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CONVEX COMBINATIONS OF MEASURABLE FUNCTIONS AND AXIOMS OF SET THEORY

GILLES GODEFROY

Abstract. We review the medial limits constructed by Christensen and Mokobodzki assuming the Continuum Hypothesis. On the other hand, we display what happens if we assume the Filter Dichotomy. We describe, under the Filter Dichotomy, a large class of non reflexive Banach spaces X such that every weak-star universally measurable linear form on X^* is actually weak-star continuous. On the other hand, we show that no such space exists if the Continuum Hypothesis is assumed.

1. Introduction

A well-known result of Sierpinski [24] asserts that a non-trivial ultrafilter on the set $\mathbb{N}$ of natural numbers is a very irregular object: more precisely, it is not measurable with respect to the canonical Haar measure on $2^{\mathbb{N}}$. It is unfortunate that such universal limits lack useful regularity properties, and it is natural to try to regain such properties with one more averaging process: if ones take an "integral" of ultrafilters - in other words, a finitely additive measure defined on $2^{\mathbb{N}}$ which vanishes on finite sets, or equivalently a Radon measure on the set $\beta\mathbb{N}\backslash\mathbb{N}$ - can it be measurable in some sense? The answer is negative in the usual set theory (ZFC) for topological regularity, as shown in [12]. On the other hand, in has been shown independently by J. P. R. Christensen [4] and G. Mokobodzki (see [19]) that the answer is positive for universal measurability, when the continuum hypothesis (CH) is assumed. One usually calls Medial limits such universally measurable finitely additive measures.

The Continuum Hypothesis has been subsequently replaced by weaker axioms such as Martin's axiom [21], but the question remained open whether Medial limits exist in (ZFC) until P. B. Larson [16] answered it negatively by showing that there is no Medial limit if the Filter Dichotomy is assumed (see Definition 4.7 below).

The first purpose of this note is to display some of the older results on the topological properties of filters and Medial limits, with simple proofs. Indeed it turns out that some of these theorems have been published in journals or reviews which are not easily accessible by now, and the reader may find it convenient to have a self-contained account of these results at hand. In this respect, this note can partly be considered as a survey. On the other hand, we state and prove some new

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results. To be precise, Proposition 4.2 and what follows Theorem 4.8 is original work. We conclude this note with questions which, to the best of our knowledge, are open.

To summarize our work, let us call an "averaging procedure" any method which consists of taking limits of successive convex combinations of functions. In short, this note demonstrates that on one hand the (bad) topological properties of averaging procedures can be established in (ZFC), and on the other hand the measure-theoretic properties of such procedures are quite sensitive to the axioms of set theory.

2. Topological lemmas

We recall in this section some well-known applications of Baire category techniques. Let P be a complete metric space. It will be called a Polish space if it is moreover separable. A subset E of P is rare if its closure has empty interior. A subset M of P is meager if it is contained in a countable union of rare sets, and then its complement $P\backslash M$ is called a comeager set. Baire's lemma states that any comeager set is dense (and in particular non empty). We recall that a subset A of a complete metric space P has the Baire property if there is an open set U such that $A\Delta U$ is meager. The collection of subsets which have the Baire property is a σ-field, which contains of course the open sets and thus the Borel sets. A map from P to a topological space S is called Baire measurable if the inverse image of every open subset of S has the Baire property.

Our first lemma is usually called the topological 0-1 law.

Lemma 2.1. *Let P be a Polish space, and G be a group of homeomorphisms of P such that for all U, V non-empty open sets in P, there is $g \in G$ such that $g(U) \cap V \neq \emptyset$. Let $A \subset P$ with the Baire Property such that $g(A) = A$ for all $g \in G$. Then A is meager or comeager.*

Proof. Let $B = P \setminus A$. If A and B are both non-meager, then there exist two non-empty open sets U and V such that $U \cap B$ and $V \cap A$ are both meager. Let $g \in G$ be such that the open set $W := g(U) \cap V$ is non-empty. Since $g(U) \cap B = g(U \cap B)$, we have that $W \cap B$ and $W \cap A$ are both meager, and this is a contradiction. □

Example 2.2. The relation E_0.

We see the Cantor set $2^{\mathbb{N}}$ as the set of subsets of $\mathbb{N}$, the set $2^{<\mathbb{N}}$ as the set of finite subsets of $\mathbb{N}$, and we define on $2^{\mathbb{N}}$ the following relation:

uE_0v if there is $n \geq 0$ such that

$$u \cap [n, +\infty) = v \cap [n, +\infty).$$

Then the equivalence classes for E_0 are the orbits of a group of homeomorphisms, namely the group G_0 of translations by finite subsets of $\mathbb{N}$, that is:

$$G_0 = \{(u\Delta .), u \in 2^{<\mathbb{N}}\}$$

where Δ denotes the symmetric difference (that is, the group law) on the Cantor set. Therefore any subset of $2^{\mathbb{N}}$ with the Baire property which is E_0-saturated, is meager or comeager. Informally, if B is a Baire measurable subset of $2^{\mathbb{N}}$ which does not depend upon finitely many coordinates, then B is meager or comeager.

Example 2.3. Free ultrafilters.

The map c defined by $c(u) = \mathbb{N}\backslash u$ is an homeomorphism of $2^{\mathbb{N}}$. If $\mathcal{U}$ is a free ultrafilter (that is, $\mathcal{U}$ contains the Fréchet filter of cofinite sets, and is maximal among filters with respect to inclusion), then $c[\mathcal{U}] = 2^{\mathbb{N}}\backslash\mathcal{U}$. If $\mathcal{U}$ has the Baire property, then by the above it is meager or comeager. But both terms of this dichotomy contradict $c[\mathcal{U}] = 2^{\mathbb{N}}\backslash\mathcal{U}$. Hence $\mathcal{U}$ fails the Baire property.

Our second lemma is a standard compactness argument (see [7], Lemma 7). The "compactness" we use is actually the trivial fact that a set with two points is compact. We recall that if A and B are subsets of $\mathbb{N}$, the notation $A < B$ means that $n < l$ for all $n \in A$ and all $l \in B$.

Lemma 2.4. *Let A be a subset of $2^{\mathbb{N}}$. The following assertions are equivalent:*

(1) *A is comeager,*
(2) *there is a sequence $I_0 < I_1 < I_2 < \cdots$ of successive subsets of $\mathbb{N}$, and $a_n \subset I_n$, such that for any $u \in 2^{\mathbb{N}}$, if the set $\{n : u \cap I_n = a_n\}$ is infinite, then $u \in A$.*

Proof. For the reverse implication, just note that

$$O_n = \{u \in 2^{\mathbb{N}} \mid \exists k \geq n, u \cap I_k = a_k\}$$

is a dense open set of $2^{\mathbb{N}}$ for any $n \geq 1$, and that

$$\cap_{n\geq 1} O_n \subset A.$$

For the direct implication, assuming A is comeager, we write

$$2^{\mathbb{N}} \setminus A \subset \cup_{n\geq 0} F_n,$$

where each F_n is closed with empty interior. An easy induction argument provides I_n and a_n such that

$$u \cap I_n = a_n \Rightarrow u \notin \cup_{i<n} F_i.$$

If $u \in F_k$, then $u \cap I_n \neq a_n$ for all $n > k$, and the conclusion follows. □

It results from the proof that we can assume without loss of generality that the I_k's constitute a partition of ω into intervals. The following corollary easily follows.

Corollary 2.5. *Let B be a subset of $2^{\mathbb{N}}$ such that:*

$$u \in B, u \subset v \Rightarrow v \in B.$$

Then B is meager if and only if there exist $I_0 < I_1 < I_2 < \cdots$ such that

$$u \in B \Rightarrow \{n; u \cap I_n = \emptyset\} \text{ is finite.}$$

Proof. The set

$$O_n = \{u \in 2^{\mathbb{N}} \mid \exists k \geq n, u \cap I_k \neq \emptyset\}$$

is a dense open set of $2^{\mathbb{N}}$ for any $n \geq 1$, and thus our condition on B implies that B is meager. Conversely, if B is meager, we have in the notation of Lemma 2.4 that if $v \in B$ then the set $\{n : v \cap I_n = a_n\}$ is finite. But since $u \in B$ and $u \subset v$ implies that $v \in B$, it follows that if $u \in B$, then $\{n; u \cap I_n = \emptyset\}$ is finite.

□

Corollary 2.5 applies of course to filters on $\mathbb{N}$. We say that a filter $\mathcal{F}$ is free if its contains the Fréchet filter. We can now state and prove a result from [27].

Theorem 2.6. *Let $\mathcal{F}$ be a free filter on $\mathbb{N}$. Then the following are equivalent:*

(1) *$\mathcal{F}$ has the Baire property.*
(2) *$\mathcal{F}$ is meager.*
(3) *there exist $I_0 < I_1 < I_2 < \cdots$ such that if $u \in \mathcal{F}$, then the set*

$$\{n; u \cap I_n = \emptyset\}$$

is finite.

Proof. It is obvious that (2) implies (1), and the converse implication follows from Lemma 2.1 since $\mathcal{F}$ does not depend upon finitely many coordinates. Indeed if we assume (1), then by Lemma 2.1, $\mathcal{F}$ is meager or comeager. Since the map c defined by $c(u) = \mathbb{N}\backslash u$ is an homeomorphism of $2^{\mathbb{N}}$, if $\mathcal{F}$ is comeager there is $u \in \mathcal{F}$ such that $\mathbb{N}\backslash u \in \mathcal{F}$ but this cannot be since $\mathcal{F}$ is a filter and this shows (2). The equivalence between (2) and (3) follows immediately from Corollary 2.5. $\square$

3. Topological properties of finitely additive measures on $\mathbb{N}$.

We observed in Example 2.3 that the topological 0-1 law easily implies that a free ultrafilter $\mathcal{U}$ fails the Baire property. The more elaborate Theorem 2.6 will allow us to show that even an "integral" of ultrafilters (that is, a purely finitely additive measure on $\mathbb{N}$) cannot have the Baire property.

Indeed let us state and prove this result from [12]. We recall that a finitely additive bounded measure on $\mathbb{N}$ is a bounded application from $2^{\mathbb{N}}$ to $\mathbb{R}$ such that $m(a \cup b) = m(a) + m(b)$ if $a \cap b = \emptyset$.

Theorem 3.1. *let m be a finitely additive bounded measure on $\mathbb{N}$, which is not identically 0. We assume that $m(f) = 0$ for every finite subset f of $\mathbb{N}$. Then the set*

$$Z = \{u \in 2^{\mathbb{N}};\ m(u) = 0\}$$

fails the Baire property.

Proof. Assume first that m takes positive values and that $m(\mathbb{N}) = 1$. We assume that Z has the Baire property. Then the set $\mathcal{F} = \{u \subset \mathbb{N};\ m(u) = 1\}$ is a filter, which is homeomorphic to Z by the complementation map c. By Theorem 2.6 this filter is meager and it satisfies condition (3).

There is a family $(Y_t)_{t\in\mathbb{R}}$ of infinite subsets of $\mathbb{N}$, indexed by the real numbers, such that if $t \neq s$ then $(Y_t \cap Y_s)$ is finite. For any $t \in \mathbb{R}$, we let

$$F(t) = \cup_{j\in Y_t} I_j.$$

If $t \neq s$ then $F(t) \cap F(s)$ is finite, and thus $m(F(t) \cup F(s)) = m(F(t)) + m(F(s))$. Since m is bounded, it follows that there is $t_0 \in \mathbb{R}$ such that $m(F(t_0)) = 0$ (actually, this condition holds for all t's outside some countable set). But then, $u = \mathbb{N}\backslash F(t_0) \in \mathcal{F}$ and $u \cap I_j = \emptyset$ for all $j \in Y_{t_0}$, and this contradicts condition (3).

Let now m be an arbitrary bounded measure. Since m is bounded, we can write $m = m^+ - m^-$, with m^+ and m^- positive measures and $|m| = m^+ + m^-$. We let

$$Z_1 = \{u \in 2^{\mathbb{N}};\ |m|(u) = 0\}.$$

If $m \neq 0$, the set Z_1 is not meager by the above, and since $Z_1 \subset Z$, the set Z is not meager. If Z has the Baire property, it is therefore comeager by Lemma 2.1 since it does not depend upon finitely many coordinates. But if $m(u_0) \neq 0$ and $u \in Z_1$, we have $m(u_0 \Delta u) = m(u_0)$ and thus $u_0 \Delta Z_1 \cap Z = \emptyset$. This cannot be since $(u_0 \Delta Z_1)$ is a translate of Z_1 and is therefore not meager, while Z is comeager. □

Example 3.2. extending the density function, I.

A subset u of $\mathbb{N}$ has density $\alpha \in [0,1]$ if

$$\lim_n (n^{-1} Card(u \cap \{1, 2, ...n\})) = \alpha.$$

It is clear that the density function is finitely additive and vanishes on finite sets, but of course it is not defined on every subset of $\mathbb{N}$. The question occurs to know if can be extended to some usable finitely additive bounded measure D defined on all subsets of $\mathbb{N}$. Theorem 3.1 above tells us that such an extension is necessarily quite irregular: it cannot be Baire-measurable (and in particular, it cannot be a Borel map from the Cantor set to $\mathbb{R}$). However, we could replace the σ-field of Baire-measurable sets by its measure theoretic analogue, namely the σ-field of universally measurable sets. We will see below that it cannot be decided in (ZFC) whether a universally measurable extension exists or not.

Along these lines, let us recall that by [26], the affine functions ϕ on the Hilbert cube $[0,1]^{\mathbb{N}}$ (equipped with the product topology) which are Baire-measurable are actually continuous. Note that the boundedness of ϕ is not part of the assumptions in [26] but follows from the conclusion.

4. Medial limits and the axioms.

We recall that a real-valued function f defined on some Polish space P is called universally measurable if for every Radon measure μ, there exists a Borel function f_μ such that $f_\mu = f$ μ- almost everywhere. The following theorem has been shown independently by J. P. R. Christensen and G. Mokobodzki (see [4], [19] or [8]), provided that the Continuum Hypothesis (CH) is assumed.

Theorem 4.1. *Assume the Continuum Hypothesis. Let P be a Polish space, and let (f_n) be a uniformly bounded sequence of universally measurable functions. Let $C_n = conv(f_k; k \geq n)$ and let D_n be the closure of C_n for the topology of pointwise convergence. Then the intersection of the sequence (D_n) contains a universally measurable function.*

Proof. Let ω_1 be the first uncountable ordinal. The Continuum Hypothesis provides a complete list $(\mu_\alpha)_{\alpha<\omega_1}$ of all bounded Radon measures on P, indexed by the countable ordinals.

We claim the existence, for all $\alpha < \omega_1$, of sequences $(c_k^\alpha)_{k\geq 1}$ of convex combinations of the sequence (f_n) such that

(1) If $\beta < \alpha$, then there exists $N = N_{\beta,\alpha}$ such that for all $n \geq N$, $conv(c_k^\alpha;\ k \geq n) \subset conv(c_k^\beta :\ k \geq n)$.

(2) For all $\alpha < \omega_1$, the sequence $(c_n^\alpha)_n$ converges μ_α- almost everywhere.

We proceed by transfinite induction. We first observe that (f_n) is a bounded sequence in the Hilbert space $L^2(P,\mu_0)$, hence by reflexivity of this Banach space there is a sequence $(c_k^0)_{k\geq 1}$ of successive convex combinations of the sequence (f_n) which converges in L^2, and thus a subsequence - still denoted in the same manner - which converges μ_0- almost everywhere.

Assume now that ψ is a limit ordinal, and that the construction has been completed for all $\beta<\psi$. We pick a sequence (ω_p) of ordinals which strictly increases to ψ. We let $g_p = c_p^{\omega_p}$, where we may and do assume that $g_p \in conv(c_k^{\omega_l} :\ k\geq p)$ for all $l<p$. Therefore if $n\geq p>l$, we have

$$g_n \in conv(c_k^{\omega_l} :\ k\geq n) \subset conv(c_k^{\omega_l} :\ k\geq p)$$

and therefore

$$conv(g_n;\ n\geq p) \subset conv(c_k^{\omega_l} :\ k\geq p)$$

for all $l<p$. Using reflexivity again, we find a sequence of convex combinations $g_p' \in conv(g_n;\ n\geq p)$ which converges μ_ψ- almost everywhere. We define $c_n^\psi = g_n'$.

We have therefore, if $n\geq p>l$,

$$c_n^\psi \in conv(c_k^{\omega_l} :\ k\geq p).$$

If $\beta<\psi$, there exists l_0 such that $\beta<\omega_{l_0}$. By our induction hypothesis, there exists $p_0>l_0$ such that if $p\geq p_0$,

$$conv(c_k^{\omega_{l_0}} :\ k\geq p) \subset conv(c_k^\beta :\ k\geq p)$$

and therefore for all $p\geq p_0$,

$$conv(c_k^\psi :\ k\geq p) \subset conv(c_k^\beta :\ k\geq p).$$

This concludes the construction in the case of limit ordinals. The case of successor ordinals is similar but simpler.

For all $\psi<\omega_1$, we let $g_\psi(x) = \lim c_n^\psi(x)$ for all $x\in P$ such that the limit exists, which is the case μ_ψ-almost everywhere. It is clear that g_ψ a μ_ψ-measurable function, defined μ_ψ-almost everywhere. Condition (1) above shows that if $\alpha<\psi$, then $g_\psi(x)$ is defined when $g_\alpha(x)$ is defined, hence μ_α- almost everywhere, and moreover $g_\psi(x)=g_\alpha(x)$.

We may consistently define $G(x)=g_\alpha(x)$, where α is the least ordinal for which $\lim c_n^\alpha(x)$ exists. The function G is defined everywhere since every Dirac measure δ_x shows up in the list $(\mu_\alpha)_{\alpha<\omega_1}$. Our construction shows that G is universally measurable, and moreover that $G=\lim c_n^\alpha$ μ_α-almost everywhere for all $\alpha<\omega_1$. In particular $G\in D_n$ for all n, and this concludes the proof. □

We note that the above proof provides, under (CH), a universally measurable function G such that for every Radon measure μ, there exists a sequence (c_k^μ) of successive convex combinations of the sequence (f_n) such that $\lim c_k^\mu = G$ μ- almost everywhere.

We recall that Komlos theorem [15] states that if (f_n) is a bounded sequence in a space $L^1(P,\mu)$, there exists an infinite subset X on $\mathbb{N}$ such that for every infinite subset Y of X, then

$$Lim_N\ (\frac{1}{N}\sum_{k\in Y_N} f_k)$$

exists μ-almost everywhere, where Y_N denotes the set consisting of the N smallest elements of Y. We refer to [9] for a small improvement of Komlos theorem, where it is shown that we can request that the convergence takes place almost everywhere and in the quasi-Banach space $L(1,\infty)$.

It is tempting to conjecture that, under the Continuum Hypothesis, one could show a universally measurable version of Komlos theorem, which would read as follows: let (f_n) be a uniformly bounded sequence of universally measurable functions. Then there exists a universally measurable function g such that for every bounded Radon measure μ on P, there exists an infinite subset X on $\mathbb{N}$ such that for every infinite subset Y of X,

$$Lim_N\ (\frac{1}{N}\sum_{k\in Y_N} f_k) = g$$

μ-almost everywhere. But even a weaker version of this property fails, as shown by the following observation.

Proposition 4.2. *Let $K = \{0,1\}^{\mathbb{N}}$ be the Cantor set equipped with the product topology, and let $f_n : K \to \{0,1\}$ be the sequence of coordinate functionals. Then there is no universally measurable function g on K such that for every measure μ with finite support and every infinite subset Y of $\mathbb{N}$, there exists an infinite subset Z of Y such that $Lim_N\ (\frac{1}{N}\sum_{k\in Z_N} f_k) = g$ μ-almost everywhere.*

Proof. It suffices to show that if g is a function on K which satisfies the conclusion of the theorem, then g is the characteristic function $\mathbf{1}_{\mathcal{U}}$ of a free ultrafilter $\mathcal{U}$. Indeed, let us recall a result that goes back to W. Sierpinski [24].

Lemma 4.3. *The characteristic function $\mathbf{1}_{\mathcal{U}}$ of a free ultrafilter $\mathcal{U}$ is not measurable for the Haar measure m_H on K.*

Indeed, by Kolmogorov's 0-1 law, if it were measurable we would have $m_H(\mathcal{U}) \in \{0,1\}$, but this cannot be since the complementation map c leaves m invariant and maps $\mathcal{U}$ to its complement.

In particular, $\mathbf{1}_{\mathcal{U}}$ is not universally measurable.

Pick now g which satisfies the conclusion of the proposition. If $supp(\mu) = F$ is finite, our convergence assumption means that

$$Lim_N\ (\frac{1}{N}\sum_{k\in Z_N} f_k(x)) = g(x)$$

for every $x \in F$. Consider the set

$$\mathcal{X}_F = \{Z \subset \mathbb{N};\ Lim_N\ (\frac{1}{N}\sum_{k\in Z_N} f_k(x)) = g(x)\ for\ all\ x \in F\}$$

where only infinite sets Z are considered. It is easily checked (see [18], p. 105) that the set $\mathcal{X}_F$ is Borel in $2^{\mathbb{N}}$. Hence it is Ramsey by Silver's theorem [25], and since every infinite subset of $\mathbb{N}$ contains an element of $\mathcal{X}_F$, the Ramsey property shows that there is $Z \in \mathcal{X}_F$ such that every infinite subset Y of Z belongs also to $\mathcal{X}_F$. But then, it is easy to show that in fact,

$$\lim_{k\in Z} f_k(x) = g(x)$$

for all $x \in F$. Since F was an arbitrary finite subset of K, we have therefore that g is a pointwise cluster point to the sequence (f_n). But these cluster points are the functions $\mathbf{1}_{\mathcal{U}}$, where $\mathcal{U}$ is a free ultrafilter. $\square$

Example 4.4. Extending the density function, II.

The density function has been defined at the end of section 3. Applying Theorem 4.1 to the sequence $f_n(u) = n^{-1}Card(u \cap \{1, 2, ...n\})$ shows that under the Continuum Hypothesis, there exists a universally measurable map D on $2^{\mathbb{N}}$ which extends the density function. Hence, although a ultrafilter is not universally measurable, an "integral" of ultrafilters can be universally measurable, at least under (CH). In other words, we can integrate the topological 0-1 law, but not the measure-theoretic one.

Example 4.5. Medial limits, I

Following the terminology used in [16], we call a Medial limit any universally measurable finitely additive measure $m : 2^{\mathbb{N}} \to [0, 1]$ which maps the singletons to 0 and $\mathbb{N}$ to 1. For instance, any universally measurable extension D of the density function is a Medial limit. Theorem 4.1 shows that Medial limits exist under (CH), and they exist as well under Martin's axiom [21], and also under the weaker axiom that the real line is not the union of less than c meager sets (see 538S in [8]). We will see below that some axiom is needed.

Example 4.6. Medial limits, II

The notation "medial limits" sometimes denotes special linear forms G on l_∞, as follows. We let $P = [-1, 1]^{\mathbb{N}}$ equipped with the product topology be the Hilbert cube, or equivalently the unit ball of l_∞ equipped with the weak* topology. The coordinate functionals f_n are continuous linear forms. Applying Theorem 4.1 provides $G \in c_0^{\perp} \subset l_\infty^*$ with $\|G\| = 1 = G(\mathbf{1})$ which is universally measurable on $(l_\infty, weak*)$. Of course, the restriction of G to $\{0, 1\}^{\mathbb{N}}$ is a Medial limit in the sense I above. Note that if (g_n) is a uniformly bounded sequence of universally measurable functions on some Polish space P, then the function $G((g_n))$ is universally measurable, and it is a limit of (g_n) in the sense that it is invariant under the change of finitely many g_n's.

G. Mokobodzki applied Medial limits to potential theory, and J. P. R. Christensen to liftings, that is, pointwise evaluations of elements of L_∞. We refer for instance to ([5], p. 194, see also [6]) for applications to Dixmier traces, and to [17] for applications to social sciences.

We will now show the existence of models of (ZFC) in which there is no medial limit (a theorem of [16]). For doing so, we first need a definition.

Definition 4.7. The Filter Dichotomy is the statement that for each non meager filter $\mathcal{F}$ on $\mathbb{N}$, there is a finite-to-one function $h : \mathbb{N} \to \mathbb{N}$ such that $\{h[x]; x \in \mathcal{F}\}$ is a ultrafilter.

Let us mention that this statement is called a dichotomy since, by Theorem **??**, when it holds then a filter $\mathcal{F}$ is mapped by a finite-to-one map either to the Fréchet filter or to an ultrafilter, and the first condition holds exactly when $\mathcal{F}$ is meager. It is shown in [1] that the Filter Dichotomy holds in models of set theory previously considered in [20], [2] and [3].

The statement below is slightly different from the formulation given in the original article [16]. However the proof follows the same lines. Note that the conclusion states that if the Filter Dichotomy is assumed, there is no Medial limit in the sense I above, and thus no Medial limit in the sense II.

Theorem 4.8. *Assume that the Filter Dichotomy holds. Let $K = \{0,1\}^{\mathbb{N}}$ be the Cantor set equipped with the product topology, and let $f_n : K \to \{0,1\}$ be the sequence of coordinate functionals. Let $C_n = conv(f_k; k \geq n)$ and let D_n be the closure of C_n for the topology of pointwise convergence. Then the intersection of the sequence (D_n) contains no universally measurable function.*

Proof. Let $g : K \to [0,1]$ be a function which belongs to all D_n's. It is clear that g is a positive finitely additive measure with $g(\mathbb{N}) = 1$, and which vanishes on finite sets. By Theorem 3.1, the set

$$Z = \{u \in 2^{\mathbb{N}};\ g(u) = 0\}$$

fails the Baire property, and since it is homeomorphic (by the complementation map c) to the filter

$$\mathcal{F} = \{u \in 2^{\mathbb{N}};\ g(u) = 1\}$$

it follows that $\mathcal{F}$ fails the Baire property and thus is not meager. By the Filter Dichotomy, there exists a finite-to-one map $h : \mathbb{N} \to \mathbb{N}$ such that $\{h[x]; x \in \mathcal{F}\}$ is a free ultrafilter $\mathcal{U}$. We let

$$S = \{h^{-1}[u] :\ u \subset \mathbb{N}\}.$$

The subset S of $2^{\mathbb{N}}$ is clearly homeomorphic (by h) to $2^{\mathbb{N}}$, and $S \cap \mathcal{F}$ is homeomorphic to $\mathcal{U}$. We know by Lemma 4.3 that $\mathcal{U}$ is not measurable for the Haar measure on $2^{\mathbb{N}}$, and it follows that $\mathcal{F}$ is not universally measurable. Therefore g is not universally measurable since $\mathcal{F} = g^{-1}(1)$. □

Corollary 4.9. *Assume the Filter Dichotomy. Let $G \in l_\infty^*$ be a universally measurable linear form on $(l_\infty, weak*)$. Then $G \in l_1$.*

Proof. It suffices to show that if $G \in c_0^\perp \subset l_\infty^*$ is universally measurable on $(l_\infty, weak*)$ then $G = 0$. The set of weak* universally measurable elements of l_∞^* is a band (see 538R in [8]) and thus we may assume that G is a positive linear form, in other words a positive Radon measure on $\beta\mathbb{N}\backslash\mathbb{N}$. But then, if $G \neq 0$, the restriction of $G/\|G\|$ to $K = \{0,1\}^{\mathbb{N}}$ belongs to all the D_n's, and this contradicts Theorem 4.8. □

We can show along the lines of the proof of Theorem 4.8 that Theorem 3.1 has a (conditional) analogue in terms of universally measurable functions. Indeed one has:

Proposition 4.10. *Assume the Filter Dichotomy. Let m be a finitely additive bounded measure on $\mathbb{N}$, which is not identically 0. We assume that $m(f) = 0$ for every finite subset f of $\mathbb{N}$. Then the set*

$$Z = \{u \in 2^{\mathbb{N}};\ m(u) = 0\}$$

is not universally measurable.

Proof. We pick $A \in 2^{\mathbb{N}}$ such that $m(A) \neq 0$. It follows that $|m|(A) \neq 0$. We denote

$$\mathcal{F} = \{u \in 2^A; |m|(u) = |m|(A)\}$$

Then $\mathcal{F}$ fails the Baire property. By the Filter Dichotomy, there exists a finite-to-one map $h : A \to \mathbb{N}$ such that $\{h[x]; x \in \mathcal{F}\}$ is a free ultrafilter $\mathcal{U}$. We let again

$$S = \{h^{-1}[u] :\ u \subset \mathbb{N}\}.$$

Then $S \subset 2^A$ satisfies: if $v \in S$ and $h(v) \in \mathcal{U}$, then $|m|(v) = |m|(A)$. If $v \in S$ and $h(v) \notin \mathcal{U}$, then $|m|(v) = 0$. But then, $h(v) \notin \mathcal{U}$ implies that $m(v) = 0$, and thus $h(v) \in \mathcal{U}$ implies $m(v) = m(A)$. The conclusion follows by Lemma 4.3. □

The last result of this section, which is again conditional to the Filter Dichotomy (and fails under (CH)), uses the same approach. It concerns the spaces which are stuck between c_0 and its bidual. It extends Lemma 6 in [10], where the result is shown in (ZFC) under the assumption that X is weak*-Suslin. The gist of the next result is that the quotient space l_∞/X is an "ergodic" space, which is somehow too large for allowing a regular embedding into a countably separated space.

Proposition 4.11. *Assume the Filter Dichotomy. Let X be a proper weak*-universally measurable subspace of l_∞ such that $c_0 \subset X \subset l_\infty$. Then there is no continuous linear injection from l_∞/X into l_∞.*

Note that this Proposition implies in particular that the space X is not complemented in l_∞, or equivalently that it is not isomorphic to l_∞. Under (CH), the kernel of a Medial limit (see Example 4.6 above) is weak*-universally measurable, contains c_0 and is isomorphic to l_∞.

Proof. Assume that such a linear injection exists. Then there is a sequence $(m_n)_n \subset c_0^\perp$ on norm-one linear forms such that $X = \cap_{n\geq 1} Ker(m_n)$. Since X is a proper subspace of l_∞, we may and do assume that $m_1 \neq 0$. We define

$$\mu = \sum_{n\geq 1} 2^{-n}|m_n|.$$

We pick $A \subset \mathbb{N}$ such that $m_1(A) \neq 0$. Then $\mu(A) > 0$. We proceed as before with the filter

$$\mathcal{G} = \{u \in 2^A; \mu(u) = \mu(A)\}$$

to find a finite-to-one map $h : A \to \mathbb{N}$ such that $\{h[x]; x \in \mathcal{G}\}$ is a free ultrafilter $\mathcal{U}$. We let again

$$S = \{h^{-1}[u] :\ u \subset \mathbb{N}\}.$$

Then $S \subset 2^A$ satisfies: if $v \in S$ and $h(v) \in \mathcal{U}$, then $\mu(v) = \mu(A)$. If $v \in S$ and $h(v) \notin \mathcal{U}$, then $\mu(v) = 0$. But then, $h(v) \notin \mathcal{U}$ implies that $m_1(v) = 0$, and thus $h(v) \in \mathcal{U}$ implies $m_1(v) = m_1(A) \neq 0$. Denote

$$V = \{v \in S;\ h(v) \notin \mathcal{U}\}.$$

If we identify in the obvious way S with a subset of $l_\infty(A) \subset l_\infty$, we have therefore that $S \cap X = V$, and then Lemma 4.3 shows that X is not universally measurable. □

5. Universally measurable linear forms

A sequence (x_n) in a Banach space is called weakly unconditionnally convergent (in short, w.u.c.) if for every $x^* \in X^*$, one has

$$\sum_{n\geq 1} |x^*(x_n)| < \infty.$$

Note that the partial sums of a w.u.c. sequence constitute a weakly Cauchy sequence. In particular, they weak* converge in any dual space E^*.

A Banach space E has Property (X) (see [13], [11]) if whenever $x^{**} \in E^{**}$ satisfies

$$\sum_{n\geq 1} x^{**}(x_n^*) = x^{**}(w^* - Lim_N \sum_{n=1}^{N} x_n^*)$$

for every w.u.c. sequence (x_n^*) in E^*, then $x^{**} \in E$. In other words, E has property (X) whenever one can check that an element $x^{**} \in E^{**}$ is weak*-continuous by testing it not on all weak*-convergent sequences - as for any separable space E - but only on partial sums of w.u.c. sequences. In practice, property (X) means that the elements of E are characterized within E^{**} by some kind of σ-additivity. Hence this property usually relies on some abstract Radon-Nikodym theorem.

It is easily seen that Property (X) is hereditary and, although it is of isomorphic nature, any space with (X) is unique isometric predual of its dual ([13], see [11]). The space L^1, and separable preduals of von Neumann algebras, have property (X). It is shown in [22] that a separable space which is L-complemented in its bidual has property (X), hence for instance quotients of L-complemented spaces by nicely placed subspaces have (X) (see [14] for definitions).

The main result of this section is the following.

Theorem 5.1. (1) *Assume the Continuum Hypothesis. Let E be a non-reflexive Banach space. Then there exists $x^{**} \in E^{**}\backslash E$ which is weak*-universally measurable on E^*.*

(2) *Assume the Filter Dichotomy. Let E be a Banach space with property (X). Then every $x^{**} \in E^{**}$ which is weak*-universally measurable on E^* belongs to E.*

Proof. For showing (1), we rely on Rosenthal's dichotomy [23]. If E is not reflexive and does not contain l_1, there exists a weakly Cauchy sequence in E which is not weakly convergent in E. Its weak* limit in E^{**} is a first Baire class function on $(E^*, weak*)$, and is therefore a universally measurable linear form which does not belong to E. If E contains a subspace Y isomorphic to l_1, we find with (CH) some $G \in Y^{\perp\perp} \subset E^{**}$ a Medial limit which is weak*-universally measurable (on Y^*, and thus on E^*) and does not belong to E.

We now assume the Filter Dichotomy. Let $x^{**} \in E^{**}$ be weak*- universally measurable, where E has property (X). If (x_n^*) is any w.u.c. sequence in E^*, we can define $G \in l_\infty^*$ by the formula

$$G((t_n)) = \sum_{n\in\mathbb{N}} x^{**}(t_n x_n^*).$$

The linear from G is weak*- universally measurable on l_∞, and thus by Corollary 4.9, $G \in l_1$. But this implies that

$$\sum_{n\geq 1} x^{**}(x_n^*) = x^{**}(w^* - Lim_N \sum_{n=1}^{N} x_n^*)$$

and thus by Property (X), $x^{**} \in E$. This shows (2). $\square$

Let us conclude this work with some related questions, which appear to be open, and hopefully not desperately hard.

Problems: 1. Let X be a norm-closed and weak*-Suslin linear subspace of l_∞. We assume that X is linearly isomorphic to l_∞. Is X weak*-closed? Note that Lemma 6 in [10] provides a positive answer if $c_0 \subset X$.

2. Let (f_n) be a uniformly bounded sequence of continuous functions on a Polish space P. Assume the Continuum Hypothesis. Does there exist a universally measurable function h such that for every Radon measure μ, there exists $Y_\mu \subset \mathbb{N}$ such that $(N^{-1}(\sum_{(Y_\mu)_N} f_j))$ converges μ-almost everywhere to h when $N \to \infty$?

3. Let m be a finitely additive bounded measure on $\mathbb{N}$, which is not identically 0, and such that $m(f) = 0$ for every finite subset f of $\mathbb{N}$. We assume the Filter Dichotomy. Does it follow that the set

$$Z = \{u \in 2^{\mathbb{N}};\ m(u) = 0\}$$

is not m_H-measurable, where m_H denotes the canonical (Haar) measure on $2^{\mathbb{N}}$?

We refer to [27–29] for relevant techniques and results on measurable filters.

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Gilles Godefroy
Institut de Mathématiques de Jussieu, 4 place Jussieu, 75252 Paris Cedex 05, France
E-mail address: `gilles.godefroy@imj-prg.fr`