^n}{1-\gamma} d(x_0, x_1)$, $n \in \mathbb{N}$. For m, $n \in \mathbb{N}\cup\{0\}$ with $m > n$, we have form (3.9) that

$$d(x_n, x_m) \le \sum_{j=n}^{m-1} d(x_j, x_{j+1}) < \alpha_n.$$

Since $\gamma \in [0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ and hence $\lim_{n\to\infty} \sup\{d(x_n, x_m) : m > n\} = 0$. This prove that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X.

Moreover, assume that (X, d) is sequentially $\preceq^{\uparrow}$-complete. So there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$. To finish the proof, we will show $v \in \mathcal{F}(T)$. For any $n \in \mathbb{N}\cup\{0\}$, by (3.8), we have

$$\begin{aligned} d(v, Tv) &\le d(v, Tx_n) + d(Tx_n, Tv) \\ &< d(v, x_{n+1}) \\ &\quad + \frac{1}{5}\left[d(x_n, v) + d(x_n, x_{n+1}) + d(v, Tv) + d(x_n, Tv) + d(v, x_{n+1})\right]. \end{aligned}$$

By taking the limit from both sides of the last inequality as $n \to \infty$, we get

$$d(v, Tv) \le \frac{2}{5} d(v, Tv),$$

which implies $d(v, Tv) = 0$ or $v \in \mathcal{F}(T)$. The uniqueness of the fixed point of T is easy to verify from (3.8). The proof is completed. □

Corollary 3.6. *Let $(X, d, \preceq)$ be a quasiordered metric space and $T : X \to X$ be a $\preceq$–nondecreasing selfmapping on X. Suppose that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and there exists $\kappa \in \left[0, \frac{1}{5}\right)$ such that*

$$d(Tx, Ty) \le \kappa S(x, y)$$

for all $x, y \in X$ with $x \preceq y$. Then $\{T^n x_0\}_{n\in\mathbb{N}\cup\{0\}}$ is a $\preceq$-nondecreasing Cauchy sequence in X, where $T^0 = I$ (the identity mapping). Moreover, if $(X, d, \preceq)$ is sequentially $\preceq^{\uparrow}$-complete, then the sequence $\{T^n x_0\}_{n\in\mathbb{N}\cup\{0\}}$ converges to the unique fixed point of T.

4. Applications to initial-value problems

In this section, by applying Corollary 3.6, we investigate the existence and uniqueness of solutions of initial-value problems for nonlinear first order ordinary differential equations.

Theorem 4.1. *Let $c, z_0 \in \mathbb{R}$ and $\zeta > 0$. Let $I = [c, c+\zeta]$ and $f : I \times \mathbb{R} \to \mathbb{R}$ be a function. Assume that*

(a) *f is continuous;*
(b) *for any $t \in I$, the function $f(t, \cdot)$ is nondecreasing;*
(c) *there exists a continuous function $g : I \to \mathbb{R}$ satisfying $\sup_{t\in I}\{g(t) - \int_c^t f(s, g(s))ds\} \le z_0$;*

(d) *there exists* $\kappa \in \left[0, \frac{1}{5}\right)$ *such that for any continuous functions* $x, y : I \to \mathbb{R}$ *with* $x(t) \le y(t)$ *for all* $t \in I$, *it holds*

$$\int_c^t |f(s,x(s)) - f(s,y(s))|\, ds$$
$$\le \kappa\, |x(t) - y(t)| + 2\kappa \left| x(t) + y(t) - 2z_0 - \int_c^t [f(s,x(s)) + f(s,y(s))] ds \right|$$

for all $t \in I$.

Then the problem

$$\left\{ \begin{array}{r} x'(t) = f(t, x(t)), \quad t \in I, \\ x(c) = z_0 \end{array} \right. \tag{4.1}$$

admits a unique solution.

Proof. Let $X := C(I, \mathbb{R})$ be the set of all continuous real-valued functions defined on the closed interval I. Define a function $\rho : X \times X \to [0, \infty)$ by

$$\rho(u, v) := \sup_{t \in I} |u(t) - v(t)| \qquad \text{for } u, v \in X.$$

Then (X, ρ) is a complete metric space. We endow the metric space X with the following quasiorder $\preceq_{(I)}$:

$$u, v \in X,\ u \preceq_{(I)} v \quad \Longleftrightarrow \quad u(t) \le v(t) \ \text{ for all } t \in I.$$

So, $(X, \rho, \preceq_{(I)})$ is a complete quasiordered metric space. Let us now introduce the operator $T : X \to X$ which is defined by

$$(Tu)(t) = z_0 + \int_c^t f(s, u(s)) ds$$

for all $u \in X$ and $t \in I$. Then $Tu \in X$ for all $u \in X$. Indeed, let $u \in X$ be given. For any $t, t_0 \in I$, we have

$$|(Tu)(t) - (Tu)(t_0)| = \left| \int_c^t f(s,u(s))ds - \int_c^{t_0} f(s,u(s))ds \right| \le \int_{t_0}^t |f(s,u(s))|\, ds$$

Since f is continuous, the last inequality implies that

$$(Tu)(t) \to (Tu)(t_0) \quad \text{as } t \to t_0.$$

So Tu is continuous and hence $Tu \in X$ for all $u \in X$. Notice that $w \in X = C(I, \mathbb{R})$ is a solution of (4.1) if and only if w is a fixed point of T. We will proceed with the following claims to find a fixed point of T.

Claim 1. $T : X \to X$ is a $\preceq_{(I)}$-nondecreasing selfmapping on X.

Let $u, v \in X$ with $u \preceq_{(I)} v$. Then $u(t) \le v(t)$ for all $t \in I$. By (b), we obtain

$$(Tv)(t) - (Tu)(t) = \int_c^t [f(s,v(s))ds - f(s,u(s))]ds \ge 0 \quad \text{for all } t \in I,$$

which implies $Tu \preceq_{(I)} Tv$. So T is $\preceq_{(I)}$-nondecreasing on X.

Claim 2. $g \in X$ and $g \preceq_{(I)} Tg$.

Indeed, from (c), we know that $g \in X$ and $g(t) \leq z_0 + \int_c^t f(s, g(s))ds = (Tg)(t)$ for all $t \in I$. Hence $g \preceq_{(I)} Tg$.

Claim 3. *$\rho(Tu, Tv) \leq \kappa[\rho(u, v) + \rho(u, Tu) + \rho(v, Tv) + \rho(u, Tv) + \rho(v, Tu)]$ for all $u, v \in X$ with $u \preceq_{(I)} v$.*

Let $u, v \in X$ with $u \preceq_{(I)} v$. Then $u(t) \leq v(t)$ for all $t \in I$. By (d), we obtain

$$\begin{aligned}
|(Tu)(t) - (Tv)(t)| &\leq \int_c^t |f(s, u(s)) - f(s, v(s))|\, ds \\
&\leq \kappa\, |u(t) - v(t)| + \kappa \left| u(t) - z_0 - \int_c^t f(s, u(s))ds \right| \\
&\quad + \kappa \left| v(t) - z_0 - \int_c^t f(s, v(s))ds \right| \\
&\quad + \kappa \left| u(t) - z_0 - \int_c^t f(s, v(s))ds \right| \\
&\quad + \kappa \left| v(t) - z_0 - \int_c^t f(s, u(s))ds \right| \\
&\leq \kappa[\rho(u, v) + \rho(u, Tu) + \rho(v, Tv) + \rho(u, Tv) + \rho(v, Tu)]
\end{aligned}$$

for all $t \in I$. So we get

$$\rho(Tu, Tv) \leq \kappa[\rho(u, v) + \rho(u, Tu) + \rho(v, Tv) + \rho(u, Tv) + \rho(v, u)]$$

for all $u, v \in X$ with $u \preceq_{(I)} v$ and Claim 3 is proved.

Now, all the hypotheses of Corollary 3.6 are fulfilled from Claims 1, 2 and 3. So, by applying Corollary 3.6, we know that the sequence $\{T^n g\}$ converges to the unique fixed point $w \in X$ of T. Hence $\mathcal{F}(T) = \{w\}$ is a singleton set and $w \in C(I, \mathbb{R})$ is the unique solution of (4.1). The proof is completed. □

Finally, we give a simple example illustrating Theorem 4.1.

Example 4.2. Consider the problem

$$\begin{cases} x'(t) = 2e^{2t} & \text{for } t \in [0, 1], \\ x(0) = 2. \end{cases} \tag{4.2}$$

It can easily be verified by direct calculation that the unique solution of the problem is

$$x(t) = e^{2t} + 1, \;\; t \in [0, 1].$$

On the other hand, let $g : [0, 1] \to \mathbb{R}$ be defined by

$$g(t) = e^{2t}, \;\; t \in [0, 1].$$

Then g is a continuous function satisfying $\sup_{t \in I}\{g(t) - \int_0^t f(s, g(s))ds\} \leq 2$, where $f(t, g(t)) = 2e^{2t}$ for $t \in I = [0, 1]$. Let $z_0 = 2$. It is obvious that the function $f(t, x) = 2e^{2t}$ satisfies the hypotheses (a), (b) and (d) in Theorem 4.1. Hence, by Theorem 4.1, we also know that the problem (4.2) has a unique solution.

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