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STRONG CONVERGENCE THEOREMS FOR A FIXED POINT OF A LIPSCHITZ PSEUDOCONTRACTIVE MULTI-VALUED MAPPING

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Abstract. It is our purpose in this paper to introduce an iterative process which converges strongly to a fixed point of a Lipschitz pseudocontractive multi-valued mapping under appropriate conditions.

1. Introduction

Let E be a nonempty real normed linear space. A subset K of E is called proximinal if for each $x \in E$ there exists $k \in K$ such that

$$\|x-k\| = \inf\{\|x-y\| : y \in K\} = d(x,K).$$

It is known that every closed convex subset of a uniformly convex Banach space is proximinal. In fact, if K is a closed and convex subset of a uniformly convex Banach space E, then for any $x \in E$ there exists a unique point $u_x \in K$ such that (see, e.g., [7, 14, 24, 25])

$$\|x-u_x\| = \inf\{\|x-y\| : y \in K\} = d(x,K).$$

Let E be a nonempty real normed space. We will denote the family of all nonempty proximinal subsets of E by $P(E)$, the family of all nonempty closed, convex and bounded subsets of E by $CBC(E)$, the family of all nonempty closed and bounded subsets of E by $CB(E)$ and the family of all nonempty subsets of E by 2^E for a nonempty normed space E.

Let D be the Hausdorff metric induced by the metric d on E, that is, for every $A, B \in CB(E)$,

$$D(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}.$$

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A multi-valued mapping $T : D(T) \subseteq E \to 2^E$ is called L-Lipschitzian if there exists $L \geq 0$ such that,

$$\forall\, x, y \in D(T),\ D(Tx, Ty) \leq L\|x - y\|. \tag{1.1}$$

In (1.1) if $L \in [0, 1)$, T is said to be a contraction, while T is nonexpansive if $L = 1$.

Let $T : D(T) \subseteq E \to 2^E$ be a multi-valued mapping on E. A point $x \in D(T)$ is called a fixed point of T if $x \in Tx$. The set $F(T) = \{x \in D(T) : x \in Tx\}$ is called a fixed point set of T. Let K be a subset of a real Hilbert space H. A mapping $T : K \to CB(K)$ is said to be *pseudocontractive* (see [18, 19, 23]), if the inequality

$$\langle u - v, x - y\rangle \leq \|x - y\|^2, \tag{1.2}$$

holds for each $x, y \in K, u \in Tx,\ v \in Ty$. In this case,

$$\|x - y - (u - v)\|^2 + 2\langle u - v, x - y\rangle \leq 2\|x - y\|^2 + \|x - y - (u - v)\|^2,$$

which implies that

$$\|u - v\|^2 \leq \|x - y\|^2 + \|x - y - (u - v)\|^2.$$

Hence, $T : K \to CB(K)$ is said to be *pseudocontractive* multi-valued mapping, if $\forall\, x, y \in K$

$$\|u - v\|^2 \leq \|x - y\|^2 + \|x - y - (u - v)\|^2, \quad \forall u \in Tx,\ v \in Ty. \tag{1.3}$$

We observe that (1.3) implies that $\forall\, x, y \in K$,

$$D^2(Tx, Ty) \leq \|x - y\|^2 + \|x - y - (u - v)\|^2, \quad \forall u \in Tx,\ v \in Ty, \tag{1.4}$$

known as *pseudocontractive-type multi-valued* mapping (see, [30]).

Now we give an example of pseudocontractive multi-valued mapping.

Example 1.1. Define $T : \mathbb{R} \to CB(\mathbb{R})$ by

$$Tx := \begin{cases} x + 1, & x < 0, \\ [-1, 1], & x = 0, \\ x - 1, & x > 0. \end{cases}$$

We observe that $F(T) = \{0\}$. One can easily show that T is pseudocontractive multi-valued mapping.

A mapping $T : K \to CB(H)$ is said to be *k-strongly pseudocontractive* (see [18,19]), if there exists $k \in (0, 1)$ such that the inequality

$$\langle u - v, x - y\rangle \leq k\|x - y\|^2, \tag{1.5}$$

holds for each $x, y \in K, u \in Tx,\ v \in Ty$. The following is an example of k-strongly pseudocontractive multi-valued mapping.

Example 1.2. Define $T : \mathbb{R} \to CB(\mathbb{R})$ by

$$Tx := \begin{cases} \{1\}, & x < 0, \\ [-1, 1], & x = 0, \\ \{-1\}, & x > 0. \end{cases}$$

Clearly, $F(T) = \{0\}$. One can easily show that T is a k-strongly pseudocontractive multi-valued mapping.

Remark 1.3. Note that the class of pseudocontractive multi-valued mappings includes the class of k-strongly pseudocontractive multi-valued mappings. The following example shows that the inclusion is proper.

Example 1.4. The mapping T given in Example 1.1 is a pseudocontractive mapping which is not k-strongly pseudocontractive multi-valued mapping. To see this, take $x = -3$ and $y = -2$. Then $u = -2, v = -1$, and $\langle u - v, x - y\rangle = 1 = |x - y|^2$. Hence, there is no $k \in [0, 1)$ such that $\langle u - v, x - y\rangle \leq k|x - y|^2, \quad \forall u \in Tx,\ v \in Ty$. Therefore, T is not k-strongly pseudocontractive mapping.

Definition 1.5. Let E be a Banach space. Let $T : D(T) \subseteq E \to 2^E$ be a multi-valued mapping. $I - T$ is said to be demiclosed at zero, if for any sequence $\{x_n\} \subseteq D(T)$ such that $\{x_n\}$ converges weakly to p and $D(\{x_n\}, Tx_n) \to 0$, then $p \in Tp$.

Multi-valued Pseudocontractive mappings are also related with the important class of nonlinear monotone mappings, where $A : K \to CB(H)$ is called *monotone*, if for any $x, y \in K$,

$$\langle u - v, x - y\rangle \geq 0, \quad \forall u \in Ax, v \in Ay. \tag{1.6}$$

A mapping $A : K \to CB(H)$ is said to be *k-strongly monotone* mapping if for all $x, y \in K$, there exists $k \in [0, 1)$, such that

$$\langle u - v, x - y\rangle \geq k\|x - y\|^2, \quad \forall u \in Ax,\ v \in Ay. \tag{1.7}$$

We note that T is pseudocontractive if and only if $A := I - T$ is monotone and hence $x \in F(T)$ if and only if $x \in N(A) := \{x \in K : 0 \in Ax\}$.

Existence of fixed points of multi-valued contractions and nonexpansive mappings via the Hausdorff metric have been proved by several authors (See for instance, Markin [17], Nadler [20], Lim [15]). Since then, the theory for nonexpansive and their generalizations has developed greatly with applications in control theory, convex optimization, differential inclusion and economics (see, for example, [9] and references therein). For early results involving fixed points of multi-valued mappings and their applications see, for example, Brouwer [2], Daffer [4], Downing and Kirk [6], Geanakoplos [8], Kakutani [12], Nash [21, 22], Cholamjiak *et al.* [3], Khan *et al.* [13], Woldeamanuel *et al.* [30] and the references therein.

In [11], Jung and Morales established a convergence theorem of *Mann-type sequence* to a unique fixed point of k-strongly pseudocontractive multi-valued mapping.

In [30], Woldeamanuel *et al.* proved that for a Lipschitz pseudocontractive-type mapping $T : K \to CB(K)$, where K is a nonempty closed and convex subset of a real Hilbert space, the sequence $\{x_n\}$ generated from an arbitrary $x_1 = w \in K$ by the scheme,

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n u_n, \ u_n \in Tx_n, \\ z_n = \gamma_n w_n + (1-\gamma_n)x_n, \ w_n \in Ty_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n)z_n, \ n \geq 1 \end{cases} \tag{1.8}$$

converges strongly to a fixed point of the mapping, under some conditions on the parameters, provided $I - T$ is demiclosed at zero. However, we observe that this demiclosed condition is strong.

Motivated by the above results, it is our purpose in this paper to prove strong convergence of Scheme (1.8) to a fixed point of a Lipschitz pseudocontractive mapping $T : K \to CB(K)$, under some mild conditions, where K is a nonempty closed and convex subset of a real Hilbert space H, without the assumption that $I - T$ is demiclosed at zero. The assumption that $T(p) = \{p\}, \forall p \in F(T)$ is not required. Our work improves most of the results that have been proved for the multi-valued case.

Let K be a subset of a real Hilbert space H. The following notations will be used in the sequel:

i. $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
ii. Given a closed convex subset K of real Hilbert space H, P_K denotes the nearest point projection from H onto K, that is, $P_K x$ is the unique point in K with the property $\|x - P_K x\| \leq \|x - y\|$, for all $y \in K$.

2. Preliminaries

We first recall some definitions, notations and results which will be needed in proving our main results.

Lemma 2.1 ([29]). *Let H be a real Hilbert space. Then, Given any x, y in H, the following equations hold:*

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \forall t \in [0,1],$$

Lemma 2.2. *[10] Let H be a real Hilbert space. Then, the following equation holds: If $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup z \in H$, then*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2, \forall y \in H.$$

Lemma 2.3 ([1]). *Let K be a nonempty, closed and convex subset of a real Hilbert space H. If $x \in H$ and $z \in K$, then, $z = P_K(x)$ if and only if $\langle x - z, y - z\rangle \leq 0, \forall y \in K$.*

Lemma 2.4 ([31]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n, n \geq n_0,$$

where $\{\alpha_n\} \subset (0,1)$ *and* $\{\delta_n\} \subset R$ *satisfying the following conditions:* $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\limsup_{n\to\infty} \delta_n \leq 0$. *Then,* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 ([16])**.** *Let* $\{a_n\}$ *be a sequence of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $a_{n_i} < a_{n_i+1}$, *for all* $i \in \mathbb{N}$. *Then, there exists a nondecreasing sequence* $\{m_k\} \subset \mathbb{N}$ *such that* $m_k \to \infty$ *and the following properties are satisfied by all (sufficiently large) numbers* $k \in \mathbb{N}$:

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}$$

In fact, $m_k := \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.6. *Let* H *be a real Hilbert space. Then,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H$$

Lemma 2.7 ([20])**.** *Let* K *be a real Hilbert space* H. *Let* $A, B \in CB(H)$ *and* $a \in A$. *If* $\gamma > 0$, *then there exists* $b \in B$ *such that* $D(a,b) \leq D(A,B) + \gamma$.

3. Main Results

Lemma 3.1. *Let* H *be a real Hilbert space. Suppose* K *is a closed, convex, nonempty subset of* H. *Assume that* $T : K \to CB(K)$ *is pseudocontractive multi-valued mapping with* $F(T) \neq \emptyset$. *Then,* $F(T)$ *is closed and convex.*

Proof. For $\lambda > 0$, define $J_\lambda : K \to K$ by $J_\lambda := (I + \lambda(I-T))^{-1}$, where J_λ is the resolvent of $A := I - T$. It is known that J_λ is a single-valued nonexpansive mapping, defined on the range of $I + \lambda(I-T)$ and hence $F(J_\lambda)$ is closed and convex (see, [23, 32]). Thus, since $F(T) \neq \emptyset$, we only need to show that $F(J_\lambda) = F(T)$. Now,

$$\begin{aligned} p \in F(J_\lambda) &\Leftrightarrow J_\lambda p = p \\ &\Leftrightarrow p = (I + \lambda(I-T))^{-1} p \\ &\Leftrightarrow p \in \big(I + \lambda(I-T)\big) p \\ &\Leftrightarrow 0 \in \lambda(I-T)p \\ &\Leftrightarrow p \in Tp \end{aligned}$$

Thus, $F(J_\lambda) = F(T)$, which shows that $F(T)$ is closed and convex. □

Lemma 3.2. *Let* H *be a real Hilbert space. Suppose* K *is a closed, convex, nonempty subset of* H. *Assume that* $T : K \to CB(K)$ *is Lipschitz pseudocontractive multi-valued mapping. Then, there is a single-valued nonexpansive mapping* $S : K \to K$, *such that for some* $\lambda > 0$ *and for any* $y \in K$, $S(y)$ *is a fixed point of* $T_y(x) := (1-\lambda)y + \lambda Tx$.

Proof. Let L be the Lipschitz constant of T, and choose $0 < \lambda < \frac{1}{2(L+1)}$. For each $y \in K$, define the mapping $T_y : K \to CB(K)$ by $T_y(x) := (1-\lambda)y + \lambda Tx$. Then, for any $x, z \in K$,

$$D(T_y(x), T_y(z)) \quad = \quad \max\{\sup_{u \in T_y} x \inf_{v \in T_y} z \|(1-\lambda)y + \lambda u - ((1-\lambda)y + \lambda v)\|,$$

$$
\begin{aligned}
& \sup_{w\in T_y} z \inf_{t\in T_y} x\|(1-\lambda)y+\lambda w-((1-\lambda)y+\lambda t)\|\} \\
= \;& \max\{\sup_{u\in T_y} x \inf_{v\in T_y} z\|\lambda u-\lambda v\|, \sup_{w\in T_y} z \inf_{t\in T_y} x\|\lambda w-\lambda t\|\} \\
= \;& \lambda \max\{\sup_{u\in Tx}\inf_{v\in Tz}\|u-v\|, \sup_{w\in Tz}\inf_{t\in Tx}\|w-t\|\} \\
= \;& \lambda D(Tx, Tz) \\
\leq \;& \lambda L\|x-z\| \\
\leq \;& \frac{L}{2(L+1)}\|x-z\|
\end{aligned}
$$

Put $k=\frac{L}{2(L+1)}$. Then, $k\in(0,1)$, which makes T_y a multi-valued contraction. Now, as K is closed and convex, by Nadler's fixed point Theorem [20], T_y has a fixed point in K, say $S(y)$, i.e., $S(y)\in(1-\lambda)y+\lambda T(S(y))$. Notice that for any $y\in K$, there exists $v\in T(S(y))$ such that $S(y)=(1-\lambda)y+\lambda v\in K$. Using the assumption that T is pseudocontractive, we next show that S is single-valued nonexpansive mapping. If $x,y\in K$, there exists $u\in T(S(x))$, $v\in T(S(y))$ such that $S(x)=(1-\lambda)x+\lambda u$ and $S(y)=(1-\lambda)y+\lambda v$. Thus,

$$
\begin{aligned}
\|S(x)-S(y)\|^2 &= \langle S(x)-S(y), S(x)-S(y)\rangle \\
&= \langle (1-\lambda)(x-y)+\lambda(u-v), S(x)-S(y)\rangle \\
&= (1-\lambda)\langle x-y, S(x)-S(y)\rangle+\lambda\langle u-v, S(x)-S(y)\rangle \\
&\leq (1-\lambda)\|x-y\|\|S(x)-S(y)\|+\lambda\|S(x)-S(y)\|^2
\end{aligned}
$$

This gives,
$(1-\lambda)\|S(x)-S(y)\|^2\leq(1-\lambda)\|x-y\|\|S(x)-S(y)\|$,
i.e.,

$$\|S(x)-S(y)\|\leq\|x-y\|,$$

which shows that S single-valued nonexpansive mapping.

□

Lemma 3.3. *Let H be a real Hilbert space. Suppose K is a closed, convex, nonempty subset of H. Assume that $T: K\to CB(K)$ is Lipschitz pseudocontractive multi-valued mapping. Then $I-T$ is demiclosed at zero.*

Proof. Let $\{x_n\}\subseteq K$ be such that $x_n\rightharpoonup p$ and suppose $D(\{x_n\}, Tx_n)\to 0$. We want to show that $0\in(I-T)p$, i.e., $p\in Tp$. Let $y_n\in Tx_n$, be such that

$$\|x_n-y_n\|\leq D(\{x_n\}, Tx_n)\to 0. \tag{3.1}$$

Now, define $f: H\to[0,\infty)$ by $f(x):=\limsup\limits_{n\to\infty}\|x_n-x\|^2$. Then, by Lemma 2.2 we get that

$$f(x)=\limsup\|x_n-p\|^2+\|p-x\|^2,\ \forall x\in H.$$

which implies that

$$f(x)=f(p)+\|p-x\|^2, \quad \forall x\in H. \tag{3.2}$$

In particular, for S as in Lemma 3.2 we get that

$$f(S(p))=f(p)+\|S(p)-p\|^2. \tag{3.3}$$

From the definition of S, we have that $S(x_n) = (1-\lambda_n)x_n + \lambda_n u_n$, for some $u_n \in T(S(x_n))$. But then, by Lemma 2.7 there exists $y_n \in Tx_n$ such that $||u_n - y_n|| \leq 2D(T(Sx_n), Tx_n)$. Thus, we have

$$\begin{aligned}
\|x_n - S(x_n)\| &= \lambda\|x_n - u_n\| \\
&= \lambda\|x_n - y_n + y_n - u_n\| \\
&\leq \lambda\|x_n - y_n\| + \lambda\|y_n - u_n\| \\
&\leq \lambda\|x_n - y_n\| + 2\lambda D(Tx_n, T(S(x_n)) \\
&\leq \lambda\|x_n - y_n\| + 2\lambda L\|x_n - S(x_n)\| \\
&\leq \lambda\|x_n - y_n\| + a\|x_n - S(x_n)\|,
\end{aligned}$$

for $a = \frac{L}{(1+L)}$. This gives that $(1-a)\|x_n - S(x_n)\| \leq \lambda\|x_n - y_n\|$, which implies,

$$\|x_n - S(x_n)\| \leq \frac{\lambda}{1-a}\|x_n - y_n\| \leq \frac{\lambda}{1-a}D(\{x_n\}, Tx_n) \to 0, \tag{3.4}$$

as $n \to \infty$. Also, using (3.4), and the fact that S is nonexpansive, from Lemma 3.2,

$$\begin{aligned}
f(S(p)) &= \limsup \|x_n - S(p)\|^2 \\
&= \limsup \|x_n - S(x_n) + S(x_n) - S(p)\|^2 \\
&\leq \limsup(\|x_n - S(x_n)\| + \|S(x_n) - S(p)\|)^2 \\
&\leq \limsup(\|x_n - S(x_n)\| + \|x_n - p\|)^2 \\
&\leq \limsup \|x_n - p\|^2 = f(p). \qquad (3.5)
\end{aligned}$$

Now, from (3.3) and (3.5), we get $\|S(p) - p\|^2 = 0$ which implies $p = S(p)$, i. e., $p \in F(S)$. It is easy to see that $F(S) = F(T)$, so, we get that $p \in Tp$. Therefore, $I - T$ is demiclosed at zero. $\square$

Theorem 3.4. *Let H be a real Hilbert space and K be a non-empty, closed and convex subset of H. Let $T : K \to CB(K)$ be a Lipschitz pseudocontractive multi-valued mapping with Lipschitz constant L. Assume that $F(T)$ is non-empty. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 = w \in K$ by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n u_n, \\ z_n = \gamma_n w_n + (1-\gamma_n)x_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n)z_n, \ n \geq 1, \end{cases} \tag{3.6}$$

where $u_n \in Tx_n, w_n \in Ty_n$ such that $||u_n - w_n|| \leq 2D(Tx_n, Ty_n)$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ satisfy the following conditions:

(i) $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \dfrac{1}{\sqrt{4L^2+1}+1}$, $\forall n \geq 1$.

Then, $\{x_n\}$ converges strongly to some point $p \in F(T)$ nearest to w.

Proof. Let $p = P_{F(T)}(w)$. Now, using Lemma 2.1 we get that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n(w-p) + (1-\alpha_n)(z_n - p)\|^2 \\
&\leq \alpha_n\|w-p\|^2 + (1-\alpha_n)\|z_n - p\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_n\|w-p\|^2+(1-\alpha_n)\|\gamma_n(w_n-p)+(1-\gamma_n)(x_n-p)\|^2\\
&= \alpha_n\|w-p\|^2+(1-\alpha_n)\gamma_n\|w_n-p\|^2+(1-\alpha_n)(1-\gamma_n)\\
&\quad \times\|x_n-p\|^2-(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n-x_n\|^2\\
&\le \alpha_n\|w-p\|^2+(1-\alpha_n)(1-\gamma_n)\|x_n-p\|^2+(1-\alpha_n)\gamma_n\\
&\quad \left[\|y_n-p\|^2+\|y_n-p-(w_n-p)\|^2\right]\\
&\quad -(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n-x_n\|^2\\
&= \alpha_n\|w-p\|^2+(1-\alpha_n)(1-\gamma_n)\|x_n-p\|^2+(1-\alpha_n)\gamma_n\\
&\quad \left[\|y_n-p\|^2+\|y_n-w_n\|^2\right]-(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n-x_n\|^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(3.7)\quad \|x_{n+1}-p\|^2 &\le \alpha_n\|w-p\|^2+(1-\alpha_n)(1-\gamma_n)\|x_n-p\|^2+(1-\alpha_n)\\
&\quad \times\gamma_n\|y_n-p\|^2+(1-\alpha_n)\gamma_n\|y_n-w_n\|^2-(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n-x_n\|^2.
\end{aligned}
$$

On the other hand, using (3.6), the fact that $\|u_n-w_n\|\le 2D(Tx_n,Ty_n)$, Lemma 2.1 and T is Lipschitz,

$$
\begin{aligned}
\|y_n-w_n\|^2 &= \|(1-\beta_n)(x_n-w_n)+\beta_n(u_n-w_n)\|^2\\
&= (1-\beta_n)\|x_n-w_n\|^2+\beta_n\|u_n-w_n\|^2-\beta_n(1-\beta_n)\|x_n-u_n\|^2\\
&\le (1-\beta_n)\|x_n-w_n\|^2+\beta_n 4D(Tx_n,Ty_n)^2-\beta_n(1-\beta_n)\|x_n-u_n\|^2\\
&\le (1-\beta_n)\|x_n-w_n\|^2+\beta_n 4L^2\|x_n-y_n\|^2-\beta_n(1-\beta_n)\|x_n-u_n\|^2\\
&= (1-\beta_n)\|x_n-w_n\|^2+4\beta_n^3L^2\|x_n-u_n\|^2-\beta_n(1-\beta_n)\|x_n-u_n\|^2.
\end{aligned}
$$

Hence,

$$
(3.8)\qquad \|y_n-w_n\|^2\le(1-\beta_n)\|x_n-w_n\|^2-\beta_n(1-\beta_n-4L^2\beta_n^2)\|x_n-u_n\|^2
$$

Again, using the assumption that T is pseudocontractive,

$$
\begin{aligned}
\|y_n-p\|^2 &= \|(1-\beta_n)x_n+\beta_n u_n-p)\|^2\\
&= \|(1-\beta_n)(x_n-p)+\beta_n(u_n-p)\|^2\\
&= (1-\beta_n)\|x_n-p\|^2+\beta_n\|u_n-p\|^2-\beta_n(1-\beta_n)\|x_n-u_n\|^2\\
&\le (1-\beta_n)\|x_n-p\|^2+\beta_n\left[\|x_n-p\|^2+\|x_n-u_n\|^2\right]\\
&\quad -\beta_n(1-\beta_n)\|x_n-u_n\|^2\\
&= \|x_n-p\|^2+\beta_n^2\|x_n-u_n\|^2.
\end{aligned}
$$

Thus,

$$
(3.9)\qquad \|y_n-p\|^2\le\|x_n-p\|^2+\beta_n^2\|x_n-u_n\|^2.
$$

Now, substituting (3.8), (3.9) into (3.7),

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\le \alpha_n\|w-p\|^2+(1-\alpha_n)(1-\gamma_n)\|x_n-p\|^2+(1-\alpha_n)\gamma_n\|x_n-p\|^2\\
&\quad +(1-\alpha_n)\gamma_n\beta_n^2\|x_n-u_n\|^2+(1-\alpha_n)\gamma_n(1-\beta_n)\|x_n-w_n\|^2\\
&\quad -\beta_n(1-\alpha_n)\gamma_n(1-\beta_n-4L^2\beta_n^2)\|u_n-x_n\|^2\\
&\quad -(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n-x_n\|^2,
\end{aligned}
$$

which reduces to

$$\|x_{n+1}-p\|^2 \le \alpha_n\|w-p\|^2+(1-\alpha_n)\|x_n-p\|^2-\beta_n(1-\alpha_n) \times \gamma_n(1-2\beta_n-4L^2\beta_n^2)\|u_n-x_n\|^2+(1-\alpha_n)\gamma_n(\gamma_n-\beta_n)\|x_n-w_n\|^2 \tag{3.10}$$

From the hypothesis (ii) in (3.6) we have that

$$1-2\beta_n-4L^2\beta_n^2 \ge 1-2\beta-4L^2\beta^2 \tag{3.11}$$

$$\gamma_n \le \beta_n. \tag{3.12}$$

Using (3.11) and (3.12) in (3.10) we get that

$$\|x_{n+1}-p\|^2 \le (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|w-p\|^2. \tag{3.13}$$

Thus, by induction

$$\|x_{n+1}-p\|^2 \le \max\{\|x_1-p\|^2, \|w-p\|^2\},\ \forall n \ge 1.$$

This implies that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are all bounded.

Furthermore, from (3.6), Lemma 2.6 and (3.10) we get that

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
= {}& \|(1-\alpha_n)\left(\gamma_n w_n+(1-\gamma_n)x_n\right)+\alpha_n w-p)\|^2\\
= {}& \|(1-\alpha_n)\left((\gamma_n w_n+(1-\gamma_n)x_n)-p\right)+\alpha_n(w-p)\|^2\\
\le {}& (1-\alpha_n)\|\gamma_n w_n+(1-\gamma_n)x_n-p\|^2+2\alpha_n\langle w-p, x_{n+1}-p\rangle\\
= {}& (1-\alpha_n)\left[\gamma_n\|w_n-p\|^2+(1-\gamma_n)\|x_n-p\|^2-\gamma_n(1-\gamma_n)\|x_n-w_n\|^2\right]\\
&+2\alpha_n\langle w-p, x_{n+1}-p\rangle\\
\le {}& (1-\alpha_n)\big[\gamma_n(\|y_n-p\|^2+\|y_n-w_n\|^2)+(1-\gamma_n)\|x_n-p\|^2\\
&-\gamma_n(1-\gamma_n)\|x_n-w_n\|^2\big]+2\alpha_n\langle w-p, x_{n+1}-p\rangle\\
\le {}& (1-\alpha_n)\gamma_n\|y_n-p\|^2+(1-\alpha_n)\gamma_n\|y_n-w_n\|^2+(1-\alpha_n)(1-\gamma_n)\\
&\times\|x_n-p\|^2-(1-\alpha_n)\gamma_n(1-\gamma_n)\|x_n-w_n\|^2+2\alpha_n\langle w-p, x_{n+1}-p\rangle\\
\le {}& (1-\alpha_n)\gamma_n\|x_n-p\|^2+(1-\alpha_n)\gamma_n\beta_n^2\|x_n-u_n\|^2+(1-\alpha_n)\gamma_n\\
&\times\left[(1-\beta_n)\|x_n-w_n\|^2-\beta_n(1-\beta_n-4L^2\beta_n^2)\|x_n-u_n\|^2\right]\\
&+(1-\alpha_n)(1-\gamma_n)\|x_n-p\|^2-(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n-x_n\|^2\\
&+2\alpha_n\langle w-p, x_{n+1}-p\rangle\\
\le {}& (1-\alpha_n)\|x_n-p\|^2-(1-\alpha_n)\gamma_n\beta_n(1-2\beta_n-4L^2\beta_n^2)\|x_n-u_n\|^2\\
&+2\alpha_n\langle w-p, x_{n+1}-p\rangle+(1-\alpha_n)\gamma_n(\gamma_n-\beta_n)\|x_n-w_n\|^2.
\end{aligned}$$

This implies that,

$$\begin{aligned}
\|x_{n+1}-p\|^2 \le {}& (1-\alpha_n)\|x_n-p\|^2-(1-\alpha_n)\gamma_n\beta_n(1-2\beta_n-4L^2\beta_n^2)\\
&\times\|x_n-u_n\|^2+2\alpha_n\langle w-p, x_{n+1}-p\rangle,
\end{aligned} \tag{3.14}$$

and hence by (i) and (ii) we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 \le {}& (1-\alpha_n)\|x_n-p\|^2-(1-c)\alpha^2(1-2\beta-4L^2\beta^2)\|x_n-u_n\|^2\\
&+2\alpha_n\langle w-p, x_{n+1}-p\rangle.
\end{aligned} \tag{3.15}$$

Now we consider the following two cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - p\|\}$ is non-increasing, $\forall n \geq n_0$. Then, we get that $\{\|x_n - p\|\}$ is convergent. So, from (3.15) we have that

$$\begin{aligned}(1-c)\alpha^2(1-2\beta-4L^2\beta^2)\|x_n - u_n\|^2 &\leq (1-\alpha_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\quad +2\alpha_n\langle w - p, x_{n+1} - p\rangle.\end{aligned}$$

Thus, from the fact that $\alpha_n \to 0$, we get $\lim_{n\to\infty} \|x_n - u_n\| = 0$ which implies that

$$d(x_n, Tx_n) \leq ||x_n - u_n|| \to 0 \text{ as } n \to \infty. \tag{3.16}$$

Now, from (3.6)

$$y_n - x_n = \beta_n(u_n - x_n) \to 0,$$

and hence we get that

$$\begin{aligned}\|z_n - x_n\| &= \gamma_n\|w_n - x_n\| = \gamma_n\|w_n - u_n + u_n - x_n\| \\ &\leq \gamma_n\|w_n - u_n\| + \gamma_n\|u_n - x_n\| \\ &\leq 2\gamma_n D(Ty_n, Tx_n) + \gamma_n\|u_n - x_n\| \\ &\leq 2\gamma_n L\|y_n - x_n\| + \gamma_n\|u_n - x_n\| \to 0. \qquad (3.17)\end{aligned}$$

Thus, from (3.6), (3.17), the fact that $\|w - z_n\|$ is bounded and $\alpha_n \to 0$, we obtain

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|x_{n+1} - z_n + z_n - x_n\| \\ &\leq \|x_{n+1} - z_n\| + \|z_n - x_n\| \\ &= \alpha_n\|w - z_n\| + \|z_n - x_n\| \to 0. \qquad (3.18)\end{aligned}$$

Now, since $\{\|x_n - p\|\}$ is bounded there exists a subsequence $\{x_{n_j+1}\}$ of $\{x_{n+1}\}$ such that

$$\limsup_{n\to\infty}\langle w - p, x_{n+1} - p\rangle = \lim_{j\to\infty}\langle w - p, x_{n_j+1} - p\rangle,$$

and $x_{n_j+1} \rightharpoonup z$, for some $z \in K$. Now, from (3.18) we get $x_{n_j} \rightharpoonup z$. Hence, from (3.16) and the fact that $I - T$ is demiclosed by Lemma 3.3, we get that $z \in F(T)$. Therefore, by Lemma 2.3 we obtain that

$$\begin{aligned}\limsup_{n\to\infty}\langle w - p, x_{n+1} - p\rangle &= \lim_{j\to\infty}\langle w - p, x_{n_j+1} - p\rangle \\ &= \langle w - p, z - p\rangle \leq 0 \qquad (3.19)\end{aligned}$$

Now, from (3.15) we have that

$$\|x_{n+1} - p\|^2 \leq (1-\alpha_n)\|x_n - p\|^2 + 2\alpha_n\langle w - p, x_{n+1} - p\rangle. \tag{3.20}$$

It then follows from (3.20), (3.19) and Lemma 2.4 that $\|x_n - p\| \to 0$ i.e., $x_n \to p$.

Case 2. Suppose there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\|x_{n_k} - p\| < \|x_{n_k+1} - p\|, \ \forall k \in \mathbb{N}.$$

Thus, by Lemma 2.5, there is a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, $\|x_{m_k} - p\| \leq \|x_{m_k+1} - p\|$ and $\|x_k - p\| \leq \|x_{m_k+1} - p\|$, $\forall k \in \mathbb{N}$. Now, from

(3.15) and the fact that $\alpha_n \to 0$ we get that $x_{m_k} - u_{m_k} \to 0$, when $u_{m_k} \in Tx_{m_k}$. Hence as in Case 1, $x_{m_k+1} - x_{m_k} \to 0$ and that

$$\limsup_{k\to\infty} \langle w - p, x_{m_k+1} - p\rangle \leq 0. \tag{3.21}$$

From (3.15) we have that

$$\|x_{m_k+1} - p\|^2 \leq (1 - \alpha_{m_k})\|x_{m_k} - p\|^2 + 2\alpha_{m_k}\langle w - p, x_{m_k+1} - p\rangle \tag{3.22}$$

and since $\|x_{m_k} - p\| \leq \|x_{m_k+1} - p\|$, (3.22) implies that

$$\begin{aligned} \alpha_{m_k}\|x_{m_k} - p\|^2 &\leq \|x_{m_k} - p\|^2 - \|x_{m_k+1} - p\|^2 + 2\alpha_{m_k}\langle w - p, x_{m_k+1} - p\rangle \\ &\leq 2\alpha_{m_k}\langle w - p, x_{m_k+1} - p\rangle, \end{aligned}$$

which implies

$$\|x_{m_k} - p\|^2 \leq 2\langle w - p, x_{m_k+1} - p\rangle.$$

So, from (3.21) we get that $\|x_{m_k} - p\| \to 0 \leq 0$ and hence this with (3.22) give that $\|x_{m_k+1} - p\| \to 0$. But, $\|x_k - p\| \leq \|x_{m_k+1} - p\|$, $\forall k \in \mathbb{N}$. Thus, $x_k \to p$. Therefore, $\{x_n\}$ converges strongly to some point $p \in F(T)$ nearest to w. □

Remark 3.5. We note that, since every Lipschitz k-strongly pseudocontractive multi-valued mapping is Lipschitz pseudocontractive multi-valued mapping the above theorem holds for a Lipschitz k-strongly pseudocontractive multi-valued mapping.

If, in Theorem 3.4 we assume that P_T is Lipschitz pseudocontractive multi-valued mapping, then we get the following corollary.

Corollary 3.6. *Let H be a real Hilbert space and K be a non-empty, closed and convex subset of H. Let $T : K \to CB(K)$ be a multi-valued mapping. Let P_T be a Lipschitz pseudocontractive mapping with Lipschitz constant L. Suppose also that $F(T)$ is non-empty. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 = w \in K$ by*

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n u_n, \\ z_n = \gamma_n w_n + (1 - \gamma_n)x_n, \\ x_{n+1} = \alpha_n w + (1 - \alpha_n)z_n, \; n \geq 1 \end{cases} \tag{3.23}$$

where $u_n \in P_T x_n, w_n \in P_T y_n$ such that $||u_n - w_n|| \leq 2D(P_T x_n, P_T y_n)$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0, 1)$ satisfy the following conditions:

i. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii. $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \dfrac{1}{\sqrt{4L^2 + 1} + 1}$, $\forall n \geq 1$.

Then, $\{x_n\}$ converges strongly to some point $p \in F(T)$ nearest to w.

If, in Theorem 3.4 we assume that $P_T : K \to CBC(K)$ is Lipschitz pseudocontractive mapping, then $P_T(x)$ is singleton and hence the following corollary follows.

Corollary 3.7. *Let H be a real Hilbert space and K be a non-empty, closed and convex subset of H. Let $T : K \to CBC(K)$, be a multi-valued mapping. Let P_T be a Lipschitz pseudocontractive mappings with Lipschitz constant L. Suppose also that $F(T)$ is non-empty. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 = w \in K$ by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n P_T x_n, \\ z_n = \gamma_n P_T y_n + (1-\gamma_n)x_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n)z_n, \ n \geq 1 \end{cases} \tag{3.24}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ satisfy the following conditions:

i. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii. $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \dfrac{1}{\sqrt{4L^2+1}+1}$, $\forall n \geq 1$.

Then, $\{x_n\}$ converges strongly to some point $p \in F(T)$ nearest to w.

Next we state and prove a convergence theorem for a zero of a monotone mapping.

Theorem 3.8. *Let H be a real Hilbert space. Let $A : H \to CB(H)$ be a Lipschitz monotone mapping with Lipschitz constant L. Assume $N(A) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 = w \in H$ by*

$$\begin{cases} y_n = x_n - \beta_n u_n, \\ z_n = x_n - \gamma_n w_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n)z_n, \ n \geq 1 \end{cases} \tag{3.25}$$

where $u_n \in Ax_n, w_n \in Ay_n$ such that $||u_n - w_n|| \leq 2D(Ax_n, Ay_n) + ||x_n - y_n||$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ satisfy the following conditions:

i. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii. $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \dfrac{1}{\sqrt{4L'^2+1}+1}$, $\forall n \geq 1$ *for* $L' := 1 + L$.

Then, $\{x_n\}$ converges strongly to a zero point of A nearest to w.

Proof. Let $Tx := (I - A)x$. Then T is Lipschitz pseudocontractive mapping with Lipschitz constant $L' := (1 + L)$ and $F(T) = N(A) \neq \emptyset$. Now replacing A with $(I - T)$ in (3.25) we get Scheme (3.6). Hence the result follows from Theorem 3.4. $\square$

Remark 3.9. Our work improves Theorem 1 and Theorem 2 of Song and Wang [28] and Theorem 2.7 of Shahzad and Zegeye [26] and extends the work of Daman and Zegeye [5] for the multi-valued case. In all our results the assumption that $T(p) = \{p\}, \forall p \in F(T)$ is not required.

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